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Local Theory of a Collocation Method for Cauchy Singular Integral Equations on an Interval

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Abstract

We consider a collocation method for Cauchy singular integral equations on the interval based on weighted Chebyshev polynomials, where the coefficients of the operator are piecewise continuous. Stability conditions are derived using Banach algebra methods, and numerical results are given.

1 Introduction

The subject of the present paper is the investigation of a collocation method based on weighted polynomials for the approximate solution of singular integral equations on $(-1,1)$ of the type

$$a(x)u(x) + \frac{b(x)}{\pi i}\int_{-1}^{1}\frac{u(t)}{t-x}\,dt = f(x), \qquad x \in (-1,1), \tag{1.1}$$

where u is the unknown function and a, b, f are given. All functions involved are assumed to be complex-valued, and we require that $b(\pm 1) = 0$.

A lot of attention has been paid to investigating polynomial collocation and quadrature methods for this and similar types of equations (see [PS, Chapter 9]). These are essentially based on special mapping properties of the operator $A\mu I$, where A denotes the operator on the left-hand side of (1.1), and μ is a generalized Jacobi weight depending on the coefficients a and b, which are required to satisfy a Hölder condition.

We are going to give a somewhat different approach using weighted polynomials as ansatz functions (a finite section (Galerkin) method with the same type of ansatz functions was investigated in [JRW]). Also in the case of variable coefficients a and b there will always occur pure Jacobi weights, which are easier to cope with than generalized Jacobi weights (for instance, the recurrence coefficients of the corresponding orthogonal polynomials, which are needed in the computer implementation of our method, are explicitly known), and we will always use the same (Chebyshev) collocation points independently of the coefficients. This is an advantage if a Newton method for a nonlinear singular integral equation results in a sequence of linear equations of type (1.1), the coefficients of which are different in each step. Furthermore, we can (apart from the additional assumption $b(\pm 1) = 0$) admit arbitrary piecewise continuous coefficients. Coefficients with jumps can occur, for example, when considering seepage problems for channels or dams with corners ([Ju2]). Finally we mention that one can introduce a scale of Sobolev-like spaces with respect to a system of weighted orthogonal polynomials analogous to [BHS] and can under suitable smoothness conditions on the coefficients obtain error estimates for the collocation method in the norms of these Sobolev-like spaces, which are continuously embedded in certain spaces of weighted continuous functions. The results we achieved here, however, still need to be improved.

We will prove the strong convergence of the approximation operators associated with our collocation method, and will investigate the stability of this method using Banach algebra techniques, which have proved to be an efficient tool in stability analysis (see for example [Si], [JS], [HRS], [JRW]). We obtain necessary and sufficient stability conditions in the case of continuous coefficients and a sufficient condition if the coefficients are piecewise continuous.

In Section 2 we give some notations and define the numerical method we are going to deal with, Section 3 provides some basic facts on Banach algebra techniques we will need for the stability analysis, Section 4 is concerned with the strong convergence of some operator sequences, and in Section 5 we will use Banach algebra techniques to derive the main result concerning the stability. Finally, we make some remarks concerning the computer implementation and present numerical results in Section 6.

2 Some notations and preliminaries

We consider equation (1.1) in the weighted Lebesgue space $L^2_\sigma := L^2_\sigma(-1,1)$ of all (classes of) measurable functions $u : (-1,1) \to \mathbb{C}$ for which

$$\|u\|^2_\sigma := \int_{-1}^{1} |u(x)|^2 \sigma(x)\, dx$$

is finite, equipped with the inner product

$$(u,v)_\sigma := \int_{-1}^{1} u(x)\overline{v(x)}\sigma(x)\, dx,$$

which turns L^2_σ into a Hilbert space. Here σ is a Jacobi weight defined by $\sigma(x) = v^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$ satisfying the conditions

$$-1 < \alpha, \beta < 1. \tag{2.2}$$

These conditions guarantee the boundedness of the Cauchy singular integral operator S defined by

$$(Su)(x) = \frac{1}{\pi i}\int_{-1}^{1} \frac{u(t)}{t-x}\, dt \tag{2.3}$$

on L^2_σ ([GK], Theorem I.4.1). The coefficients a, b are assumed to belong to the algebra PC of all piecewise continuous functions. The latter is defined as the closure (in the space of all bounded functions, equipped with the supremum norm) of the set of those functions being continuous on $[-1,1]$ with the possible exception of a finite number of jumps in $(-1,1)$, where the value of the function coincides with the left-sided limit. Note that PC-functions possess finite one-sided limits at all points. Under these assumptions, the operator on the left-hand side of (1.1), which in the following will be briefly referred to as $aI + bS$, is bounded on L^2_σ.

Let φ denote the Chebyshev weight of second kind,

$$\varphi(x) = \sqrt{1-x^2},$$

and let U_n be the orthonormal polynomial of degree n (with positive leading coefficient) with respect to the inner product $(.,.)_\varphi$. For these polynomials we have the well-known trigonometric representation

$$U_n(\cos s) = \sqrt{\frac{2}{\pi}}\frac{\sin(n+1)s}{\sin s}.$$

If σ is a Jacobi weight satisfying (2.2), we define the function

$$w_\sigma := \sqrt{\sigma\varphi}.$$

Obviously, $w_{\sigma^{-1}}I$ is an isometric isomorphism from L^2_φ onto L^2_σ. Thus, the functions

$$\tilde{u}_n := w_{\sigma^{-1}}U_n, \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

form an orthonormal basis in L^2_σ, because the same is true for U_n in the space L^2_φ. For the approximation method we want to apply to (1.1), the functions (2.4) will be used as ansatz functions. We need two sequences of projections with respect to the system $\{\tilde{u}_n\}_{n=0}^\infty$. The first are the Fourier projections P_n^σ given by

$$P_n^\sigma \sum_{k=0}^{\infty} \xi_k \tilde{u}_k := \sum_{k=0}^{n-1} \xi_k \tilde{u}_k.$$

Evidently, P_n^σ converges strongly to the identity operator I as $n \to \infty$. The second is the weighted interpolation operator $\widetilde{L_n^\sigma}$ assigning to a Riemann integrable function f the uniquely determined weighted polynomial of degree less than n (that is, the element of $X_n := \text{span}\,\{\tilde{u}_k\}_{k=0}^{n-1}$) that coincides with f in the collocation points $x_{kn}^\varphi = \cos\frac{k\pi}{n+1}$ $(k = 1,\ldots,n)$, which are the zeros of U_n. Thus we can write

$$\widetilde{L_n^\sigma} = w_{\sigma^{-1}} L_n^\varphi (w_{\sigma^{-1}})^{-1} I,$$

where L_n^φ is the usual (polynomial) Lagrangian interpolation operator with respect to these nodes. A class of functions f for which $\|\widetilde{L_n^\sigma} f - f\|_\sigma \to 0$ will be described in Corollary 4.6.

We now consider a collocation method which replaces equation (1.1) by the discrete approximate equations

$$\widetilde{L_n^\sigma}(aI + bS)v_n = \widetilde{L_n^\sigma} f, \tag{2.5}$$

where $v_n \in \text{span}\,\{\tilde{u}_k\}_{k=0}^{n-1}$ is sought. Our main concern is the *applicability* of this method to the original equation. In the following, we are going to define in a somewhat more general situation what we understand by this concept: Let X be a Banach space, let $\{P_n\} \subset \mathcal{L}(X)$ (where $\mathcal{L}(X,Y)$ means the set of all linear bounded operators between two Banach spaces X and Y, and $\mathcal{L}(X) := \mathcal{L}(X,X)$) be a sequence of projections with $P_n \to I$ (strongly), $X_n = \text{im}\, P_n$, $A \in \mathcal{L}(X)$ and $A_n \in \mathcal{L}(X_n)$, where we require that $A_n P_n$ converges strongly to A.

Definition 2.1 *The projection method*

$$A_n u_n = P_n f \tag{2.6}$$

is said to be applicable to the equation

$$Au = f \tag{2.7}$$

if the following conditions are fulfilled:

(i) The equations (2.6) have a unique solution for all sufficiently large n.

(ii) Their solutions u_n converge to a solution of (2.7) for $n \to \infty$.

Definition 2.2 *The sequence $\{A_n\}$ is said to be stable if there is an n_0 such that A_n is invertible for all $n \geq n_0$ and $\sup\limits_{n\geq n_0} \|A_n^{-1} P_n\| < \infty$.*

The problem of the applicability of (2.6) to (2.7) can be reduced to the stability of $\{A_n\}$ by the following lemma.

Lemma 2.1 ([HRS], Prop. 1.1) *Let $A_n P_n \to A$ strongly. Then (2.6) is applicable to (2.7) if and only if A is invertible and the sequence $\{A_n\}$ is stable.*

In our concrete situation we have $X = L_\sigma^2$, $P_n = P_n^\sigma$, $A = aI + bS$ and $A_n = \widetilde{L_n^\sigma} A|_{X_n}$. The strong convergence of these operators will be dealt with in section 4.

Remark 2.2 *Note that in the following we consider the projection method (2.6) rather than (2.5). If, however, (2.6) is applicable to (1.1) and the right-hand side f satisfies $\|f - \widetilde{L_n^\sigma} f\|_\sigma \to 0$ $(n \to \infty)$, the solutions v_n of (2.5) converge to the solution u of (1.1).*

Proof. We have

$$\begin{aligned}\|v_n - u\| &\leq \|A_n^{-1}\widetilde{L_n^\sigma} f - A_n^{-1} A_n P_n^\sigma u\| + \|P_n^\sigma u - u\| \\ &\leq \|A_n^{-1} P_n^\sigma\| \left(\|\widetilde{L_n^\sigma} f - f\| + \|Au - A_n P_n^\sigma u\|\right) + \|P_n^\sigma u - u\| \longrightarrow 0 \quad \blacksquare\end{aligned}$$

3 Banach algebra techniques

3.1 Basic facts

In this subsection we compile some basic facts that will be used later on to investigate the stability problem for our approximation method by Banach algebra techniques.

Definition 3.1 *Let $\mathcal{B}$, $\mathcal{C}$ be unital Banach algebras, $\mathcal{J} \subset \mathcal{B}$ a closed two-sided ideal. A unital homomorphism $W : \mathcal{B} \to \mathcal{C}$ is called $\mathcal{J}$-lifting, if $W(\mathcal{J})$ is a closed two-sided ideal in $\mathcal{C}$ and $W|_{\mathcal{J}}$ is an isomorphism between $\mathcal{J}$ and $W(\mathcal{J})$.*

The following theorem is usually called 'lifting theorem', its first version appears in [Si].

