Ill-Posedness Aspects of Some Nonlinear Inverse Problems and their Linearizations

G. FLEISCHER* and B. HOFMANN*

ABSTRACT

In this paper we deal with aspects of characterizing the ill-posedness of nonlinear inverse problems based on the discussion of specific examples. In particular, a parameter identification problem to a second order differential equation and its ill-posed linear components are under consideration. A new approach to the classification of ill-posedness degrees for multiplication operators completes the paper.

Key words: degree of ill-posedness, multiplication operators, nonlinear inverse problems, parameter identification. **AMS Subject Classifications**: 65 J 20, 47 B 38.

1 INTRODUCTION

In the last few years the accelerated coupling of applied mathematics, natural sciences and engineering has enormously stimulated the inverse problem theory and practice. The rapidly growing number of inverse problems occurring in numerous practical situations also implied a qualitative and quantitative development of the mathematical modelling of inverse problems. This process was accompanied with a large number of new articles and books on the mathematics of inverse problems from analytical and numerical viewpoints (see, e.g., Anger et al. [1], Engl [5], Engl, Hanke and Neubauer [6] and Kirsch [10]). In the framework of many authors the different aspects of ill-posedness for linear and nonlinear inverse problems were intensively studied. For the choice of appropriate regularization methods aimed at solving inverse problems in a unique and stable manner, the characterization of the kind and degree of ill-posedness of any specific inverse problem plays an important role. In the class of nonlinear inverse problems, written as an operator equation

$$F(x) = y\,, \quad x \in D(F) \subset X,\ y \in Y \tag{1}$$

*Faculty of Mathematics, Technical University Chemnitz, D-09107 Chemnitz, Germany.

in Banach spaces X and Y, the ill-posedness behaviour may depend on the location of the solution point x^* under consideration inside the domain $D(F)$ (cf. [7] and [8]), whereas linear inverse problems

$$A\,x = y\,, \quad x \in X,\ y \in Y \tag{2}$$

have a global degree of ill-posedness associated with the smoothing properties of the linear operator $A : X \to Y$ characterizing the corresponding direct problem (cf. [9]).

In our paper we consider the specific situation that the nonlinear inverse problem (1) can be decomposed into a well-posed nonlinear part and an ill-posed linear part such that the degree of ill-posedness of (1) is also of global nature, since it is determined by the linear operators A with non-closed range that characterize the linear part in the decomposition. For that purpose we are going to discuss some aspects of ill-posedness based on an example concerning a parameter identification problem to a second order differential equation. This example will be given in Section 2. Identification problems of this type were treated by many authors (cf., e.g., Banks and Kunisch [2] and Colonius and Kunisch [4]). It can be seen that the mentioned inverse problem leads to a nonlinear operator equation (1), where we in general will choose pairs of separable Hilbert spaces X and Y for our considerations. For the example we will show also in Section 2 that a decomposition of the inverse problem into a well-posed nonlinear part and an ill-posed linear part is possible for appropriately chosen spaces X and Y. For the linear part we find linear ill-posed problems with compact embedding operators A and the fewer ill-posed linear problems, where A is a non-compact multiplication operator. We will give some contributions to the discussion of this latter type in Section 3. In detail, we define the degree of ill-posedness for the multiplication operators and give some assertions on the convergence rate of Tikhonov regularization for linear ill-posed problems with multiplication operators.

2 AN ORDINARY DIFFERENTIAL EQUATION PROBLEM

To motivate our ideas presented below we consider as an example the ordinary differential equation

$$-\frac{d}{dt}\left(q(t)\frac{d}{dt}u(t)\right) = f(t), \quad t \in (0,1), \tag{3}$$

where the parameter function $q \in Q$, the state function $u \in U$, and the right-hand side $f \in V$ are elements of Banach spaces Q, U and V, respectively, which

will be specified below. Moreover, for parameter functions $q \in Q$, we want to denote by $L(q) : D(L) \subset U \to V$ the operator defined by

$$[L(q)u](t) := -\frac{d}{dt}\left(q(t)\frac{d}{dt}u(t)\right) = -\left(q\,u_t\right)_t . \tag{4}$$

Then the differential equation (3) can be rewritten as an operator equation

$$L(q)\,u \;=\; f. \tag{5}$$

We note that the domain $D(L)$ of the linear differential operator L is chosen in such a way that the elements of this domain automatically satisfy imposed boundary conditions, in our case homogeneous Dirichlet conditions

$$u(0) = u(1) = 0. \tag{6}$$

Additionally we assume that

$$q(t) \geq \underline{q} > 0, \quad t \in [0,1]. \tag{7}$$

The aim of our considerations is to identify the parameter function q from observations of the state u. Setting $X := Q$, $Y := U$, $x := q$ and $y := u$, this corresponds with the solution of a nonlinear inverse problem (1), where F expresses the parameter-to-state mapping $q \mapsto u$. Note that the parameter-to-state mapping F is nonlinear in general, although the corresponding differential equation (3) is linear. Taking into account the bilinear structure of the differential equation we formulate a lemma which can be found in a similar form in Tautenhahn's paper [13]:

