Simulation of Weakly Correlated Functions and its Application to Random Surfaces and Random Polynomials*

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Abstract

The paper is dedicated to the modeling and the simulation of random processes and fields. Using the concept and the theory of weakly correlated functions a consistent representation of sufficiently smooth random processes will be derived. Special applications will be given with respect to the simulation of road surfaces in vehicle dynamics and to the confirmation of theoretical results with respect to the zeros of random polynomials.

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1 Introduction

The concept of so-called weakly correlated functions $f_\varepsilon(x,\omega)$, $x \in R^n$ and $\omega \in \Omega$, as random functions "without distant effect" can be used for the modeling and simulation of random variables, processes and fields. In this paper this will be explicitely demonstrated for random surfaces, e. g. of roads, as well as for random coefficients of algebraic polynomials.

Weakly correlated functions are zero-mean random functions $f_\varepsilon(x,\omega)$ having decomposition properties with respect to all their stochastic moments $\langle f_\varepsilon(x_1)f_\varepsilon(x_2)\dots f_\varepsilon(x_k)\rangle$, $k = 2, 3, \dots$, (cf. [3]). Especially, the values at two points x and y do not influence each other if the distance $|x-y|$ exceeds the so-called correlation length $\varepsilon > 0$, i. e. the correlation function $R(x,y)$ vanishes if $|x-y| > \varepsilon$. A typical example of a correlation function of weakly correlated processes is given by the following hat-like function

$$R(x,y) = \langle f_\varepsilon(x)f_\varepsilon(y)\rangle = \begin{cases} \sigma^2(1-\frac{|x-y|}{\varepsilon}) & \text{for } |x-y| \le \varepsilon \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{1}$$

Firstly, in section 2 a general model for smooth weakly correlated processes $f_\varepsilon(x,\omega)$ is described which can be easily used for simulation and be extended to fields $f_\varepsilon(x_1,...,x_n,\omega)$. A subsequent analysis provides necessary stochastic characteristics for fitting the model to real data.

Considering additionally linear functionals of the form

$$f(t,\omega) = \int_{D(t)} G(t,x)f_\varepsilon(x,\omega)dx \tag{2}$$

the field of application will be extended in section 3 to the modeling and simulation of random surfaces which can be used e. g. in random vibration systems (cf. [4]) and in section 4 to the solution of algebraic equations

$$a_0(\omega) + a_1(\omega)z + ... + a_n(\omega)z^n = 0 \ ,$$

where especially the accuracy of theoretical approximations of the distribution of the zeros (cf. [2]) will be verified by simulation. Thereby, the following limit theorem proved in [3]

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^n}\langle f(t_1)f(t_2)\rangle = \int_{D(t_1)\cap D(t_2)} G(t_1,x)G(t_2,x)a(x)dx \tag{3}$$

with the so-called intensity $a(x)$ of $f_\varepsilon(x,\omega)$ and expansions of the distributions play an essential rule.

2 Modeling of weakly correlated functions

The simplest way to model weakly correlated processes $f_\varepsilon(x,\omega), x \in [\alpha,\beta]$ is given by a random step function on the decomposition $\{I_i\}_{i=0,1,\ldots,n-1}$ of the interval $[\alpha,\beta]$ in equidistant and disjoint intervals $I_i = [a_i, a_{i+1})$ of length $h = (\beta-\alpha)/n$, where $f_\varepsilon(x,\omega) = \xi_i(\omega)$ for $x \in I_i$ with independent zero-mean random variables $\xi_i(\omega), i = 0,1,\ldots$ Obviously, the so defined process is weakly correlated with correlation length $\varepsilon = h$ but not stationary in the (wide) sense (i. e. $R(x,y) \neq r(x-y)$). Its correlation function for $x \in I_i$ is given by

$$R(x,y) = \langle f_\varepsilon(x) f_\varepsilon(y)\rangle = \begin{cases} <\xi_i^2> & \text{for } \ y \in I_i, \\ 0 & \text{otherwise }. \end{cases}$$

But, considering an "averaged version" of the correlation function on I_i

$$\bar{R}(\tau) = \frac{1}{h}\int\limits_{I_i} R(x, x+\tau)dx =< \xi_i^2 > (1 - \frac{|\tau|}{\varepsilon})$$

it can be stated that the averaged correlation function corresponds for $< \xi_i^2 >= \sigma^2$, $i = 0,1,\ldots$ to a wide-sense stationary process with correlation function (1).