Theorem 3.1 (see [HRS], Theorem 1.8) *Let $\mathcal{B}$, $\mathcal{C}_t$ be unital Banach algebras, where t belongs to an arbitrary index set T, let $\mathcal{J}_t$ be closed two-sided ideals in $\mathcal{B}$ and let $W_t : \mathcal{B} \to \mathcal{C}_t$ be $\mathcal{J}_t$-lifting homomorphisms. By $\mathcal{J}$ we denote the smallest closed two-sided ideal in $\mathcal{B}$ that contains all $\mathcal{J}_t$, $t \in T$. Then an element $b \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if $W_t(b)$ is invertible in $\mathcal{C}_t$ for all $t \in T$ and the coset $b + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{B}/\mathcal{J}$.*

The invertibility of the coset $b + \mathcal{J}$ is usually investigated by the help of local principles.

Definition 3.2 *Let $\mathcal{B}$ be a unital Banach algebra. A subset $M \subset \mathcal{B}$ is called a localizing class if $0 \notin M$ and if for all a_1, $a_2 \in M$ there exists an element $a \in M$ such that*

$$aa_j = a_j a = a \qquad (j = 1, 2).$$

In the following let M be a localizing class. Two elements x, $y \in \mathcal{B}$ are called M-equivalent (in symbols: $x \overset{M}{\sim} y$), if

$$\inf_{a \in M} \|a(x-y)\| = \inf_{a \in M} \|(x-y)a\| = 0.$$

Further, $x \in \mathcal{B}$ is called M-invertible if there exist a_1, $a_2 \in M$, z_1, $z_2 \in \mathcal{B}$ such that

$$z_1 x a_1 = a_1, \qquad a_2 x z_2 = a_2.$$

A system $\{M_t\}_{t \in T}$ of localizing classes (T is an arbitrary index set) is said to be covering, if for each system $\{a_t\}_{t \in T}$, $a_t \in M_t$, there exists a finite subsystem $a_{t_1}, \ldots, a_{t_n}$ such that $a_{t_1} + \cdots + a_{t_n}$ is invertible in $\mathcal{B}$.

Now we can formulate the local principle of Gohberg and Krupnik:

Theorem 3.2 ([GK], Theorem XII.1.1) *Let $\mathcal{B}$ be a unital Banach algebra, $\{M_t\}_{t \in T}$ a covering system of localizing classes in $\mathcal{B}$, $x \in \mathcal{B}$ and $x \overset{M_t}{\sim} x_t$ for all $t \in T$. Further, assume that x commutes with all elements from $\bigcup_{t \in T} M_t$. Then x is invertible in $\mathcal{B}$ if and only if x_t is M_t-invertible for all $t \in T$.*

Another local principle is due to Allan and Douglas:

Theorem 3.3 ([BS], Theorem 1.34) *Let $\mathcal{B}$ be a unital Banach algebra and $\mathcal{C} \subset \mathcal{B}$ a closed central subalgebra (that is, all elements of $\mathcal{C}$ commute with all elements of $\mathcal{B}$) that contains the unit element. For every maximal ideal t of $\mathcal{C}$ we introduce the local ideal $\mathcal{J}_t$ as the smallest closed two-sided ideal of $\mathcal{B}$ that contains t. Then*

(i) An element $x \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if the cosets $x + \mathcal{J}_t$ are invertible in $\mathcal{B}/\mathcal{J}_t$ for all t.

(ii) $\bigcap\limits_t \mathcal{J}_t$ *is contained in the radical of* $\mathcal{B}$.

Remark 3.1 (see [PS], proof of Theorem 1.21) *If* $\mathcal{B}$, $\mathcal{C}$ *are* C^**-algebras, there is a close relation between the two local principles. Let* $M(\mathcal{C})$ *denote the maximal ideal space of* $\mathcal{C}$. *For* $t \in M(\mathcal{C})$ *we define* $M_t := \{a \in \mathcal{C} : 0 \le (Ga)(s) \le 1,\ (Ga)(s) \equiv 1$ *in some neighbourhood of* $t\}$, *where* $G : \mathcal{C} \to C(M(\mathcal{C}))$ *denotes the Gelfand map. Then* $\{M_t\}_{t \in M(\mathcal{C})}$ *forms a covering system of localizing classes in* $\mathcal{B}$, *and the local ideals occurring in the principle of Allan and Douglas can be described by* $\mathcal{J}_t = \{x \in \mathcal{B} : x \overset{M_t}{\sim} 0\}$.

3.2 Application to stability analysis

We want to investigate the applicability of the approximation method (2.6) to equation (2.7). In all what follows we assume that A_nP_n converges strongly to A, which allows us to reduce the problem to the question if $\{A_n\}$ is stable. Furthermore, we specify X to be a Hilbert space.

By $\mathcal{E}$ we denote the set of all operator sequences $\{A_nP_n\}$, where $A_n \in \mathcal{L}(X_n)$ and $\sup\limits_n \|A_nP_n\| < \infty$. Endowed with componentwise algebraic operations and the norm $\|\{B_n\}\|_{\mathcal{E}} = \sup\limits_n \|B_n\|$, $\mathcal{E}$ becomes a C^*-algebra. The set $\mathcal{N} := \{\{C_n\} \in \mathcal{E} : \|C_n\| \to 0\}$ is a closed ideal in $\mathcal{E}$. In the sequel we will use the notation $\mathcal{GB}$ for the set of all invertible elements of a Banach algebra $\mathcal{B}$. The following well-known result identifies the question of stability with an invertibility problem.

Lemma 3.2 ([HRS], Proposition 1.2) *A sequence* $\{A_n\} \in \mathcal{E}$ *is stable if and only if* $\{A_n\} + \mathcal{N} \in \mathcal{G}(\mathcal{E}/\mathcal{N})$.

We introduce a further family of operators $W_n \in \mathcal{L}(X)$, where we assume that $W_n = W_n^*$ converges weakly to 0, $W_nP_n = W_n$ and $W_n^2 = P_n$. Let $\mathcal{A}$ denote the set of all sequences $\{A_n\} \in \mathcal{E}$ for which A_n, A_n^*, $\widetilde{A}_n := W_nA_nW_n$ and $\widetilde{A}_n^*$ are strongly convergent.

Lemma 3.3 *The set* $\mathcal{A}$ *is a* C^**-algebra.*

Proof. Evidently, $\mathcal{A}$ is a linear space. If $\{A_n\} \in \mathcal{A}$, then so is $\{A_n^*\}$ (note that $\{W_nA_n^*W_n\} = \{(W_nA_nW_n)^*\}$). Let $\{A^{(n)}\}$ be a fundamental sequence in $\mathcal{A}$, where $A^{(n)} = \{A_k^{(n)}\}_{k=1}^\infty$. If $\varepsilon > 0$ is given, we have

$$\|A_k^{(n)} - A_k^{(m)}\| < \varepsilon$$

for all k if m, n are large enough. Hence, there is a sequence $\{A_k\} \in \mathcal{L}(X)$ such that $\|A_k^{(n)} - A_k\| \to 0 \quad (n \to \infty)$ uniformly with respect to k. If we choose $x \in X$, we can estimate

$$\|(A_k - A_l)x\| \le \|A_k - A_k^{(n)}\|\,\|x\| + \|A_l - A_l^{(n)}\|\,\|x\| + \|(A_k^{(n)} - A_l^{(n)})x\|.$$

The first two terms can be made arbitrarily small by the choice of n, and the third one goes to zero if n is fixed and k, $l \to \infty$, since $\{A_k^{(n)}\}_{k=1}^\infty \in \mathcal{A}$. Hence, A_k is strongly convergent. The sequences A_k^*, $\widetilde{A}_k$ and $\widetilde{A}_k^*$ are treated in the same way (note that $\|\widetilde{A}_k^{(n)} - \widetilde{A}_k^{(m)}\| \le$ *const* $\|A_k^{(n)} - A_k^{(m)}\|$ because of the weak convergence of W_n). ■

In the following, the coset $\{A_n\} + \mathcal{N}$ of a sequence $\{A_n\} \in \mathcal{E}$ will be denoted by $\widehat{\{A_n\}}$. The ideal $\mathcal{N}$ is contained in $\mathcal{A}$, and thus $\widehat{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ is a C^*-subalgebra of $\widehat{\mathcal{E}} := \mathcal{E}/\mathcal{N}$ and is therefore inverse-closed (that means $\widehat{\mathcal{A}} \cap \mathcal{G}\widehat{\mathcal{E}} = \mathcal{G}\widehat{\mathcal{A}}$). Hence, the stability of $\{A_n\} \in \mathcal{A}$ is equivalent to $\widehat{\{A_n\}} \in \mathcal{G}\widehat{\mathcal{A}}$.

In all what follows, $\mathcal{K}(X,Y)$ denotes the space of all compact linear operators between two Banach spaces X and Y. If $X = Y$, we briefly write $\mathcal{K}(X)$ instead of $\mathcal{K}(X,X)$.

Lemma 3.4 ([PS], 1.1.h) *Let X, Y, Z, V be Banach spaces, $\{A_n\} \subset \mathcal{L}(Z,V)$, $\{B_n\} \subset \mathcal{L}(X,Y)$ such that $A_n \to A \in \mathcal{L}(Z,V)$, $B_n^* \to B^* \in \mathcal{L}(Y^*, X^*)$ strongly. If $K \in \mathcal{K}(Y,Z)$, then $\|A_nKB_n - AKB\|_{\mathcal{L}(X,V)} \to 0 \quad (n \to \infty)$.*
If $K \in \mathcal{K}(Y,Z)$ and $B_n \rightharpoonup B \in \mathcal{L}(X,Y)$ (weakly), then $KB_n \to KB$ strongly.

Let $\mathcal{J}_0 := \{\widehat{\{P_nKP_n\}} : K \in \mathcal{K}(X)\}$, $\mathcal{J}_1 := \{\widehat{\{W_nKW_n\}} : K \in \mathcal{K}(X)\}$. By Lemma 3.4, $\mathcal{J}_0$, $\mathcal{J}_1$ are subsets of $\widehat{\mathcal{A}}$.