Lemma 1 *For Hilbert spaces Q, U and V let $\hat{L} : Q \times U \to V$ be a bilinear operator. Then for the operator equation*

$$\hat{L}(q,u) = f, \quad q \in Q,\ u \in U,\ f \in V \tag{8}$$

we assume the existence of positive constants M and m with

$$\|\hat{L}(q,u)\|_V \leq M\|q\|_Q\|u\|_U \quad \forall q \in Q, \forall u \in U, \tag{9}$$

$$\|\hat{L}(q,u)\|_V \geq m\|u\|_U \quad \forall q \in Q, \forall u \in U. \tag{10}$$

If we denote by $F : Q \to U$ the well-defined parameter-to-state operator with $u := F(q)$, then the following assertions hold:
1. F is Fréchet-differentiable with $F'(q) : Q \to U$.
2. If we define $L(q)\,u := \hat{L}(q,u)$ for all $q \in Q$, then the Fréchet-derivative can be expressed by

$$F'(q^*)\,q = -(L(q^*))^{-1}\hat{L}(q,u^*) \tag{11}$$

for elements q^* and u^* satisfying $F(q^*) = u^*$.
3. Moreover, we have the estimation

$$\|F(q) - F(q^*) - F'(q^*)(q - q^*)\|_U \le \frac{M}{m}\|q - q^*\|_Q \|u - u^*\|_U \tag{12}$$

whenever $F(q) = u$.

Using Sobolev spaces we may apply Lemma 1 to our operator equation (5). In detail we set $Q := X := H^1(0,1)$ with norm

$$\|q\|_{H^1(0,1)} := \sqrt{\|q\|^2_{L^2(0,1)} + \|q_t\|^2_{L^2(0,1)}},$$

$U := Y := H_0^1(0,1)$ with norm

$$\|u\|_{H_0^1(0,1)} := \|u_t\|_{L^2(0,1)},$$

$V := (H_0^1(0,1))^* = H^{-1}(0,1)$ with norm

$$\|v\|_{H^{-1}(0,1)} = \sup_{w \in H_0^1(0,1)} \frac{|(v,w)_{L^2(0,1)}|}{\|w\|_{H_0^1(0,1)}}$$

and $\hat{L}(q,u) := L(q)u$. Then it can easily be proven that the assumptions of Lemma 1 are fulfilled. Namely, we have

$$\begin{aligned} &\|\hat{L}(q,u)\|_{H^{-1}(0,1)} \\ &= \sup_{w \in H_0^1(0,1)} \frac{|(qu_t, w_t)_{L^2(0,1)}|}{\|w\|_{H_0^1(0,1)}} \le \frac{\|qu_t\|_{L^2(0,1)}\|w\|_{H_0^1(0,1)}}{\|w\|_{H_0^1(0,1)}} \\ &\le \|q\|_{L^\infty(0,1)}\|u_t\|_{L^2(0,1)} \le c_E\|q\|_{H^1(0,1)}\|u\|_{H_0^1(0,1)}. \end{aligned} \tag{13}$$

Here is c_E the embedding constant from $H^1(0,1)$ into $L^\infty(0,1)$. Further it holds

$$\begin{aligned} \|\hat{L}(q,u)\|_{H^{-1}(0,1)} &= \sup_{w \in H_0^1(0,1)} \frac{|(qu_t, w_t)_{L^2(0,1)}|}{\|w\|_{H_0^1(0,1)}} \ge \frac{|(qu_t, u_t)_{L^2(0,1)}|}{\|u\|_{H_0^1(0,1)}} \\ &\ge \underline{q}\frac{\|u_t\|^2_{L^2(0,1)}}{\|u\|_{H_0^1(0,1)}} = \underline{q}\|u\|_{H_0^1(0,1)}. \end{aligned} \tag{14}$$

Now the main result of this lemma is for our differential equation problem the estimation

$$\|F(q) - F(q^*) - F'(q^*)(q - q^*)\|_{H_0^1(0,1)} \le k\|q - q^*\|_{H^1(0,1)}\|F(q) - F(q^*)\|_{H_0^1(0,1)}. \tag{15}$$

Furthermore, we see that the Fréchet-derivative can be expressed by

$$F'(q^*)\,q = -L(q^*)^{-1}\hat{L}(q,u^*) = L(q^*)^{-1}(qu_t^*)_t. \tag{16}$$

Now we cite a proposition which can be found as Proposition 3 in [8].