Now, we generalize this model to smooth processes of any given order $K > 0$, where we in the following suppose that the random variables ξ_i, $i = 0,1,\ldots,n$ are identically distributed with variance σ^2. Putting

$$f_\varepsilon(x,\omega) = g_i(x)\xi_i(\omega) + h_i(x)\xi_{i+1}(\omega) \text{ for } x \in I_i \ , \tag{4}$$

where $g_i(x) = p\left(\frac{a_{i+1}-x}{h}\right)$, $h_i(x) = 1 - g_i(x)$ and $p(\cdot)$ are polynomials in $[0,1]$ with $p(0) = 0$, $p(1) = 1$, $p^{(m)}(0) = p^{(m)}(1) = 0$, $m = 1,2,\ldots,K$, a weakly correlated process with correlation length $\varepsilon = 2h$ and a. s. differentiability of order K is obtained. In Fig. 1 trajectories of $f_\varepsilon(x,\omega)$ are drawn with uniformly distributed $\xi_i(\omega)$ on $[-1,1]$. Moreover, the structure of the polynomial $p(x)$ introduced above results in a pointwise convergence (for $K \to \infty$) of the trajectories of $f_\varepsilon(x,\omega)$ to the trajectories of a step function $l(x,\omega)$ with correlation length h (cf. Fig. 1).

To fit such models to real data stochastic characteristics as intensity, variance, (averaged) correlation function and spectral density with respect to the corresponding stationary process are determined. For example, the variance function is given by

$$\left\langle f_\varepsilon^2(x)\right\rangle = (g_i^2(x) + h_i^2(x))\sigma^2 \text{ for } x \in I_i.$$

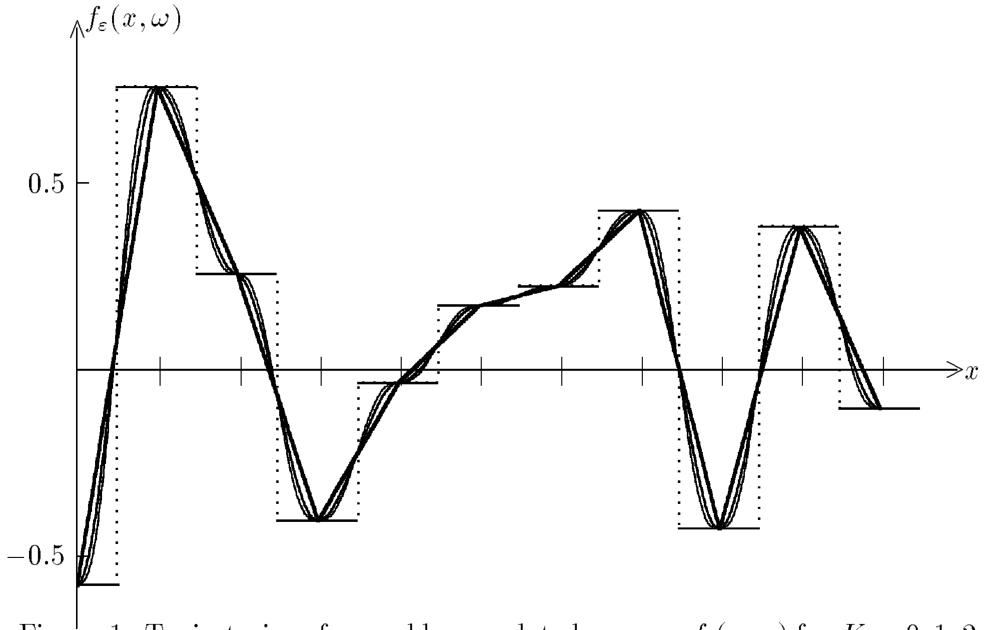

Figure 1: Trajectories of a weakly correlated process $f_\varepsilon(x,\omega)$ for $K = 0, 1, 2$ in comparison with the trajectory of $l(x,\omega)$