Lemma 3.5 ([Si], Satz 2) *$\mathcal{J}_0$, $\mathcal{J}_1$ are closed ideals in $\widehat{\mathcal{A}}$, and the smallest closed ideal containing $\mathcal{J}_0$ and $\mathcal{J}_1$ equals*

$$\mathcal{J} = \{\{P_nK_1P_n + W_nK_2W_n\} + \mathcal{N} : K_1,\ K_2 \in \mathcal{K}(X)\}.$$

Proof. We show that $\mathcal{I}_0 := \{\{P_nKP_n + C_n\} : K \in \mathcal{K}(X),\ \|C_n\| \to 0\}$ is a closed ideal in $\mathcal{A}$, whence the corresponding property of $\mathcal{J}_0$ in $\widehat{\mathcal{A}}$ follows immediately. Obviously, $\mathcal{I}_0$ is a linear space. Let $\{B_n\} \subset \mathcal{I}_0$ be a fundamental sequence, $B_n = \{A_k^{(n)}\}_{k=1}^{\infty}$, $A_k^{(n)} = P_kT^{(n)}P_k + C_k^{(n)}$, where $T^{(n)} \in \mathcal{K}(X)$, $\|C_k^{(n)}\| \to 0 \quad (k \to \infty)$. We have $A_k^{(n)} \to T^{(n)}$ (strongly), and hence for $\varepsilon > 0$ the relation $\|T^{(n)} - T^{(m)}\| \le \sup_k \|A_k^{(n)} - A_k^{(m)}\| < \varepsilon$ holds for n, m large enough. Therefore, $T^{(n)}$ converges uniformly to some $T \in \mathcal{K}(X)$. Besides, we can estimate

$$\|C_k^{(n)} - C_k^{(m)}\| \le \|P_k(T^{(n)} - T^{(m)})P_k\| + \|A_k^{(n)} - A_k^{(m)}\| < \varepsilon$$

for sufficiently large m, n independently of k. Thus, there exists a sequence $\{C_k\}$ with $\|C_k^{(n)} - C_k\| \to 0 \quad (n \to \infty)$, and since $\|C_k\| \le \|C_k^{(n)} - C_k\| + \|C_k^{(n)}\|$, we have $\{C_k\} \in \mathcal{N}$. If we put $B := \{P_kTP_k + C_k\}$, we have

$$\|B_n - B\|_{\mathcal{A}} \le const\ \|T^{(n)} - T\| + \sup_k \|C_k^{(n)} - C_k\| \longrightarrow 0 \qquad (n \to \infty),$$

which proves the closedness of $\mathcal{I}_0$. To show (for instance) that $\mathcal{I}_0$ is a left ideal, let $\{A_k\} \in \mathcal{A}$, $A_k \to A$, $\{B_k\} = \{P_kTP_k + C_k\} \in \mathcal{I}_0$. Then

$$\{A_k\}\{B_k\} = \{P_kA_kTP_k + A_kC_k\} = \{P_kATP_k + \underbrace{P_k(A_k - A)TP_k + A_kC_k\}}_{\in \mathcal{N}} \in \mathcal{I}_0$$

(cf. Lemma 3.4). The proof for $\mathcal{J}_1$, $\mathcal{J}$ is analogous. Obviously, $\mathcal{J}$ is contained in every ideal that contains $\mathcal{J}_0$ and $\mathcal{J}_1$, which completes the proof. ■

For $\widehat{\{A_n\}} \in \widehat{\mathcal{A}}$ we define

$$\mathcal{W}_0\widehat{\{A_n\}} := \text{s-}\lim_{n\to\infty} A_n, \qquad \mathcal{W}_1\widehat{\{A_n\}} := \text{s-}\lim_{n\to\infty} W_nA_nW_n.$$

Note that $\mathcal{W}_0$, $\mathcal{W}_1$ are correctly defined. Evidently, $\mathcal{W}_i \quad (i = 0, 1)$ are continuous *-homomorphisms from $\widehat{\mathcal{A}}$ into $\mathcal{L}(X)$ (remember that $\|\text{s-}\lim A_n\| \le \liminf \|A_n\|$). Further, it is easy to see that $\mathcal{W}_i|_{\mathcal{J}_i}$ is an isomorphism between $\mathcal{J}_i$ and the ideal $\mathcal{K}(X)$ of all compact linear operators on X, in other words, $\mathcal{W}_i$ is $\mathcal{J}_i$-lifting. If we apply Theorem 3.1 with $T = \{0, 1\}$ to this situation, we obtain the original version of the lifting theorem:

Theorem 3.4 ([Si], Satz 3) *Let $\{A_n\} \in \mathcal{A}$, $A_n \to A$, $\widetilde{A_n} \to \widetilde{A}$ strongly. Then $\{A_n\}$ is stable if and only if A, $\widetilde{A} \in \mathcal{GL}(X)$ and $\widehat{\{A_n\}} + \mathcal{J} \in \mathcal{G}(\widehat{\mathcal{A}}/\mathcal{J})$.*

Remark 3.6 *The third condition of the preceding theorem can be written in the equivalent form $\{A_n\} + \mathcal{I} \in \mathcal{G}(\mathcal{A}/\mathcal{I})$, with the ideal $\mathcal{I} = \{\{P_nK_1P_n + W_nK_2W_n + C_n\} : K_1,\ K_2 \in \mathcal{K}(X),\ \|C_n\| \to 0\}$.*

In the following two sections we are going to apply the tools from this section to the stability analysis of the collocation method. There we choose $P_n = P_n^\sigma$ and introduce the operators $W_n = W_n^\sigma$ defined by

$$W_n^\sigma \sum_{k=0}^{\infty} \alpha_k \tilde{u}_k = \sum_{k=0}^{n-1} \alpha_{n-1-k} \tilde{u}_k,$$

which obviously possess the properties required above. The approximation operators under consideration here are $A_n = \widetilde{L_n^\sigma}(aI + bS)P_n^\sigma$, $a, b \in PC$, in the space $X = L_\sigma^2$. We will show that, under the additional condition $b(\pm 1) = 0$, we have $\{A_n\} \in \mathcal{A}$, compute $\widetilde{A}$ and apply Theorem 3.4 (with Remark 3.6) to the stability problem. The invertibility of the coset $\{A_n\} + \mathcal{I}$ will be investigated by the local principle of Gohberg and Krupnik.

4 Strong convergence of the operator sequences

The strong convergence of $A_n = \widetilde{L_n^\sigma}(aI + bS)P_n^\sigma$ and of A_n^* will be shown for Riemann integrable coefficients. Unfortunately, the authors did not succeed in proving the convergence of $\widetilde{A_n}$ and $\widetilde{A_n}^*$ without the additional condition $b \in PC$ and $b(\pm 1) = 0$.

4.1 Strong convergence of A_n

First we give sufficient conditions for the weighted interpolation polynomial $\widetilde{L_n^\sigma} f$ to converge in the L_σ^2-norm. For this end, we provide some material from [Fr] concerning the convergence of Gaussian quadrature rules and Lagrangian interpolation operators.

Consider a Jacobi weight v. Let x_{kn}^v $(k = 1, \ldots, n)$ be the zeros of the orthogonal polynomial of degree n related to v, and L_n^v the Lagrangian interpolation operator with respect to x_{kn}^v. By Q_n^v we denote the Gaussian quadrature rule

$$Q_n^v f := \int_{-1}^{1} (L_n^v f)(x) v(x)\, dx = \sum_{k=1}^{n} A_{kn}^v f(x_{kn}^v).$$

Lemma 4.1 ([Fr], Hilfssatz III.1.5) *Let $g : (-1, 1) \to [0, \infty)$, $\quad g^{(2\nu)}(x) \geq 0$ for all $x \in (-1, 1)$ and $\nu = 0, 1, 2, \ldots$, and let $\int_{-1}^{1} g(x)v(x)\, dx < \infty$. Then*

$$Q_n^v g \leq \int_{-1}^{1} g(x) v(x)\, dx.$$

Lemma 4.2 ([Fr], Satz III.1.4) *Let f be bounded on $(-1, 1)$. Then*

$$Q_n^v f \to \int_{-1}^{1} f(x) v(x)\, dx \qquad (n \to \infty)$$

provided that this integral exists in the Riemann sense.

Lemma 4.3 (cf. [Fr], Satz III.1.6b) *Assume that f is bounded on every compact subinterval of $(-1, 1)$ and the (improper) Riemann integral $\int_{-1}^{1} f(x)v(x)\, dx$ exists. Suppose that there exist functions g_{-1}, g_1 satisfying the conditions of Lemma 4.1 and the relations*

$$\lim_{x \to -1+0} \frac{f(x)}{g_{-1}(x)} = \lim_{x \to 1-0} \frac{f(x)}{g_1(x)} = 0. \tag{4.1}$$

Then $Q_n^v f \to \int_{-1}^{1} f(x)v(x)\, dx \quad (n \to \infty)$.

Proof. Splitting the interval, we can restrict ourselves to the case $f(x) = 0$ in some neighbourhood of 1, $g := g_{-1}$. Let $\varepsilon > 0$ be arbitrary and $\delta > 0$ such that $|f(x)| \le \varepsilon g(x)$ for $-1 < x \le -1 + \delta$. Lemma 4.2 yields

$$\lim_{n\to\infty} \sum_{x_{kn}^v \ge -1+\delta} A_{kn}^v f(x_{kn}^v) = \int_{-1+\delta}^1 f(x)v(x)\,dx.$$

Furthermore,

$$\left| \sum_{x_{kn}^v < -1+\delta} A_{kn}^v f(x_{kn}^v) \right| \le \varepsilon \sum_{k=1}^n A_{kn}^v g(x_{kn}^v) \le \varepsilon \int_{-1}^1 g(x)v(x)\,dx$$

by Lemma 4.1, and

$$\left| \int_{-1}^{1+\delta} f(x)v(x)\,dx \right| \le \varepsilon \int_{-1}^1 g(x)v(x)\,dx. \quad \blacksquare$$

Remark 4.4 *If $v = v^{\gamma,\delta}$ with γ, δ satisfying (2.2) we can choose $g_{-1} = (1+x)^{-1-\delta+\varepsilon}$, $g_1(x) = (1-x)^{-1-\gamma+\varepsilon}$ with some $\varepsilon > 0$. Hence, the Gaussian quadrature rule converges if f is locally Riemann integrable and satisfies*

$$|f(x)| \le const(1-x)^{-1-\gamma+\varepsilon}(1+x)^{-1-\delta+\varepsilon}.$$

Lemma 4.5 ([Fr], Satz III.2.1) *Assume the hypotheses of Lemma 4.3 are fulfilled with $|f|^2$ instead of f and $\lim_{x\to-1+0} g_{-1}(x) = \lim_{x\to 1-0} g_1(x) = \infty$. Then $\lim_{n\to\infty} \|L_n^v f - f\|_{L_v^2} = 0$.*