Proposition 2 *Let Q and U be Hilbert spaces and let exist a ball $B_{\tilde{\rho}}(q^*)$ with center $q^* \in Q$ and radius $\tilde{\rho} > 0$ such that the operator $F : D(F) \subset Q \to U$ with $B_{\tilde{\rho}}(q^*) \subset D(F)$ satisfies the inequality*

$$\|F(q) - F(q^*) - F'(q^*)(q - q^*)\|_U \leq K\|F(q) - F(q^*)\|_U \tag{17}$$

for all $q \in B_{\tilde{\rho}}(q^*)$, where $0 < K < 1$ is a constant. Then we have a radius $0 < \rho \leq \tilde{\rho}$ and constants $0 < \underline{c} \leq \overline{c} < \infty$ such that

$$\underline{c}\,\|F'(q^*)(q - q^*)\|_U \leq \|F(q) - F(q^*)\|_U \leq \overline{c}\,\|F'(q^*)(q - q^*)\|_U \tag{18}$$

for all $q \in B_\rho(q^*)$.

As a consequence of Proposition 2 and formula (15) we obtain:

Corollary 3 *For the differential equation (3) with boundary conditions (6) an estimation of the form*

$$c_1\|u_t^*(q - q^*)\|_{L^2(0,1)} \leq \|u - u^*\|_{H_0^1(0,1)} \leq c_2\|u_t^*(q - q^*)\|_{L^2(0,1)} \tag{19}$$

is valid for all q from a ball $B_\rho(q^*) := \{q \in H^1(0,1) : \|q - q^*\|_{H^1(0,1)} \leq \rho\}$, where ρ does not depend on u. In this context, c_1, c_2 are positive real constants.

Proof: Obviously Proposition 2 applied to formula (15) yields for $\|q - q^*\|_{H^1(0,1)}$ sufficiently small

$$\underline{c}\|F'(q^*)(q - q^*)\|_{H_0^1(0,1)} \leq \|F(q) - F(q^*)\|_{H_0^1(0,1)} \leq \overline{c}\|F'(q^*)(q - q^*)\|_{H_0^1(0,1)}. \tag{20}$$

Now we estimate the Fréchet-derivative from below and above under consideration of Lemma 1. This gives

$$\begin{aligned} &\|F'(q^*)(q - q^*)\|_{H_0^1(0,1)} = \|L(q^*)^{-1}(u_t^*(q - q^*))_t\|_{H_0^1(0,1)} \\ &\geq M^{-1}\|q^*\|_{H^1(0,1)}^{-1}\|(u_t^*(q - q^*))_t\|_{H^{-1}(0,1)} \geq \bar{c}_1\|u_t^*(q - q^*)\|_{L^2(0,1)} \end{aligned} \tag{21}$$

as well as

$$\begin{aligned} &\|F'(q^*)(q - q^*)\|_{H_0^1(0,1)} = \|L(q^*)^{-1}(u_t^*(q - q^*))_t\|_{H_0^1(0,1)} \\ &\leq m^{-1}\|(u_t^*(q - q^*))_t\|_{H^{-1}(0,1)} \leq \bar{c}_2\|u_t^*(q - q^*)\|_{L^2(0,1)}. \end{aligned} \tag{22}$$

Note that the last estimation in (21) is not true in general. This is due to the fact that q can not be identified completely from u. Namely, if (q^*, u^*) solves the differential equation (3), then $(q^* + \frac{c}{u_t^*}, u^*)$ is also a solution (cf. [2, p. 98]). However, if u_t^* has a zero or is not continuous, then $q^* + \frac{c}{u_t^*}$ is not an element of $H^1(0,1)$. Consequently, q^* can be uniquely identified from u^* and the norms $\|\cdot_t\|_{H^{-1}(0,1)}$ and $\|\cdot\|_{L^2(0,1)}$ are equivalent. Now in our case of homogeneous

Dirichlet conditions u_t^* is either continuous and has then a zero point or u_t^* is not continuous. Then the element $\frac{1}{u_t^*}$ is not in $H^1(0,1)$. ∎

By modified specifications of the Hilbert spaces Q and U other variants of inequalities of the form (15) can be proven that hold at least in a local sense. In the paper [3] with $Q := H^1(0,1), U := L^2(0,1)$ the inequality

$$\|F(q) - F(q^*) - F'(q^*)(q-q^*)\|_{L^2(0,1)} \le \hat{k}\|q-q^*\|_{H^1(0,1)}\|F(q)-F(q^*)\|_{L^2(0,1)} \tag{23}$$

is shown.

Moreover, if we choose $U := H^1(0,1)$, but with a domain $D(L) := H_0^1(0,1)$, then as a consequence of Friedrich's inequality the estimations in (13) and (14) can be done also for the space $H^1(0,1)$ instead of $H_0^1(0,1)$.