Some averaged correlation functions are plotted in Fig. 2, the spectral density e. g. for $K = 0$, where $p(x) = x$, is determined as

$$\bar{S}(\alpha) = \frac{1}{\pi}\int\limits_0^\varepsilon \cos(\alpha\tau)\bar{R}(\tau)d\tau = \frac{8\sigma^2}{\pi\alpha^4\varepsilon^3}\left(\cos(\alpha\varepsilon) - 4\cos(\frac{\alpha\varepsilon}{2}) + 3\right)$$

and the intensity is for every K given by the constant value

$$a = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon} R(x, x+\tau)d\tau = \sigma^2/2 \quad .$$

Further it can be stated that for the derivatives $f_\varepsilon^{(j)}(x,\omega)$, $j = 1, 2, \ldots$, the usual relations

$$\bar{R}_j(\tau) = (-1)^j\bar{R}^{(2j)}(\tau) \text{ and } \bar{S}_j(\alpha) = \alpha^{2j}\bar{S}(\alpha)$$

are fulfilled with respect to the averaged functions $\bar{R}_j$ and $\bar{S}_j$ of the derivatives.

Finally, several extensions of the described model are possible, e. g.

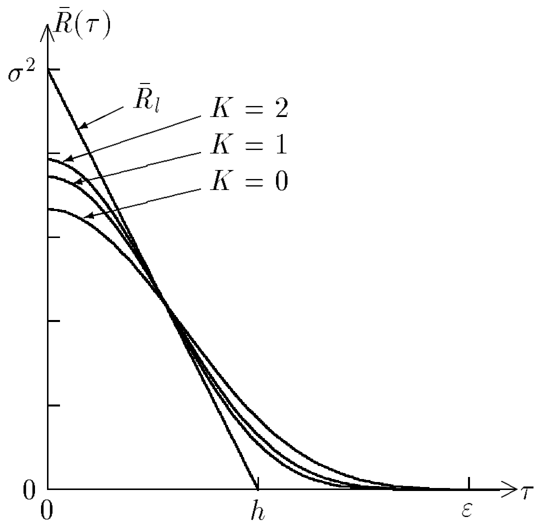

Figure 2: Averaged correlation functions for $K = 0, 1, 2$ in comparison with the correlation function $\bar{R}_l$ of $l(x, \omega)$

- different distributions of the weakly correlated process $f_\varepsilon(x, \omega)$ by simulating different distributions of $\xi_i(\omega)$,
- non-stationary processes $h(x) f_\varepsilon(x, \omega)$ by multiplying by a deterministic non-constant shape function $h(x)$,
- superpositions $h_\varepsilon(x, \omega) = f_{\varepsilon_f}(x, \omega) + g_{\varepsilon_g}(x, \omega)$ with different correlation lengths ε_f and ε_g,
- random fields, e. g.

$$\begin{aligned} f_\varepsilon(x_1, x_2, \omega) &= g_{1i}(x_1) g_{2j}(x_2) \xi_{ij} + g_{1i}(x_1) h_{2j}(x_2) \xi_{ij+1} \\ &\quad + h_{1i}(x_1) g_{2j}(x_2) \xi_{i+1j} + h_{1i}(x_1) h_{2j}(x_2) \xi_{i+1j+1} \end{aligned}$$

using the same ideas with respect to a grid decomposition of a rectangular domain of R^2.