Proof. According to [Fr], Satz III.4.3, the polynomials are dense in L_v^2. (This remains true in the complex case, since real and imaginary part can be approximated separately.) Let $\varepsilon > 0$, and let p be a polynomial with $\|p - f\|_{L_v^2} < \varepsilon$. For $n > \deg p$ we have

$$\begin{aligned} \|f - L_n^v f\|^2 &\le 2(\|f-p\|^2 + \|L_n^v(p-f)\|^2) \\ &< 2(\varepsilon^2 + Q_n^v(|p-f|^2)) \end{aligned}$$

(note that the Gaussian quadrature rule with n nodes is exact for polynomials of degree less than $2n$). Given $r > 0$, in a suitable neighbourhood of -1 the relation

$$|p(x) - f(x)|^2 \le 2(|p(x)|^2 + |f(x)|^2) < r g_{-1}(x)$$

holds (as well as the analogous relation in a neighbourhood of 1). Thus, (4.1) is satisfied with $|p-f|^2$ instead of f. We further have

$$\int_{-1}^1 |p(x) - f(x)|^2 v(x)\,dx \le 2\int_{-1}^1 (|p(x)|^2 + |f(x)|^2)v(x)\,dx < \infty.$$

Lemma 4.3 now yields $Q_n^v(|p-f|^2) \to \int_{-1}^1 |p(x)-f(x)|^2 v(x)\,dx < \varepsilon^2$. $\blacksquare$

Corollary 4.6 *Let $\sigma = v^{\alpha,\beta}$. If f is locally Riemann integrable on $(-1,1)$ and*

$$|f(x)| \le const(1-x)^{(-1-\alpha)/2+\varepsilon}(1+x)^{(-1-\beta)/2+\varepsilon}$$

with some $\varepsilon > 0$, then

$$\|\widetilde{L_n^\sigma} f - f\|_\sigma \to 0 \qquad (n \to \infty).$$

Proof. Using the isometric isomorphism $w_{\sigma^{-1}}I : L^2_\varphi \to L^2_\sigma$, we have

$$\|\widetilde{L^\sigma_n} f - f\|_\sigma = \|L^\varphi_n w^{-1}_{\sigma^{-1}} f - w^{-1}_{\sigma^{-1}} f\|_\varphi.$$

Now the assertion follows from Lemma 4.5 and Remark 4.4. ■

In the sequel we will investigate the behaviour of the operators A_n. First we do this for the multiplication operator $A = aI$ separately.

Proposition 4.1 *Let a be Riemann integrable. Then $\widetilde{L^\sigma_n} a P^\sigma_n \to aI$ strongly on L^2_σ. Furthermore, we have the estimation $\|\widetilde{L^\sigma_n} a P^\sigma_n\|_{\mathcal{L}(L^2_\sigma)} \le \|a\|_\infty$.*

Proof. First we show the convergence on the dense subset span $\{\tilde{u}_m\}_{m=0}^\infty$. Clearly, the functions $a\tilde{u}_m$ satisfy the conditions of Corollary 4.6. Thus, for $n > m$ we have $\|A_n \tilde{u}_m - A\tilde{u}_m\|_\sigma = \|\widetilde{L^\sigma_n} a\tilde{u}_m - a\tilde{u}_m\|_\sigma \to 0$. For showing the uniform boundedness, let $u \in L^2_\sigma$ and write $P^\sigma_n u = w_{\sigma^{-1}} q_n$ with a polynomial q_n of degree less than n. If we again note the exactness of the Gaussian quadrature rule, we have

$$\begin{aligned} \|\widetilde{L^\sigma_n} a P^\sigma_n u\|^2_\sigma &= \|L^\varphi_n a q_n\|^2_\varphi = Q^\varphi_n(|a q_n|^2) \\ &\le \|a\|^2_\infty Q^\varphi_n(|q_n|^2) = \|a\|^2_\infty \|q_n\|^2_\varphi = \|a\|^2_\infty \|P^\sigma_n u\|^2_\sigma \le \|a\|^2_\infty \|u\|^2_\sigma. \end{aligned}$$

Now the assertion follows from the Banach-Steinhaus theorem. ■

Let ϱ be a Jacobi weight and $\mu \in (0,1)$. By $H^\mu_0(\varrho)$ we denote the Banach space of all functions f for which $\varrho f \in C^{0,\mu}[-1,1]$ and $(\varrho f)(\pm 1) = 0$. The norm in this space is defined by $\|f\|_{H^\mu_0(\varrho)} := \|\varrho f\|_{C^{0,\mu}}$, where, as usually, $\|g\|_{C^{0,\mu}} = \|g\|_\infty + \sup\limits_{x \ne y} \dfrac{|g(x) - g(y)|}{|x-y|^\mu}$.

Lemma 4.7 ([GK], Theorem I.6.2) *Let $\varrho = v^{\gamma,\delta}$, $\mu \in (0,1)$ and $\mu < \gamma, \delta < \mu + 1$. Then the Cauchy singular integral operator S defined by (2.3) is bounded on $H^\mu_0(\varrho)$.*

Proposition 4.2 *The operators $\widetilde{L^\sigma_n} S P^\sigma_n$ converge to S on* span $\{\tilde{u}_m\}_{m=0}^\infty$.

Proof. Let $\varrho = v^{\gamma,\delta}$, where we choose γ, δ such that

$$\max\left\{0, \frac{\alpha}{2} - \frac{1}{4}\right\} < \gamma < \frac{1+\alpha}{2}, \qquad \max\left\{0, \frac{\beta}{2} - \frac{1}{4}\right\} < \delta < \frac{1+\beta}{2},$$

and let

$$0 < \mu < \min\left\{\gamma, \delta, \gamma + \frac{1}{4} - \frac{\alpha}{2}, \delta + \frac{1}{4} - \frac{\beta}{2}\right\}.$$

Then we have $\mu < \gamma, \delta < \mu + 1$ and $\tilde{u}_m \in H^\mu_0(\varrho)$. The latter relation follows from

$$\varrho \tilde{u}_m(x) = U_m(x)(1-x)^{\gamma + 1/4 - \alpha/2}(1+x)^{\delta + 1/4 - \beta/2},$$

the exponents being greater than μ. By Lemma 4.7, we also have $S\tilde{u}_m \in H^\mu_0(\varrho)$, which means that $h := \varrho S \tilde{u}_m \in C^{0,\mu}$. Thus, we can estimate

$$\begin{aligned} |(S\tilde{u}_m)(x)| &= |(\varrho^{-1} h)(x)| = |h(x)(1-x)^{-\gamma}(1+x)^{-\delta}| \\ &\le \mathit{const}\,(1-x)^{-(1+\alpha)/2 + \varepsilon}(1+x)^{-(1+\beta)/2+\varepsilon} \end{aligned}$$

if $\varepsilon > 0$ is small enough. Corollary 4.6 now gives the assertion. ■

In all what follows we exclude the cases $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ since they bring about some technical difficulties in the proofs that we could only partially overcome. Some remarks concerning these cases will be given in a separate subsection.

Lemma 4.8 (comp. [PS], 9.7 and 9.9) *Assume that (2.2) is satisfied. Let* $g, \tilde{b} \in C^{0,\eta}[-1,1]$ $(\eta \in (0,1))$ *be real-valued functions for which* $g(x) - i\tilde{b}(x) \neq 0$ *for all* $x \in [-1,1]$. *Let* λ, ν *be integers such that* $\alpha_0 := \lambda + \tilde{g}(1), \beta_0 := \nu - \tilde{g}(-1) \in (-1,1)$, *where* $g(x) - i\tilde{b}(x) = \sqrt{\tilde{b}^2(x) + g^2(x)}e^{i\pi\tilde{g}(x)}$ *with a continuous function* $\tilde{g}$. *Then there exists a positive function* $c \in C^{0,\eta}[-1,1]$ *such that the operator* $(gI + iS\tilde{b}I)v^{\alpha_0,\beta_0}cI$ *transforms every polynomial of degree* n *into a polynomial of degree* $n - \kappa$, *where* $\kappa = -(\lambda + \nu)$. *(If* $n < \kappa$, *this is to be understood in the sense that a polynomial of negative degree is identically zero.)*

If, as we assumed, $\alpha \neq \frac{1}{2}$ and $\beta \neq \frac{1}{2}$, we can choose $\tilde{b} \equiv 1$ and $g \in C^1$ such that $\lambda := \frac{1}{4} - \frac{\alpha}{2} - \tilde{g}(1)$ and $\nu := \frac{1}{4} - \frac{\beta}{2} + \tilde{g}(-1)$ are integers, whence we get the mapping properties described in Lemma 4.8 for the operator $(gI + iS)w_{\sigma^{-1}}cI$ with some positive function c, $c \in C^{0,\eta}$ for all $\eta \in (0,1)$. Furthermore, we can always achieve $\kappa \geq -1$ (we need the latter relation to guarantee the exactness of the Gaussian quadrature rule): Evidently, we always have $\tilde{g}(x) \in (-1,0)$.

- In case $\alpha_0 := \frac{1}{4} - \frac{\alpha}{2}$, $\beta_0 := \frac{1}{4} - \frac{\beta}{2} < 0$ we choose $\tilde{g}(1) = \alpha_0, \tilde{g}(-1) = -1 - \beta_0$ and have $\kappa = 1$.
- If $\alpha_0\,, \beta_0 > 0$, let $\tilde{g}(1) = \alpha_0 - 1$, $\tilde{g}(-1) = -\beta_0$.
- Finally, take $\tilde{g}(1) = \alpha_0$, $\tilde{g}(-1) = -\beta_0$ if $\alpha_0 < 0 < \beta_0$, and $\tilde{g}(1) = \alpha_0 - 1$, $\tilde{g}(-1) = -1 - \beta_0$ if $\beta_0 < 0 < \alpha_0$.

Our proof of the uniform boundedness of $\widetilde{L_n^\sigma} S P_n^\sigma$ will be based on the following decomposition of the operator S:

$$S = igI - ic^{-1}(gI + iS)cI + c^{-1}(cS - ScI). \tag{4.2}$$

Here $g \in C^1$ and c are the same as in Lemma 4.8. In particular, $c \in C^{0,\eta}$ for all $\eta \in (0,1)$. Now we are going to estimate the three summands of $\widetilde{L_n^\sigma} S P_n^\sigma$ separately.

By virtue of Proposition 4.1 we have $\|\widetilde{L_n^\sigma} g P_n^\sigma\| \leq \|g\|_\infty$, so we are done with the first term.