In the following we consider the case $U := H^2(0,1)$ with norm

$$\|u\|_{H^2(0,1)} := \sqrt{\|u\|^2_{L^2(0,1)} + \|u_t\|^2_{L^2(0,1)} + \|u_{tt}\|^2_{L^2(0,1)}},$$

$V := L^2(0,1)$, $Q := H^2(0,1)$, $D(L) := H^2(0,1) \cap H_0^1(0,1)$. Then the assumptions of Lemma 1 can also be shown:

$$\begin{aligned} \|\hat{L}(q,u)\|_{L^2(0,1)} &= \|(qu_t)_t\|_{L^2(0,1)} \\ \le \|qu_t\|_{H^1(0,1)} &\le \|q\|_{L^\infty(0,1)}\|u_t\|_{H^1(0,1)} \le c\|q\|_{H^1(0,1)}\|u\|_{H^2(0,1)}. \end{aligned} \tag{24}$$

For a lower bound we refer to Colonius and Kunisch ([4]). There we have

$$(\hat{L}(q,u),u)_{L^2(0,1)} = (-(qu_u)_t, u)_{L^2(0,1)} = (qu_t, u_t)_{L^2(0,1)} \ge \underline{q}\|u\|^2_{H^1(0,1)} \tag{25}$$

due to Friedrich's inequality. This implies

$$\|\hat{L}(q,u)\|_{L^2(0,1)} \ge \underline{q}\|u\|_{H^1(0,1)} \tag{26}$$

Further it follows

$$\|\hat{L}(q,u)\|_{L^2(0,1)} = \|qu_{tt} + q_t u_t\|_{L^2(0,1)} \ge \underline{q}\|u_{tt}\|_{L^2(0,1)} - \|q_t\|_{L^\infty(0,1)}\|u\|_{H^1(0,1)}. \tag{27}$$

Choosing a constant $K > \frac{\|q_t\|_{L^\infty(0,1)}}{\underline{q}}$, multiplying (25) with K and adding (27) gives

$$(K+1)\|\hat{L}(q,u)\|_{L^2(0,1)} \ge \underline{q}\|u_{tt}\|_{L^2(0,1)} + (K\underline{q} - \|q_t\|_{L^\infty(0,1)})\|u\|_{H^1(0,1)}$$

and

$$\|\hat{L}(q,u)\|_{L^2(0,1)} \ge \tilde{q}\|u\|_{H^2(0,1)} \tag{28}$$

with a positive constant $\tilde{q}$.

If we summarize the considerations above, then we have in analogy to (15)

$$\|F(q)-F(q^*)-F'(q^*)(q-q^*)\|_{H^1(0,1)} \le \tilde{k}\|q-q^*\|_{H^1(0,1)}\|F(q)-F(q^*)\|_{H^1(0,1)} \tag{29}$$

for all $u = F(q)$ from $H^1(0,1)$ with imposed homogeneous Dirichlet conditions as well as

$$\|F(q)-F(q^*)-F'(q^*)(q-q^*)\|_{H^2(0,1)} \le \overline{k}\|q-q^*\|_{H^2(0,1)}\|F(q)-F(q^*)\|_{H^2(0,1)} \tag{30}$$

for all $u = F(q) \in H^2(0,1) \cap H_0^1(0,1)$.

Now we can proceed as in the H_0^1-case: From (29) in analogy to (19) we obtain with respect to Proposition 2

$$c_1 \|u_t^*(q-q^*)\|_{L^2(0,1)} \le \|u-u^*\|_{H^1(0,1)} \le c_2 \|u_t^*(q-q^*)\|_{L^2(0,1)}. \tag{31}$$

On the other hand, (30) implies the inequality

$$c_1 \|u_t^*(q-q^*)\|_{H^1(0,1)} \le \|u-u^*\|_{H^2(0,1)} \le c_2 \|u_t^*(q-q^*)\|_{H^1(0,1)}. \tag{32}$$

To show this estimation one uses the facts that u_t^* has a zero point because of the homogeneous Dirichlet conditions and that $\|(u_t^*(q-q^*))_t\|_{L^2(0,1)} = 0$ implies $\|u_t^*(q-q^*)\|_{L^2(0,1)} = 0$. Namely, we then have $[u_t^*(q-q^*)]_t \equiv 0$ and $u_t^*(q-q^*) = \text{const}$. Consequently, if u_t^* has a zero and $q-q^*$ lies in $H^1(0,1)$, this constant must be 0.

In order to get stability estimates as in (19), (31) and (32), the deviations $u-u^*$ are evaluated by a stronger norm than the weighted deviations $u_t^*(q-q^*)$. In such pairs of topologies the parameter-to-data mapping is continuously invertible and we get a stable reconstruction of q for sufficiently smooth data u. Really, inequalities of the form (32) cannot be found in the case that an equivalent topology is used for both deviations. For a specific counterexample we refer to Example 2.6 in [2]. This is a specific kind of ill-posedness for the identification problem under consideration.