3 Random surfaces

Considering the linear functional (2) in the form

$$f(t,\omega) = \int\limits_{-\infty}^{t} e^{-\gamma(t-x)} f_\varepsilon(x,\omega)dx \tag{5}$$

with a weakly correlated process $f_\varepsilon(x,\omega)$ having a constant intensity a the limit theorem (3) results in

$$\langle f(t_1)f(t_2)\rangle \approx \varepsilon a \int_{-\infty}^{min(t_1,t_2)} e^{-\gamma(t_1+t_2-2x)}dx = \frac{\varepsilon a}{2\gamma} e^{-\gamma|t_2-t_1|} \tag{6}$$

for small values of ε. The obtained approximation is often used as model of random road surfaces (cf. [1]) with the spectral density

$$S(\alpha) = \frac{\varepsilon a}{2\pi(\gamma^2+\alpha^2)} \ .$$

To simulate (5) the infinite integral is truncated and determined as a finite series, i. e.

$$f(t,\omega) \approx \int_{\alpha}^{t} e^{-\gamma(t-x)} f_\varepsilon(x,\omega)dx = \sum_{i=0}^{m-1} c_i e^{-\gamma(t-a_i)} + c_m(t)e^{-\gamma(t-a_m)}$$

where α has to be chosen such that the error of truncation is neglectably small. Further, the values c_i as well as $c_m(t)$ depend on $f_\varepsilon(x,\omega)$ according to (4) and m is given by $t \in [a_m, a_{m+1}]$. To obtain sufficiently smooth realizations of $f(t,\omega)$ the case $K = 1$ is recommended. To simulate derivatives of $f(t,\omega)$, which are necessary in mathematical models of random vibration systems we have for instance

$$\ddot{f}(t,\omega) = \dot{f}_\varepsilon(t,\omega) - \gamma f_\varepsilon(t,\omega) + \gamma^2 \int\limits_{-\infty}^{t} e^{-\gamma(t-x)} f_\varepsilon(x,\omega)dx$$

and the simulation procedure is obviously. A large variety of concrete models and applications to vehicle dynamics can be found in [4]. The property of (5) to be a solution of the differential equation

$$\dot{f}(t,\omega) + \gamma f(t,\omega) = f_\varepsilon(t,\omega)$$

opens another possibility to use the model (4) directly in simulating dynamic systems.

4 Random algebraic polynomials

A random algebraic polynomial of degree n is a random function of the form

$$P_n(z, \omega) = a_0(\omega) + a_1(\omega)z + \ldots + a_n(\omega)z^n, \quad z \in C, \tag{7}$$

where the coefficients $a_i(\omega)$, $i = 0, 1, \ldots, n$, are random variables with $\mathbf{P}(a_n(\omega) \neq 0) = 1$. Such polynomials arise in various applications, for example as characteristic polynomials of random matrices and random difference equations.

Under the condition that the first and second order moments of the random coefficients $a_i(\omega)$ are known,

$$\langle a_i \rangle = a_{i0} \text{ and } \langle a_i a_j \rangle = \sigma_{ij}, \quad i, j = 0, 1, \ldots, n, \tag{8}$$

the aim consists in the approximation of the distribution of the zeros of (7).

The averaged polynomial associated with the random polynomial (7) is the function

$$P_{n0}(z) = a_{00} + a_{10}z + \ldots + a_{n0}z^n, \tag{9}$$

which is obtained by substitution of the random variables $a_i(\omega)$ by their expectations. The zeros of (9) are denoted by z_{k0}, $k = 1, \ldots, n$, and we consider here only the case of simple real zeros of the averaged polynomial.