Lemma 4.9 ([Ne], Theorem 9.25) *Let* v, v^* *be Jacobi weights with* $vv^* \in L^1$. *Let* $l \in \mathbb{N}$ *be fixed and* q *a polynomial with* $\deg q \leq ln$. *Then*

$$\sum_{k=1}^{n} A_{kn}^v |q(x_{kn}^v)| v^*(x_{kn}^v) \leq const \int_{-1}^{1} |q(x)| v(x) v^*(x)\, dx,$$

the constant being independent of n *and* q.

Proposition 4.3 *We have the estimation*

$$\|\widetilde{L_n^\sigma} c^{-1}(gI + iS)cP_n^\sigma\|_{\mathcal{L}(L^2_\sigma)} \leq const\, \|c^{-1}\|_\infty\, \|gcI + iScI\|_{\mathcal{L}(L^2_\sigma)}.$$

Proof. According to Lemma 4.8, $q_{n-\kappa} := (gI + iS)cP_n^\sigma u$ is a polynomial of degree less than $n - \kappa$ for all $u \in L^2_\sigma$. Now we have

$$\|\widetilde{L_n^\sigma} c^{-1}(gI + iS)cP_n^\sigma u\|_\sigma^2 = \|L_n^\varphi c^{-1}(w_{\sigma^{-1}})^{-1} q_{n-\kappa}\|_\varphi^2$$

$$= \sum_{k=1}^{n} A_{kn}^\varphi |c^{-1}(x_{kn}^\varphi)|^2 (w_{\sigma^{-1}}(x_{kn}^\varphi))^{-2} |q_{n-\kappa}(x_{kn}^\varphi)|^2$$

$$
\begin{aligned}
&\le \|c^{-1}\|_\infty^2 \sum_{k=1}^n A_{kn}^\varphi (w_{\sigma^{-1}}(x_{kn}^\varphi))^{-2} |q_{n-\kappa}(x_{kn}^\varphi)|^2 \\
&\le const\ \|c^{-1}\|_\infty^2 \int_{-1}^1 |q_{n-\kappa}(x)|^2 (w_{\sigma^{-1}}(x))^{-2} \varphi(x)\, dx \\
&= const\ \|c^{-1}\|_\infty^2\ \|q_{n-\kappa}\|_\sigma^2 \\
&\le const\ \|c^{-1}\|_\infty^2\ \|gcI + iScI\|_{\mathcal{L}(L^2_\sigma)}^2\ \|u\|_\sigma^2,
\end{aligned}
$$

where we used Lemma 4.9 with $v = \varphi$, $v^* = w_{\sigma^{-1}}^{-2}$ for the estimation in the fourth line. ■

The following lemma is a generalization of [Ju1, Lemma 2.3].

Lemma 4.10 *Let $\sigma = v^{\alpha,\beta}$ and $0 < \gamma < m := \frac{1-\max\{\alpha,\beta,0\}}{2}$. Then*

$$
\int_{-1}^1 \left| \frac{1}{|t-x|^\gamma} - \frac{1}{|t-y|^\gamma} \right|^2 \sigma^{-1}(t)\, dt \le const\ |x-y|^{2\lambda}
$$

for all $\lambda \in (0,1)$ with $\lambda + \gamma < m$.

Proof. Let $\lambda_0 := \lambda + \gamma$. We have

$$
\begin{aligned}
|t-x|^{-\gamma} - |t-y|^{-\gamma} \quad = \quad & \frac{|t-x|^{\lambda_0-\gamma} - |t-y|^{\lambda_0-\gamma}}{|t-x|^{\lambda_0}} + \frac{|t-x|^{\lambda_0-\gamma} - |t-y|^{\lambda_0-\gamma}}{|t-y|^{\lambda_0}} \\
& + \frac{|t-y|^{2\lambda_0-\gamma} - |t-x|^{2\lambda_0-\gamma}}{|t-x|^{\lambda_0}|t-y|^{\lambda_0}}.
\end{aligned}
$$

Hence, we can estimate

$$
\begin{aligned}
&\left| |t-x|^{-\gamma} - |t-y|^{-\gamma} \right|^2 \\
&\qquad \le 3 \left(|t-x|^{\lambda_0-\gamma} - |t-y|^{\lambda_0-\gamma} \right)^2 \left(\frac{1}{|t-x|^{2\lambda_0}} + \frac{1}{|t-y|^{2\lambda_0}} \right) \\
&\qquad\quad + 3 \frac{\left(|t-y|^{2\lambda_0-\gamma} - |t-x|^{2\lambda_0-\gamma} \right)^2}{|t-x|^{2\lambda_0}|t-y|^{2\lambda_0}} \\
&\qquad \le const \left[|x-y|^{2(\lambda_0-\gamma)} \left(\frac{1}{|t-x|^{2\lambda_0}} + \frac{1}{|t-y|^{2\lambda_0}} \right) + \frac{|x-y|^{4\lambda_0-2\gamma}}{|t-x|^{2\lambda_0}|t-y|^{2\lambda_0}} \right] \\
&\qquad = const\ |x-y|^{2\lambda} \left[\frac{1}{|t-x|^{2\lambda_0}} + \frac{1}{|t-y|^{2\lambda_0}} + \left| \frac{1}{t-x} - \frac{1}{t-y} \right|^{2\lambda_0} \right] \\
&\qquad \le const\ |x-y|^{2\lambda} \left(\frac{1}{|t-x|^{2\lambda_0}} + \frac{1}{|t-y|^{2\lambda_0}} \right).
\end{aligned}
$$

Obviously, we have $\sigma^{-1} \in L^p$ for $p < \frac{1}{\max\{\alpha,\beta,0\}}$. Then for the adjoint exponent q the relation $q > \frac{1}{1-\max\{\alpha,\beta,0\}} = \frac{1}{2m}$ holds. Since $\lambda_0 < m$, we can guarantee $2\lambda_0 q < 1$. Hence, the Hölder inequality gives

$$
\int_{-1}^1 \frac{\sigma^{-1}(t)}{|t-x|^{2\lambda_0}}\, dt \le \|\sigma^{-1}\|_{L^p} \left(\int_{-1}^1 \frac{dt}{|t-x|^{2\lambda_0 q}} \right)^{\frac{1}{q}} \le const
$$

independently of x, and the lemma is proved. ■

Lemma 4.11 *Let $\sigma = v^{\alpha,\beta}$ and $c \in C^{0,\eta}$ with $\eta > \frac{1+\max\{\alpha,\beta,0\}}{2}$. Then $K := cS - ScI \in \mathcal{K}(L^2_\sigma, C^{0,\lambda})$ for some $\lambda > 0$.*

Proof. Choose $1-\eta < \gamma < \frac{1-\max\{\alpha,\beta,0\}}{2}$ and put $k(t,x) := \frac{c(t)-c(x)}{t-x}|t-x|^\gamma$. If $\lambda \le \eta-(1-\gamma)$, we have $k \in C^{0,\lambda}$ in both variables, uniformly with respect to the other ([Mu], §5). Moreover, we require $\gamma + \lambda < \frac{1-\max\{\alpha,\beta,0\}}{2}$. Let now $u \in L^2_\sigma$. We can write

$$\begin{aligned} |(Ku)(x) - (Ku)(y)| &\le \int_{-1}^1 \left| \frac{k(t,x)}{|t-x|^\gamma} - \frac{k(t,y)}{|t-y|^\gamma} \right| |u(t)|\, dt \\ &\le \left(\int_{-1}^1 \left| \frac{k(t,x)}{|t-x|^\gamma} - \frac{k(t,y)}{|t-y|^\gamma} \right|^2 \sigma^{-1}(t)\, dt \right)^{\frac{1}{2}} \|u\|_\sigma, \end{aligned}$$

and, using Lemma 4.10, the following estimation holds:

$$\begin{aligned} &\int_{-1}^1 \left| \frac{k(t,x)}{|t-x|^\gamma} - \frac{k(t,y)}{|t-y|^\gamma} \right|^2 \sigma^{-1}(t)\, dt \le \\ &\le \int_{-1}^1 \left(\left| \frac{k(t,x)-k(t,y)}{|t-x|^\gamma} \right| + |k(t,y)| \left| \frac{1}{|t-x|^\gamma} - \frac{1}{|t-y|^\gamma} \right| \right)^2 \sigma^{-1}(t)\, dt \\ &\le 2 \int_{-1}^1 \left(\left| \frac{k(t,x)-k(t,y)}{|t-x|^\gamma} \right|^2 + |k(t,y)|^2 \left| \frac{1}{|t-x|^\gamma} - \frac{1}{|t-y|^\gamma} \right|^2 \right) \sigma^{-1}(t)\, dt \\ &\le const\, |x-y|^{2\lambda} \int_{-1}^1 \frac{\sigma^{-1}(t)\, dt}{|t-x|^{2\gamma}} + const\, |x-y|^{2\lambda} \\ &\le const\, |x-y|^{2\lambda}. \end{aligned}$$

Hence, all functions in $\{Ku : \|u\|_\sigma \le 1\}$ uniformly satisfy a Hölder condition with the exponent λ. It remains to show the uniform boundedness of these functions. Using Lemma 2.4 from [CJLM], we get

$$\begin{aligned} |(Ku)(x)| &\le \frac{1}{\pi} \int_{-1}^1 \left| \frac{c(t)-c(x)}{t-x} \right| |u(t)|\, dt \\ &\le const \left(\int_{-1}^1 \frac{\sigma^{-1}(t)\, dt}{|t-x|^{2(1-\eta)}} \right)^{\frac{1}{2}} \|u\|_\sigma \\ &\le const\, (1-x)^{-\alpha^+/2}(1+x)^{-\beta^+/2} \|u\|_\sigma, \end{aligned}$$

where $\alpha^+ := \max\{0,\alpha\}, \beta^+ := \max\{0,\beta\}$. In particular, $\|u\|_\sigma \le 1$ implies $|(Ku)(0)| \le const$, which together with the uniform Hölder condition results in $\|Ku\|_\infty \le const$. Thus, we have $K \in \mathcal{L}(L^2_\sigma, C^{0,\lambda})$ for some $\lambda > 0$, and the assertion follows if we note that the embedding $C^{0,\lambda} \subset C^{0,\lambda'}$ is compact for $\lambda > \lambda'$. ∎

Using Lemma 4.11 and Corollary 4.6, we can now estimate the third summand:

$$\|\widetilde{L_n^\sigma} c^{-1} K P_n^\sigma\|_{\mathcal{L}(L^2_\sigma)} \le \|\widetilde{L_n^\sigma}\|_{\mathcal{L}(C^{0,\lambda}, L^2_\sigma)}\, \|c^{-1}\|_{C^{0,\lambda}}\, \|K\|_{\mathcal{L}(L^2_\sigma, C^{0,\lambda})} \le const.$$