If, however, the natural topology for measuring errors in the real data $\tilde{u}$ is not strong enough, for example an L^2-topology, we have additionally to solve an ill-posed linear operator equation

$$\mathcal{E}u = \tilde{u}, \tag{33}$$

where $\mathcal{E} : H^1(0,1) \to L^2(0,1)$ in (31) and $\mathcal{E} : H^2(0,1) \to L^2(0,1)$ in (32) are compact embedding operators. Then the nonlinear inverse problem of recovering q from $\tilde{u}$ is decomposed into a linear ill-posed problem (33) and a nonlinear well-posed problem characterized by (31) or (32) (cf. [7]). In such a case the degree of ill-posedness of the whole problem is determined by the decay rate of the singular values of the embedding operators $\mathcal{E}$.

As the inequalities (31) and (32) show, the stable identification of q from u is possible only in weighted L^2 and H^1 parameter space norms. The weighting factor is the function u_t^*. Hence, the reconstruction of q from $\tilde{q} := u_t^* q$ corresponds to the solution of a linear operator equation

$$\mathcal{M}q = \tilde{q} \tag{34}$$

with a multiplication operator $\mathcal{M}$. Under more general assumptions this equation can be both well-posed and ill-posed. To obtain well-posedness and stability estimations for $\|q - q^*\|$ instead of $\|u_t^*(q - q^*)\|$, it is sufficient to assume that u_t^* is bounded and has no zeros, i.e.,

$$0 < \inf_{t \in (0,1)} \operatorname{ess} u_t^*(t) \leq \sup_{t \in (0,1)} \operatorname{ess} u_t^*(t) < \infty. \tag{35}$$

Otherwise, ill-posed situations arise whenever u_t^* has zeros, i.e. always in the case of our imposed homogeneous Dirichlet boundary conditions.

We see that there occur two different types of ill-posedness in our identification problem. On the one hand, we have to solve an operator equation (33) with compact embedding operator $\mathcal{E}$. Nashed (cf. [11]) calls this ill-posedness of type II. On the other hand, the multiplication operator $\mathcal{M}$ is for no function u_t^* a compact operator (except in the trivial case $u_t^* \equiv 0$). Since the operator equation (34) is ill-posed in our case, we have ill-posedness of type I in Nashed's sense. Whenever the data and parameter spaces are chosen in such a way that we need no embedding, then the fewer ill-posed situation of multiplication operators is characteristic for the whole identification problem.

3 MULTIPLICATION OPERATORS

Now we want to analyze multiplication operators $\mathcal{M}$ more in detail. In this context, we would like to use the discussions of the previous paragraph as a good motivation to do so. We define an operator $\mathcal{M} : L^2(0,1) \to L^2(0,1)$ of multiplication with $\varphi \in L^\infty(0,1)$ by

$$[\mathcal{M}x](t) := \varphi(t)x(t), \quad t \in (0,1) \; a.e. \tag{36}$$

It is clear that φ has to be chosen from $L^\infty(0,1)$ in order to guarantee that the product $\mathcal{M}x$ is in $L^2(0,1)$ for all $x \in L^2(0,1)$. Usually, such a φ is a piecewise continuous function. It can easily be shown that the adjoint $\mathcal{M}^*$ of the operator $\mathcal{M}$ is given as

$$[\mathcal{M}^*x](t) = \overline{\varphi(t)}x(t), \quad t \in (0,1) \; a.e. \tag{37}$$

From this representation it follows

$$[\mathcal{M}^*\mathcal{M}x](t) = |\varphi(t)|^2 x(t), \quad t \in (0,1)\ a.e. \tag{38}$$

and

$$[(\mathcal{M}^*\mathcal{M})^{\frac{1}{2}}x](t) = |\varphi(t)|x(t), \quad t \in (0,1)\ a.e. \tag{39}$$

Therefore, we can change between these operators as it is usual in the case of compact operators or we can simply assume that φ is a real and non-negative function. This will be done throughout this paragraph. Consequently, $\mathcal{M}$ is always assumed to be a self-adjoint and non-negative operator.

Every self-adjoint operator of multiplication can be written in the spectral form

$$\mathcal{M}x = \int_0^1 \lambda dE_\lambda x, \tag{40}$$

where the spectral family E_λ is given by

$$[E_\lambda x](t) = \begin{cases} x(t), & \varphi(t) \le \lambda \\ 0, & \varphi(t) > \lambda \end{cases}. \tag{41}$$

If we consider the multiplication operator with the independent variable by

$$[\tilde{\mathcal{M}}x](t) := tx(t), \quad t \in (0,1)\ a.e., \tag{42}$$

then its spectral family has the structure

$$[\tilde{E}_\lambda x](t) = \begin{cases} x(t), & t \le \lambda \\ 0, & t > \lambda \end{cases}. \tag{43}$$