To avoid large deviations of the zeros $z_k(\omega)$ of (7) from the zeros z_{k0} of (9) it is usefull to demand

$$\sum_{i=0}^{n} (a_i(\omega) - a_{i0})^2 \leq \gamma^2 \text{ a. s.}$$

for a sufficiently small γ. Under these assumptions it is possible to find an enumeration of the zeros in the sense of

$$\sum_{k=1}^{n} |z_{k0} - z_k(\omega)| \to \min \qquad \text{a. s.}$$

To approximate the moments and the distribution of the random zeros the coefficients of the polynomial (7) are substituted by functionals of a weakly correlated process so that the first and second order moments of the original coefficients are kept; i. e. we investigate a substitution polynomial

$$\widetilde{P_n}(z, \omega) = \widetilde{a_0}(\omega) + \widetilde{a_1}(\omega)z + \ldots + \widetilde{a_n}(\omega)z^n, \tag{10}$$

$$\begin{aligned} \text{where} \qquad \widetilde{a_i}(\omega) &= a_{i0} + \int_0^1 F_i(x) f_\varepsilon(x,\omega)\, dx \qquad & i = 0,1,\ldots,n \\ \langle \widetilde{a_i}\,\widetilde{a_j} \rangle &= \sigma_{ij} & i,j = 0,1,\ldots,n. \end{aligned} \tag{11}$$

Using (3) it is possible to choose the non-random functions $F_i(x)$, $i = 0,1,\ldots,n$, in such a way, that (11) and subsequently (8) can be approximately fulfilled for small values of ε (an example is given below).

Applying the theory of weakly correlated functions it is a well-known result, that the normalized differences $\frac{1}{\sqrt{\varepsilon}}(z_k(\omega) - z_{k0})$ between the random zeros and the corresponding zeros of the averaged polynomial are Gaussian distributed as $\varepsilon \downarrow 0$. Expansions (cf. [3]) of probability densities $p_k(u)$ of $(z_k(\omega) - z_{k0})$ with respect to the correlation length $\varepsilon > 0$ lead to the following 2^{nd} order approximation (cf. [2])

$$\begin{aligned} p_k(u) \quad \approx \quad & \frac{d_k}{\sqrt{2\pi}} \exp\left(-\frac{d_k^2}{2}u^2\right) \left[1 + \left(-d_{k1}\, H_1(d_k u) + \frac{1}{2}\, d_{k2}\, H_2(d_k u)\right.\right. \\ & \left.\left. -\frac{1}{6}\, d_{k3}\, H_3(d_k u) + \frac{1}{24}\, d_{k4} H_4(d_k u) + \frac{1}{720}\, d_{k6}\, H_6(d_k u)\right)\right], \end{aligned} \tag{12}$$

where the values of d_k, d_{k1}, d_{k2}, d_{k3}, d_{k4} and d_{k6} for $k = 1,\ldots,n$ can be computed from the given first and second order moments (8) and $H_m(\cdot)$, $m = 0,1,\ldots$, denote the Chebyshev-Hermite polynomials of order m. It should be noted that the approximation (12) is the better the more the coefficients are Gaussian distributed. Now, we verify the described substitution and approximation by simulation.

Example: Let $n = 2$

$$P_2(z,\omega) = z^2 + a_1(\omega) z + a_0(\omega) \tag{13}$$

with Gaussian distributed coefficients $a_0(\omega)$ and $a_1(\omega)$:

$$\begin{pmatrix} a_0(\omega) \\ a_1(\omega) \end{pmatrix} \sim N\left(\begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 0.0100 & 0.0025 \\ 0.0025 & 0.0050 \end{pmatrix} \right)$$

Then, the corresponding averaged polynomial has the zeros $z_{10} = 1$, $z_{20} = 3$ and we consider here explicitely the probability density function $p_1(u)$ of the difference $(z_1(\omega) - 1)$.

Firstly, simulation results of $p_1(u)$ with respect to the original polynomial (13) are compared with simulation results of the probability density $\widetilde{p_1}(u)$ of the corresponding substituted polynomial

$$\widetilde{P_2}(z,\omega) = z^2 + \widetilde{a_1}(\omega) z + \widetilde{a_0}(\omega) \tag{14}$$

with $\widetilde{a_0}(\omega) = 3 + \int_0^1 F_0(x) f_\varepsilon(x,\omega)\,dx$ and $\widetilde{a_1}(\omega) = -4 + \int_0^1 F_1(x) f_\varepsilon(x,\omega)\,dx$.