Thus, we have proved the uniform boundedness of $\widetilde{L_n^\sigma} S P_n^\sigma$. If we note the obvious identity $\widetilde{L_n^\sigma} b S P_n^\sigma = \widetilde{L_n^\sigma} b P_n^\sigma\ \widetilde{L_n^\sigma} S P_n^\sigma$, we can summarize the results of this subsection as follows:

Theorem 4.1 *Let a, b be Riemann integrable on $[-1,1]$. Then the operators $A_n = \widetilde{L_n^\sigma}(aI + bS)P_n^\sigma$ converge strongly to $aI + bS$ on L^2_σ.*

4.2 The cases $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$

If one of the numbers $\alpha_0 := \frac{1}{4} - \frac{\alpha}{2}$, $\beta_0 := \frac{1}{4} - \frac{\beta}{2}$ is zero, the decomposition (4.2), on which our proof of the uniform boundedness of $\widetilde{L_n^\sigma} SP_n^\sigma$ was based, is not possible. We cannot choose $\tilde{b} \equiv 1$ in Lemma 4.8 since in this case $\alpha_0 - \tilde{g}(1)$ (or $\beta_0 + \tilde{g}(-1)$, respectively) cannot be an integer. We can, however, to some extent overcome the difficulties connected with this fact if we estimate the whole term $\widetilde{L_n^\sigma} bSP_n^\sigma$ instead of considering $\widetilde{L_n^\sigma} SP_n^\sigma$ separately.

Assume for instance $\alpha_0 = 0$, $\beta_0 \neq 0$. In this case we can proceed as follows. Let b be a function satisfying the following conditions:

$$\begin{cases} b \in C^{0,\eta} \text{ for some } \eta > \frac{1+\max\{\alpha,\beta,0\}}{2} \\ b(1) = 0 \\ b(-1) \neq 0. \end{cases} \tag{4.3}$$

(If $\beta_0 = 0$, we would also require $b(-1) = 0$). For the following argument we can assume without loss of generality that b is real-valued and $b(-1) < 0$. Now we choose some $g \in C^1$, $g(1) = 1$ (that is, $\tilde{g}(1) = 0$) such that the operator $(gI + iSbI)w_{\sigma^{-1}}cI$ possesses the mapping properties described in Lemma 4.8 with $\kappa \geq -1$. If $\beta_0 > 0$, we can choose g such that $\tilde{g}(-1) = 1 - \beta_0$, whence $\kappa = -1$, if $\beta_0 < 0$ we choose $\tilde{g}(-1) = -\beta_0$ and have $\kappa = 0$. (If we had $\beta_0 = 0$, we would simply take $g \equiv 1$, which means $\kappa = 0$.)

Instead of (4.2) we now use the decomposition

$$bS = SbI + K = igI - ic^{-1}(gI + iSbI)cI + c^{-1}(cS - ScI)bI + K,$$

where $K = bS - SbI \in \mathcal{K}(L^2_\sigma, C^{0,\mu})$ for some $\mu > 0$ (compare Lemma 4.11). This enables us to show in the same way as before that $A_n = \widetilde{L_n^\sigma} bSP_n^\sigma$ and A_n^* converge strongly if b satisfies (4.3). In particular, we have $\|\widetilde{L_n^\sigma}(1-x)^\eta SP_n^\sigma\| \leq const$ if $\eta > \frac{1+\max\{\alpha,\beta,0\}}{2}$.

Proposition 4.4 *Let $\alpha_0 = 0$, $\beta_0 \neq 0$. Let $b \in PC$ and $b(x) = o((1-x)^\eta)$ for $x \to 1$ with some $\eta > \frac{1+\max\{\alpha,\beta,0\}}{2}$. Then $\|\widetilde{L_n^\sigma} bSP_n^\sigma\| \leq const$.*

Proof. Let χ be Riemann integrable, and let $\tilde{b} \in P_1$, which denotes the set of all polynomials vanishing in 1. Then $\widetilde{L_n^\sigma}\chi\tilde{b}SP_n^\sigma = \widetilde{L_n^\sigma}\chi(1+k)^k P_n^\sigma \widetilde{L_n^\sigma}\tilde{b}_1 SP_n^\sigma$ with some nonnegative integer k, where $\tilde{b}_1$ satisfies (4.3). Hence, the uniform boundedness of $\widetilde{L_n^\sigma} bSP_n^\sigma$ continues to hold if b is the product of a polynomial from P_1 with a Riemann integrable function.

Now let b be as in the hypothesis with a finite number of jumps. Then $(1-x)^{-\eta}b$ is piecewise continuous and can therefore be approximated uniformly by a piecewise polynomial $\tilde{b} = \sum_{j=1}^m \chi_j \tilde{b}_j$, where $\tilde{b}_j \in P_1$, χ_j is the characteristic function of $[x_j, x_{j+1}]$ and $-1 = x_1 < x_2 < \ldots < x_{m+1} = 1$. (Note that P_1 is dense in $C[x_j, x_{j+1}]$ for all $j = 1, \ldots m$ due to the Stone-Weierstraß theorem.) Then we have

$$\begin{aligned} \left\|\widetilde{L_n^\sigma}\left((1-x)^\eta\tilde{b} - b\right)SP_n^\sigma\right\| &\leq \left\|\widetilde{L_n^\sigma}\left(\tilde{b} - (1-x)^{-\eta}b\right)P_n^\sigma\right\| \left\|\widetilde{L_n^\sigma}(1-x)^\eta SP_n^\sigma\right\| \\ &\leq const \left\|\tilde{b} - (1-x)^{-\eta}b\right\|_\infty, \end{aligned}$$

which can be made as small as desired by the choice of $\tilde{b}$. If we note the fact that the set of all sequences from $\mathcal{E}$ which, together with their adjoint operators, are strongly convergent is a closed subalgebra of $\mathcal{E}$ (compare the proof of Lemma 3.3), we get the assertion for b with a finite number of jumps. If $b \in PC$ is arbitrary, $b(x) = o((1-x)^\eta)$, we approximate $(1-x)^{-\eta}b$ uniformly by a function $\tilde{b}$ with the same properties and only finitely many jumps and repeat the same arguments as above to get the assertion for b. ■

Remark 4.12 *If $\alpha_0 = \beta_0 = 0$, we would have to require $b = o((1-x^2)^\eta)$ for $x \to \pm 1$ in the preceding proposition.*

4.3 Strong convergence of A_n^*

In all what follows we identify the dual space of L^2_σ with L^2_σ itself and consider A_n^* as an element of $\mathcal{L}(L^2_\sigma)$. As an auxiliary relation, we deduce a formula for the Fourier coefficients of $\widetilde{L_n^\sigma} f$: We have $\widetilde{L_n^\sigma} f = \sum_{k=0}^{n-1} \alpha_k \tilde{u}_k$, where, because of the exactness of the Gaussian quadrature rule,

$$
\begin{aligned}
\alpha_k &= \sum_{s=0}^{n-1} \alpha_s \overbrace{\sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma \tilde{u}_k(x_{jn}^\varphi)\tilde{u}_s(x_{jn}^\varphi)}^{=\delta_{ks}} \\
&= \sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma \tilde{u}_k(x_{jn}^\varphi) \sum_{s=0}^{n-1} \alpha_s \tilde{u}_s(x_{jn}) \\
&= \sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma \tilde{u}_k(x_{jn}^\varphi) f(x_{jn}^\varphi),
\end{aligned}
\tag{4.4}
$$

where $\widetilde{A}_{jn}^\sigma := w_{\sigma^{-1}}^{-2}(x_{jn}^\varphi) A_{jn}^\varphi$ and δ_{ks} denotes the Kronecker symbol.

Now we compute $(\widetilde{L_n^\sigma} a P_n^\sigma)^*$ with Riemann integrable a. For $u = \sum_{k=0}^\infty u_k \tilde{u}_k$, $v = \sum_{k=0}^\infty v_k \tilde{u}_k$ we have due to (4.4)

$$
\begin{aligned}
&(\widetilde{L_n^\sigma} a P_n^\sigma u, v)_\sigma = \\
&\quad = \sum_{k=0}^{n-1} \overline{v_k} \left(\sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma a(x_{jn}^\varphi) \sum_{s=0}^{n-1} u_s \tilde{u}_s(x_{jn}^\varphi)\, \tilde{u}_k(x_{jn}^\varphi) \right) \\
&\quad = \sum_{s=0}^{n-1} u_s \overline{\left(\sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma \overline{a}(x_{jn}^\varphi) \sum_{k=0}^{n-1} v_k \tilde{u}_k(x_{jn}^\varphi)\, \tilde{u}_s(x_{jn}^\varphi) \right)} \\
&\quad = (u, \widetilde{L_n^\sigma} \overline{a} P_n^\sigma v)^\sigma,
\end{aligned}
$$

that means

$$
(\widetilde{L_n^\sigma} a P_n^\sigma)^* = \widetilde{L_n^\sigma} \overline{a} P_n^\sigma,
$$

which is strongly convergent due to Proposition 4.1.