It is well-known that any function $\varphi(\tilde{\mathcal{M}})$ of this operator can be written as

$$\varphi(\tilde{\mathcal{M}})x = \int_0^1 \varphi(\lambda) d\tilde{E}_\lambda x \tag{44}$$

whenever the integral exists. Obviously we then have

$$\mathcal{M} = \varphi(\tilde{\mathcal{M}}). \tag{45}$$

For compact linear operators A the degree ν of ill-posedness can be defined by the decay rate of the ordered sequence

$$s_1(A) \ge s_2(A) \ge \ldots \ge s_i(A) \ge \ldots > 0$$

of singular values to zero, where $s_j^2(A)$ is the j-th eigenvalue of A^*A. If for example

$$s_j(A) \sim j^{-\nu}, \tag{46}$$

then the positive exponent ν expresses the degree of ill-posedness. Note that such a proportionality need not be valid in general (cf. [9]). For a compact operator A with a degree $\nu = \nu(A)$ of ill-posedness, however, any power A^α has a degree $\alpha\nu$ of ill-posedness.

To generalize this statement to non-compact operators we give the following definition:

Definition 4 *Let $\varphi \in L^\infty(0,1)$. Then we call*

$$p(\lambda) := \mathrm{meas}\{t : \varphi(t) \in [0,\lambda]\} \tag{47}$$

the distribution function of φ. This is the measure of the pre-image of the interval $[0,\lambda]$ with respect to φ. Moreover, we call

$$\tilde{\varphi}(s) = \sup\{\lambda : p(\lambda) \le s\} \tag{48}$$

the increasing rearrangement of φ.

It is clear that the function $\tilde{\varphi}$ is in general the inverse function of p. Furthermore, $\tilde{\varphi}$ is a monotonically increasing function. As it can be seen, this function becomes zero in the point $t = 0$, if the multiplication operator $\mathcal{M}$ implies ill-posedness, i.e. if φ has zeros. Now we want to assume that the multiplication operator $\mathcal{M}$ is injective. This property is equivalent to the fact that $\varphi(t) > 0$ a.e. in $(0,1)$. In this case we find that $\tilde{\varphi}(t) > 0$ whenever $t > 0$. Therefore we can define the degree of ill-posedness for multiplication operators as follows:

Definition 5 *Let $\mathcal{M}$ be the operator of multiplication with the function φ, $\varphi \in L^\infty(0,1)$. Then we define the real value $\nu = \nu(\mathcal{M})$ to be the degree of ill-posedness of this multiplication operator if we have*

$$\tilde{\varphi}(t) \sim t^\nu, \tag{49}$$

i.e., if there are two positive constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}t^\nu \le \tilde{\varphi}(t) \le \overline{c}t^\nu, \quad t \in (0,1)\ a.e. \tag{50}$$

Note that such a value ν also need not exist. It is quite obvious that the operator $\tilde{\mathcal{M}}$ of the multiplication with t gets a degree $\nu = 1$ of ill-posedness. Moreover, its αth powers (i.e., the multiplication operators with t^α) have by definition a degree $\nu = \alpha$ of ill-posedness.

Now we want to give a characterization of the degree of ill-posedness by the properties of the function φ. First we need another definition:

Definition 6 *If $\varphi \in L^\infty(0,1)$ and if, moreover, the value α_0 with*

$$\alpha_0 := \sup\left\{\alpha : \operatorname*{inf\,ess}_{B_\varepsilon(t_0)} \left|\frac{\varphi(t)}{(t-t_0)^\alpha}\right| = 0\right\} \tag{51}$$

exists and is positive for a value $t_0 \in (0,1)$, then we call t_0 a zero point of φ and α_0 its order.

The essential infimum has to be taken over all balls $B_\varepsilon(t_0)$ around the point t_0 with sufficiently small radius $\varepsilon > 0$. Note that this definition implies the classical definition of the order of a zero for continuous functions.

Lemma 7 *If α_0 is the order of a zero point t_0 of the function φ, then there exists, for sufficiently small $\varepsilon > 0$ and all $\alpha > \alpha_0$, a positive value δ such that*

$$|\varphi(t)| > \delta |t - t_0|^\alpha \quad \text{for a.e. } t \in B_\varepsilon(t_0). \tag{52}$$

Proof: The essential infimum of a function ψ over a set S is defined by

$$\inf_{t \in S} \operatorname{ess} \psi(t) = \sup \{\delta : \operatorname{meas}\{t \in S : \psi(t) \le \delta\} = 0\}. \tag{53}$$

Therefore we can rewrite the definition of the order of a zero by

$$\alpha_0 = \sup\{\alpha : \sup\{\delta : \operatorname{meas}\{t \in B_\varepsilon(t_0) : \left| \frac{\varphi(t)}{(t-t_0)^\alpha} \right| \le \delta\} = 0\} = 0\} \tag{54}$$

or by

$$\begin{aligned} \alpha_0 &= \sup\{\alpha : \operatorname{meas}\{t \in B_\varepsilon(t_0) : \left| \frac{\varphi(t)}{(t-t_0)^\alpha} \right| \le \delta\} \ne 0\} \ \forall \delta > 0\} \\ &= \sup\{\alpha : \operatorname{meas}\{t \in B_\varepsilon(t_0) : |\varphi(t)| \le \delta |t-t_0|^\alpha \ \forall \delta > 0\}. \end{aligned} \tag{55}$$