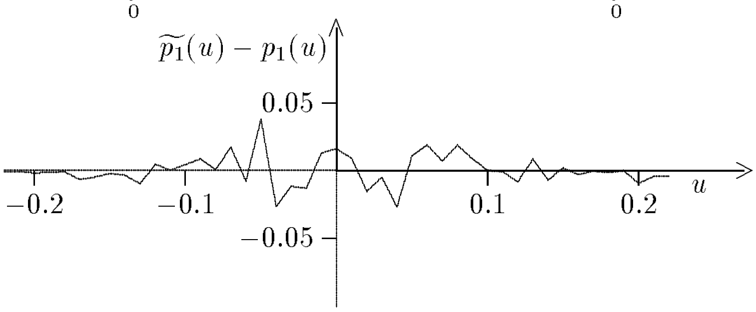

Figure 3: Difference function $\widetilde{p_1}(u) - p_1(u)$

Thereby, the parameters of $f_\varepsilon(x,\omega)$ are chosen as $K = 0$, $\varepsilon = 0.02$ and $\sigma^2 = 0.0625$, the deterministic functions $F_0(x)$ and $F_1(x)$ are determined according (11) with

$$F_0(x) = \frac{0.2}{\sqrt{\varepsilon}\sigma}\sin(2\pi x) \text{ and } F_1(x) = \frac{1}{\sqrt{\varepsilon}\sigma}\left(0.05\sin(2\pi x) + 0.1323\sin(4\pi x)\right) .$$

The small values of the plotted differences in Fig. 3 confirm the described substitution method (see also the absolute values in Fig. 4).

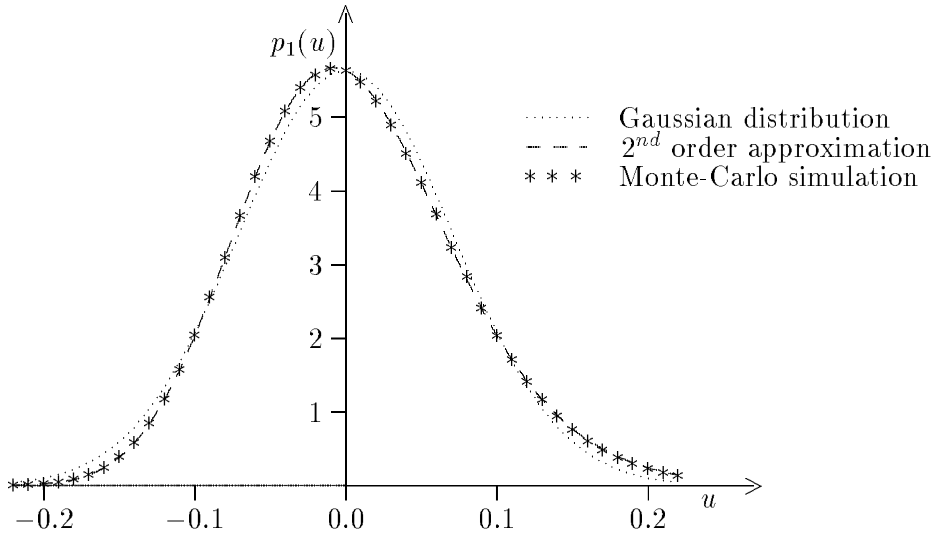

Figure 4: Approximations of the probability density function of $(z_1(\omega) - 1)$

In addition, in Fig.4 the theoretical approximation (12) of the probability density function $p_1(u)$ is confirmed by simulation of the linear functionals.

In summary it may be said that the substitution method leads to a sufficiently accurate approximation of the probability density function of the zeros of the considered random polynomial.

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