Since we have $(\widetilde{L_n^\sigma} b S P_n^\sigma)^* = (\widetilde{L_n^\sigma} S P_n^\sigma)^* \, (\widetilde{L_n^\sigma} b P_n^\sigma)^*$, we can now restrict ourselves to investigating $(\widetilde{L_n^\sigma} S P_n^\sigma)^*$. This will be done again by using a three-term decomposition according to (4.2). The multiplication operator was already considered. To deal with $(gI + iS)cI$ (we can obviously neglect the factor c^{-1}), we note that in Lemma 4.8 there is always $\kappa \in \{-1, 0, 1\}$, which implies that

$$
q_{n-\kappa} := (gI + iS) c P_n^\sigma u
$$

is always a polynomial of degree at most n. If $u, v \in L^2_\sigma$ and $\widetilde{A}_{jn}^\sigma$ are as above, we can write

$$
\begin{aligned}
&(\widetilde{L_n^\sigma} (gI + iS) c P_n^\sigma u, v)_\sigma = \\
&\quad = \sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma q_{n-\kappa}(x_{jn}^\varphi) \overline{(P_n^\sigma v)(x_{jn}^\varphi)} \\
&\quad = (w_{\sigma^{-1}} q_{n-\kappa}, \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} P_n^\sigma v)_\sigma
\end{aligned}
$$

$$= (w_{\sigma^{-1}}(gI + iS)cP_n^\sigma u, \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} P_n^\sigma v)_\sigma$$

$$= (u, P_n^\sigma c(gI + iS)^* w_{\sigma^{-1}} \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} P_n^\sigma v)_\sigma.$$

(Note that $gI + iS \in \mathcal{L}(L^2_\sigma)$ and $w_{\sigma^{-1}} \widetilde{L_n^\sigma} f \in L^2_\sigma$.) Hence, we have

$$(\widetilde{L_n^\sigma}(gI + iS)cP_n^\sigma)^* = P_n^\sigma c(gI + iS)^* w_{\sigma^{-1}} \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} P_n^\sigma.$$

Lemma 4.13 (Cf. [MR], Cor. 3.2 and the following remark; [DT], Th. 6.2.1) *Let u, w be Jacobi weights satisfying*

$$\frac{u}{\sqrt{w\varphi}}, \quad \frac{\sqrt{w\varphi}}{u} \in L^2.$$

If f is a function with $f'\varphi u \in L^2$, then the following estimation holds:

$$\|u(L_n^w f - f)\|_{L^2} \le \frac{const}{n} \|f'\varphi u\|_{L^2}.$$

Lemma 4.14 *The operators $P_n^\sigma c(gI + iS)^* w_{\sigma^{-1}} \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} P_n^\sigma$ converge strongly in L^2_σ.*

Proof. Since the uniform boundedness is trivial in view of Lemma 4.3, we only have to show the convergence on span $\{\tilde{u}_m\}_{m=0}^\infty$. First we consider the term $w_{\sigma^{-1}} \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} I$. Let $m \ge n$. We can write

$$\|w_{\sigma^{-1}} \widetilde{L_n^\sigma} w_{\sigma^{-1}}^{-1} \tilde{u}_m - \tilde{u}_m\|_\sigma = \|\widetilde{L_n^\sigma} U_m - U_m\|_\varphi = \|\varphi^{1/2} w_{\sigma^{-1}} (L_n^\varphi w_{\sigma^{-1}}^{-1} U_m - w_{\sigma^{-1}}^{-1} U_m)\|_{L^2}.$$

We now apply Lemma 4.13 with $f = w_{\sigma^{-1}}^{-1} U_m$, $u = \varphi^{1/2} w_{\sigma^{-1}}$ and $w = \varphi$, which allows us to estimate the last expression by

$$\frac{const}{n} \|(w_{\sigma^{-1}}^{-1} U_m)' \, \varphi^{1/2} w_{\sigma^{-1}} \, \varphi\|_{L^2} \le \frac{const}{n} \longrightarrow 0 \qquad (n \to \infty).$$

Since $P_n^\sigma \to I$ and $(gI + iS)^* \in \mathcal{L}(L^2_\sigma)$, the assertion follows. ∎

Due to Lemma 4.11, we have $K := cS - ScI \in \mathcal{K}(L^2_\sigma, C^{0,\lambda})$ for some $\lambda > 0$. Furthermore, $(P_n^\sigma)^* = P_n^\sigma \to I$ in L^2_σ and $\widetilde{L_n^\sigma} \to E$ (cf. Corollary 4.6), where E denotes the continuous embedding of $C^{0,\lambda}$ into L^2_σ. Thus, if we write again K instead of EK, Lemma 3.4 gives

$$\|\widetilde{L_n^\sigma} K P_n^\sigma - K\|_{\mathcal{L}(L^2_\sigma)} \longrightarrow 0 \qquad (n \to \infty),$$

and hence

$$\|(\widetilde{L_n^\sigma} K P_n^\sigma)^* - K^*\|_{\mathcal{L}(L^2_\sigma)} \longrightarrow 0 \qquad (n \to \infty).$$

Thus, we have proved the following theorem:

Theorem 4.2 *If a, b are Riemann integrable, then $(\widetilde{L_n^\sigma}(aI + bS)P_n^\sigma)^*$ is strongly convergent.*

4.4 Strong convergence of $\widetilde{A_n}$

First we consider the case of a multiplication operator $A = aI$ with a Riemann integrable function a. The following lemma together with Proposition 4.1 shows the convergence of $\widetilde{A_n}$ in this case.

Lemma 4.15 *We have* $W_n^\sigma \widetilde{L_n^\sigma} a W_n^\sigma \equiv \widetilde{L_n^\sigma} a P_n^\sigma$.

Proof. It is sufficient to show the identity on span $\{\tilde{u}_m\}$. For $n > m$ we have (compare relation (4.4))

$$\begin{aligned}
W_n^\sigma \widetilde{L_n^\sigma} a W_n^\sigma \tilde{u}_m &= W_n^\sigma \widetilde{L_n^\sigma} a \tilde{u}_{n-1-m} \\
&= W_n^\sigma \sum_{k=0}^{n-1} \left(\sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma \tilde{u}_k(x_{jn}^\varphi) \tilde{u}_{n-1-m}(x_{jn}^\varphi) a(x_{jn}^\varphi) \right) \tilde{u}_k \\
&= \sum_{k=0}^{n-1} \left(\sum_{j=1}^{n} \widetilde{A}_{jn}^\sigma \tilde{u}_{n-1-k}(x_{jn}^\varphi) \tilde{u}_{n-1-m}(x_{jn}^\varphi) a(x_{jn}^\varphi) \right) \tilde{u}_k =: \sum_{k=0}^{n-1} \beta_{km} \tilde{u}_k.
\end{aligned}$$

If we remember that $x_{jn}^\varphi = \cos \frac{j\pi}{n+1} \quad (j = 1, \ldots, n)$ and $A_{jn}^\varphi = \pi \frac{1-(x_{jn}^\varphi)^2}{n+1}$, we get

$$\begin{aligned}
\beta_{km} &= \frac{2}{n+1} \sum_{j=1}^{n} \sin \frac{(n-k)j\pi}{n+1} \sin \frac{(n-m)j\pi}{n+1} a(x_{jn}^\varphi) \\
&= \frac{2}{n+1} \sum_{j=1}^{n} \sin \frac{(k+1)j\pi}{n+1} \sin \frac{(m+1)j\pi}{n+1} a(x_{jn}^\varphi) \\
&= \beta_{n-1-k,n-1-m},
\end{aligned}$$

which means $W_n^\sigma \widetilde{L_n^\sigma} a W_n^\sigma \tilde{u}_m = \widetilde{L_n^\sigma} a P_n^\sigma \tilde{u}_m$. ■

Let $V := xI - iw_{\sigma^{-1}} S w_\sigma I$, that is, $(Vu)(x) = xu(x) - iw_{\sigma^{-1}}(x)(Sw_\sigma u)(x)$ for $u \in L_\sigma^2$. The following lemma is a generalization of a result from [RR] (cf. also [PS, Th. 4.123]), which is formulated there for the case $\alpha = \beta = 0$.

Lemma 4.16 ([RR], see also [PS], Theorem 4.123) *V is a shift operator with respect to the system $\{\tilde{u}_n\}_{n=0}^\infty$, more precisely, the relation*

$$V \sum_{k=0}^{\infty} \alpha_k \tilde{u}_k = \sum_{k=0}^{\infty} \alpha_k \tilde{u}_{k+1}$$

holds. The adjoint operator, which satisfies

$$V^* \sum_{k=0}^{\infty} \alpha_k \tilde{u}_k = \sum_{k=0}^{\infty} \alpha_{k+1} \tilde{u}_k,$$

is given by

$$V^* = xI + iw_{\sigma^{-1}} S w_\sigma I.$$

Proof. For the shift property of V compare the proof of [PS, Theorem 4.123]. To verify the representation of V^*, note that $\sigma^{1/2} I : L_\sigma^2 \to L^2$ is an isometric isomorphism and $S^* = S$ in L^2. ■

Lemma 4.17 ([Lu], Lemma 3.10, cf. also [Mu], §5) *Assume that*

$$b \in C^{p,\eta}[-1,1] \quad (0 < \eta \leq 1) \qquad \text{and} \qquad b^{(j)}(\pm 1) = 0 \quad (j = 0, \ldots, p).$$

Let further $v = v^{-\gamma,-\delta}$ *and* $\lambda := \eta - \max\{\gamma, \delta, 0\} > 0$. *Then*

$$bv \in C^{p,\lambda}[-1,1] \qquad \text{and} \qquad (bv)^{(j)}(\pm 1) = 0 \quad (j = 0, \ldots, p).$$

We introduce the notations

$$PC_0 := \{b \in PC[-1,1] : b(\pm 1) = 0\},$$

$$C_0 := \{b \in C[-1,1] : b(\pm 1) = 0\}$$

and

$$C_{0,0}^{1,\eta} := \left\{b \in C^{1,\eta}[-1,1] : b(\pm 1) = b'(\pm 1) = 0\right\}.$$

Proposition 4.5 *Let* $b \in PC_0$. *Then* $W_n^\sigma \widetilde{L_n^\sigma} b S W_n^\sigma$ *converges strongly to* $-b w_\sigma^{-1} S w_\sigma I$.

Proof. First let $b \in C_{0,0}^{1,\eta}$, where $\eta > \max\{\frac{1}{4} + \frac{\alpha}{2}, \frac{1}{4} + \frac{\beta}{2}, 0\}$. We use the following decomposition of bS (cf. Lemma 4.16):

$$\begin{aligned} bS &= b w_\sigma^{-1} S w_\sigma I + \underbrace{bS - SbI + (S b w_\sigma^{-1} I - b w_\sigma^{-1} S) w_\sigma I}_{=:K} \\ &= i b \varphi^{-1} (xI - V^*) + K. \end{aligned} \tag{4.5}$$

Using Lemma 4.15, we can manage the first two summands (note that $b\varphi^{-1}$ is continuous):

$$i W_n^\sigma \widetilde{L_n^\sigma} b \varphi^{-1} x W_n^\sigma = i \widetilde{L_n^\sigma} b \varphi^{-1} x P_n^\sigma \longrightarrow i b \varphi^{-1} x I$$

and

$$\begin{aligned} -i W_n^\sigma \widetilde{L_n^\sigma} b \varphi^{-1} V^* W_n^\sigma &= -i W_n^\sigma \widetilde{L_n^\sigma} b \varphi^{-1} W_n^\sigma \cdot W_n^\sigma V^* W_n^\sigma \\ &= -i \widetilde{L_n^\sigma} b \varphi^{-1} P_n^\sigma \cdot P_n^\sigma V \longrightarrow -i b \varphi^{-1} V. \end{aligned}$$