This is equivalent to the representation

$$\alpha_0 = \sup\{\alpha : \exists K_\varepsilon \subset B_\varepsilon(t_0), \operatorname{meas}(K_\varepsilon) > 0 : |\varphi(t)| \le \delta |t-t_0|^\alpha \ \forall \delta > 0 \, \forall t \in K_\varepsilon\}. \tag{56}$$

Now take $\alpha > \alpha_0$. Then it follows that there exists a $\delta > 0$ such that we have no subset K_ε of $B_\varepsilon(t_0)$ with $\operatorname{meas} K_\varepsilon > 0$ and

$$|\varphi(t)| \le \delta |t - t_0|^\alpha \quad \forall t \in K_\varepsilon. \tag{57}$$

This gives the assertion of the lemma. ∎

Proposition 8 *If $\varphi \in L^\infty(0,1)$ has only a finite number of zero points, then the degree of ill-posedness of the associated operator of multiplication is not greater than the maximum of all the orders of the zero points of φ.*

Proof: Obviously, it suffices to consider only two zeros of φ. Let t_1 and t_2 be two zeros and α_1 and α_2 the corresponding orders of the zeros. We assume

$\alpha_1 \geq \alpha_2$ and consider two balls $B_{\varepsilon_1}(t_1)$ and $B_{\varepsilon_2}(t_2)$ around the zeros. Now we choose a sufficiently small λ such that

$$\{t : \varphi(t) \in [0,\lambda]\} \subset B_{\varepsilon_1}(t_1) \cup B_{\varepsilon_2}(t_2). \tag{58}$$

This is always possible due to the assumption that t_1 and t_2 are the only zeros of φ. Now we consider $\alpha > \alpha_1 \geq \alpha_2$. From Lemma 7 we conclude that $\varphi(t) > \delta_i |t - t_i|^{\alpha_i}$ for $t \in B_{\varepsilon_i}$. Here δ_i $(i = 1, 2)$ are the constants appearing in Lemma 7. Now we have for $\alpha_1 \geq \alpha_2$

$$\begin{aligned} & p(\lambda) = \operatorname{meas}\{t : \varphi(t) \in [0,\lambda]\} \\ \leq\ & \operatorname{meas}\{t \in B_{\varepsilon_1}(t_1) : \varphi(t) \in [0,\lambda]\} + \operatorname{meas}\{t \in B_{\varepsilon_2}(t_2) : \varphi(t) \in [0,\lambda]\} \\ \leq\ & \operatorname{meas}\{t \in B_{\varepsilon_1}(t_1) : \delta_1 |t - t_1|^{\alpha}\} + \operatorname{meas}\{t \in B_{\varepsilon_2}(t_2) : \delta_2 |t - t_2|^{\alpha}\} \\ & \leq 2 \left(\tfrac{\lambda}{\delta_1}\right)^{\frac{1}{\alpha}} + 2 \left(\tfrac{\lambda}{\delta_2}\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{59}$$

The estimation above is true for all $\alpha > \alpha_1$. Obviously, it holds for $\alpha = \alpha_1$, too. Therefore we have

$$p(\lambda) \leq 2 \left(\frac{\lambda}{\delta_1}\right)^{\frac{1}{\alpha_1}} + 2 \left(\frac{\lambda}{\delta_2}\right)^{\frac{1}{\alpha_1}} \leq C \lambda^{\frac{1}{\alpha_1}} \tag{60}$$

with a constant C depending only on δ_1 and δ_2. Using the fact that $\tilde{\varphi}$ is in practice the inverse function of p, we see that

$$\tilde{\varphi}(s) \geq C^{-1} s^{\alpha_1} \tag{61}$$

for sufficiently small s, which proves the lemma. ∎

Note that the converse assertion to this proposition is not true. Namely, if we take

$$\varphi(t) := \begin{cases} t, & \frac{1}{2^n+1} < t \leq \frac{1}{2^n}\ \forall n \\ 1 & \text{elsewhere} \end{cases}, \tag{62}$$

then it has a zero at $t = 0$ of order 1, but the degree of ill-posedness is only $\frac{1}{2}$.

Finally, we want to give some brief remarks on Tikhonov regularization for multiplication operators. This methods works here in a very simple manner. Namely, we have to solve the equation

$$(\mathcal{M}^*\mathcal{M} + \alpha I) x_\alpha = \mathcal{M}^* y \tag{63}$$

instead of

$$\mathcal{M} x^\dagger = y. \tag{64}$$

So we have only to solve pointwise

$$(|\varphi(t)|^2 + \alpha) x_\alpha(t) = \overline{\varphi(t)} y(t), \tag{65}$$

where $\alpha > 0$ is the regularization parameter. In [12], Neubauer gives a necessary and sufficient condition for the convergence of Tikhonov regularized solutions to the exact solution. For $0 < \gamma < 1$ we find there the following assertion:

$$\|x^\dagger - x_\alpha\| = \mathcal{O}(\alpha^\gamma) \iff \int_0^\mu d\|E_\lambda x^\dagger\|^2 = \mathcal{O}(\mu^{2\gamma}). \tag{66}$$

Here E_λ is the spectral family of the Gauss-transformation of the linear operator under consideration. A completely analogous expression can be found for "o" instead of "$\mathcal{O}$".
In the context of this statement we can also give a proposition, which shows connections between the degree of ill-posedness and the order of convergence of Tikhonov regularized solutions:

Proposition 9 *If $\nu = \nu(\mathcal{M}) > \frac{1}{4}$ is the degree of ill-posedness of a multiplication operator $\mathcal{M}$, then the Tikhonov regularized solutions x_α converge to the exact solution $x^\dagger \in L^\infty(0,1)$, where $x^\dagger \geq 0$ a.e., at least with the order $\frac{1}{4\nu(\mathcal{M})}$, i.e.,*

$$\|x^\dagger - x_\alpha\| = \mathcal{O}(\alpha^{\frac{1}{4\nu(\mathcal{M})}}). \tag{67}$$

Proof: To prove our proposition, we only have to show the assumptions of Neubauer's theorem. In (41) the spectral family of the multiplication operator was given. To obtain the spectral family of $\mathcal{M}^*\mathcal{M}$, we have to substitute φ^2 for φ. Now we can estimate

$$\int_0^\mu d\|E_\lambda x^\dagger\|^2 = \int\limits_{\{t:\varphi^2(t)\leq\mu\}} (x^\dagger(t))^2\, dt \leq \int\limits_{\{t:\varphi^2(t)\leq\mu\}} dt \cdot \operatorname{sup\,ess}(x^\dagger(t))^2 \leq c\mu^{\frac{1}{2\nu}}\|x^\dagger\|_{L^\infty(0,1)}. \tag{68}$$

Here c is a constant value. If we have a positive lower bound b for $x^\dagger(t)$, then we can also estimate

$$\textstyle\int_0^\mu d\|E_\lambda x^\dagger\|^2 = \int\limits_{\{t:\varphi^2(t)\leq\mu\}} (x^\dagger(t))^2\, dt \geq \int\limits_{\{t:\varphi^2(t)\leq\mu\}} b^2\, dt = \hat{c}\mu^{\frac{1}{2\nu}} \tag{69}$$

with a certain constant $\hat{c}$. This proves the assumption of Neubauer's theorem. Consequently, we obtain the assertion of our proposition, namely a convergence order of $\frac{1}{4\nu(\mathcal{M})}$. ∎

Finally we add some remarks concerning Proposition 9 and its proof:

- If $x^\dagger$ is not bounded from below by a positive number b, then the estimation (69) from below is not true. But the estimation (68) from above is always valid. That means, we may have a convergence order of $\frac{1}{4\nu}$ or higher.

- The result of Proposition 9 bases on the gap between the spaces $L^2(0,1)$ and $L^\infty(0,1)$. Therefore the assumption $x^\dagger \in L^\infty(0,1)$ cannot be reduced to $x^\dagger \in L^2(0,1)$.

- It can be shown that the formula (67) remains valid if the symbol "$\mathcal{O}$" is replaced by "o", where in our context the Landau symbol "$\mathcal{O}$" is used in a sense that does not exclude the case "o". Norms in this paragraph are always $L^2(0,1)$ norms.

- In the case $\nu(\mathcal{M}) \leq \frac{1}{4}$ we detect a so-called saturation effect for the convergence rate of Tikhonov regularization. That means, the convergence order will not be better than α^1, even if $\nu(\mathcal{M})$ becomes arbitrarily small (cf. [12]).

4 SUMMARY

In this paper, we have presented a new approach of analyzing the degree of ill-posedness of linear operator equations with multiplication operators. The main idea of this approach consists in using the increasing rearrangements of the multiplier functions as a tool for comparing different kinds of multiplication operators. By studying a specific identification problem to a second order differential equation its is shown that linear ill-posed problems with multiplication operators arise as subproblems in the context of nonlinear inverse problems in a natural manner. Our example under consideration has the property that the inverse problem can be decomposed into a well-posed nonlinear problem, a linear ill-posed problem with a compact embedding operator and a linear ill-posed problem with a non-compact multiplication operator. Such a decomposition structure occurs not only in the mentioned example, but also for other classes of inverse problems in differential and integral equations.

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