The multiplication operator $w_\sigma I$ is an isometric isomorphism from L_σ^2 onto $L_{\varphi^{-1}}^2$. Since $b \in C_{0,0}^{1,\eta}$, we get $b\varphi^{-1} \in C^1$ from Lemma 4.17, and Lemma 4.11 gives us $S b w_\sigma^{-1} I - b w_\sigma^{-1} S \in \mathcal{K}(L_{\varphi^{-1}}^2, C^{0,\lambda})$ as well as $bS - SbI \in \mathcal{K}(L_\sigma^2, C^{0,\lambda})$ with some $\lambda > 0$. Hence, $K \in \mathcal{K}(L_\sigma^2, C^{0,\lambda})$. Since W_n^σ converges weakly to 0, we have $K W_n^\sigma \to 0$ strongly in $\mathcal{L}(L_\sigma^2, C^{0,\lambda})$ by virtue of Lemma 3.4. Moreover, we have $\|W_n^\sigma \widetilde{L_n^\sigma}\| \leq \|W_n^\sigma\| \|\widetilde{L_n^\sigma}\|_{\mathcal{L}(C^{0,\lambda}, L_\sigma^2)} \leq const$, which results in $W_n^\sigma \widetilde{L_n^\sigma} K W_n^\sigma \to 0$ (strongly). Thus, we can conclude

$$W_n^\sigma \widetilde{L_n^\sigma} b S W_n^\sigma \to i b \varphi^{-1} (xI - V) = -b w_\sigma^{-1} S w_\sigma I \qquad (b \in C_{0,0}^{1,\eta}).$$

If χ is an arbitrary Riemann integrable function and $b \in C_{0,0}^{1,\eta}$, we get (using Lemma 4.15)

$$W_n^\sigma \widetilde{L_n^\sigma} \chi b S W_n^\sigma = W_n^\sigma \widetilde{L_n^\sigma} \chi W_n^\sigma \cdot W_n^\sigma \widetilde{L_n^\sigma} b S W_n^\sigma \to -\chi b w_\sigma^{-1} S w_\sigma I. \tag{4.6}$$

Now we consider the general case $b \in PC_0$. Without loss of generality we can restrict ourselves to investigating functions with a finite number of jumps, that is, we can write $b = \sum_{j=1}^m \chi_j b_j$, where $b_j \in C_0$, $\chi_j = \chi_{[x_j, x_{j+1}]}$ is the characteristic function of the subinterval $[x_j, x_{j+1}]$, and $-1 = x_1 < x_2 < \ldots < x_m < x_{m+1} = 1$ is an arbitrary partition of $[-1,1]$.

Now we approximate b by piecewise $C^{1,\eta}_{0,0}$-functions: Let $\varepsilon > 0$ and choose numbers $y_1 \in (-1, x_2)$, $y_{m+1} \in (x_m, 1)$ such that

$$|b_1(x)| < \frac{\varepsilon}{2} \text{ for } x \le y_1, \qquad |b_m(x)| < \frac{\varepsilon}{2} \text{ for } x \ge y_{m+1}.$$

Let (for instance) $\tilde{b}_0(x) = \tilde{b}_{m+1}(x) := \frac{\varepsilon}{2}(1-x)^2(1+x)^2$. Then $\tilde{b}_0, \tilde{b}_{m+1} \in C^{1,\eta}_{0,0}$,

$$|b_1(x) - \tilde{b}_0(x)| < \varepsilon \text{ for } x \in [-1, y_1], \qquad |b_m(x) - \tilde{b}_{m+1}(x)| < \varepsilon \text{ for } x \in [y_{m+1}, 1].$$

Put $y_j := x_j \quad (j = 2, \dots, m)$, $y_0 := -1$, $y_{m+2} := 1$. For $j = 1, \dots, m$ we choose functions $\tilde{b}_j \in C^{1,\eta}_{0,0}$ such that

$$|b_j(x) - \tilde{b}_j(x)| < \varepsilon, \qquad x \in [y_j, y_{j+1}].$$

If we define $\tilde{\chi}_j := \chi_{[y_j, y_{j+1}]}$, $\tilde{b} := \sum_{j=0}^{m+1} \tilde{\chi}_j \tilde{b}_j$, we obviously have $\|b - \tilde{b}\|_\infty < \varepsilon$. Since

$$\|\{\widetilde{L^\sigma_n}(b - \tilde{b}) S P^\sigma_n\}\|_{\mathcal{E}} \le \|\widetilde{L^\sigma_n}(b - \tilde{b}) P^\sigma_n\| \, \|\widetilde{L^\sigma_n} S P^\sigma_n\| \le const \, \|b - \tilde{b}\|_\infty < \varepsilon$$

and since $\mathcal{A}$ is a closed subalgebra of $\mathcal{E}$, we can conclude from (4.6) and Proposition 4.6 from the following subsection that $W^\sigma_n \widetilde{L^\sigma_n} b S W^\sigma_n$ is strongly convergent. The continuity of the homomorphism $\mathcal{W}_1$ and the relation

$$\|(b - \tilde{b}) w_\sigma^{-1} S w_\sigma I\| \le const \, \|b - \tilde{b}\|_\infty$$

imply that $\mathcal{W}_1 \{\widetilde{L^\sigma_n} b S P^\sigma_n\} = -b w_\sigma^{-1} S w_\sigma I$ for $b \in PC_0$ with finitely many jumps. An arbitrary PC_0-function can be approximated uniformly by such functions, and we can repeat the same arguments as above to get the assertion. ■

Remark 4.18 *If we have one of the special cases from subsection 4.2, say* $\alpha = \frac{1}{2}$, *we require* $b(x) = o((1-x)^\xi)$, $\xi > \frac{1+\max\{\alpha,\beta,0\}}{2}$ *and choose* $\tilde{b}$ *in the preceding proof such that* $\|(b - \tilde{b})(1-x)^{-\xi}\|_\infty$ *becomes small. Thus we can estimate* $\|\{\widetilde{L^\sigma_n}(b - \tilde{b}) S P^\sigma_n\}\|_{\mathcal{E}} \le \|(b - \tilde{b})(1-x)^{-\xi}\|_\infty \, \|\widetilde{L^\sigma_n}(1-x)^\xi S P^\sigma_n\| < \varepsilon$.

Remark 4.19 (cf. [GK], Theorem IX.4.1) *If* $a \in PC$, $b \in PC_0$, *the operator* $\widetilde{A} = aI - b w_\sigma^{-1} S w_\sigma I$ *is invertible in* L^2_σ *if and only if* $A = aI + bS$ *is so.*

4.5 Strong convergence of $\widetilde{A_n}^*$

Proposition 4.6 *If* a *is Riemann integrable and* $b \in PC_0$, *then* $\widetilde{A_n}^* = (W^\sigma_n \widetilde{L^\sigma_n}(aI + bS) W^\sigma_n)^*$ *is strongly convergent.*

Proof. For the multiplication operator we have

$$(W^\sigma_n \widetilde{L^\sigma_n} a W^\sigma_n)^* = (\widetilde{L^\sigma_n} a P^\sigma_n)^* = \widetilde{L^\sigma_n} \overline{a} P^\sigma_n \longrightarrow \overline{a} I$$

by Lemma 4.15.

The investigation of bS will again be based on (4.5), where we first assume $b \in C^{1,\eta}_{0,0}$. The expression $b\varphi^{-1} x I$ is already covered by the preceding arguments. Further we have

$$(W^\sigma_n \widetilde{L^\sigma_n} b \varphi^{-1} V^* W^\sigma_n)^* = (W^\sigma_n \widetilde{L^\sigma_n} b \varphi^{-1} W^\sigma_n \cdot W^\sigma_n V^* W^\sigma_n)^*$$

$$= W^\sigma_n V W^\sigma_n \widetilde{L^\sigma_n} \overline{b} \varphi^{-1} P^\sigma_n = V^* P^\sigma_n \widetilde{L^\sigma_n} \overline{b} \varphi^{-1} P^\sigma_n \longrightarrow V^* \overline{b} \varphi^{-1} I.$$

Since $K \in \mathcal{K}(L^2_\sigma, C^{0,\lambda})$ and hence $\|\widetilde{L^\sigma_n} K - K\|_{\mathcal{L}(L^2_\sigma)} \to 0$, we also have $\|(\widetilde{L^\sigma_n} K)^* - K^*\| \to 0$. Thus, we can write

$$(W^\sigma_n \widetilde{L^\sigma_n} K W^\sigma_n)^* = W^\sigma_n \left((\widetilde{L^\sigma_n} K)^* - K^*\right) W^\sigma_n + W^\sigma_n K^* W^\sigma_n,$$

the first summand uniformly and the second strongly converging to 0 (compare Lemma 3.4). If χ is Riemann integrable (in particular a characteristic function of a subinterval) and $b \in C^{1,\eta}_{0,0}$, we have

$$(W^\sigma_n \widetilde{L^\sigma_n} \chi b S W^\sigma_n)^* = (W^\sigma_n \widetilde{L^\sigma_n} b S W^\sigma_n)^* (W^\sigma_n \widetilde{L^\sigma_n} \chi W^\sigma_n)^*,$$

which allows us to apply the preceding reflections. By approximation we can finally get the assertion for arbitrary $b \in PC_0$ (compare the proof of Proposition 4.5). ■

5 Application of the local principle and main result

Having shown $\{A_n\} = \{\widetilde{L^\sigma_n}(aI + bS)P^\sigma_n\} \in \mathcal{A}$ and having computed $\widetilde{A}$, we are left with investigating the invertibility of the coset $\{A_n\} + \mathcal{I} \in \mathcal{A}/\mathcal{I}$ (comp. Theorem 3.4), which will be done by the local principle of Gohberg and Krupnik.

For $t \in [-1, 1]$ let

$$m_t := \{f \in C[-1,1] : 0 \le f(x) \le 1,\ f(x) \equiv 1 \text{ in some neighbourhood of } t\}$$

and define

$$M_t := \{\{\widetilde{L^\sigma_n} f P^\sigma_n\} + \mathcal{I} : f \in m_t\}.$$

Lemma 5.1 (comp. [JS], Lemma 2.6) *$\{M_t\}_{t\in[-1,1]}$ is a covering system of localizing classes in $\mathcal{A}/\mathcal{I}$.*

Now we have to show that $\{A_n\} + \mathcal{I}$ commutes with all elements of $\bigcup_{t\in[-1,1]} M_t$. For this end, we consider an analogous problem in the space $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle $\{t \in \mathbb{C} : |t| = 1\}$. If $f \in L^2(\mathbb{T})$, $f(t) = \sum_{k=-\infty}^{\infty} f_k t^k$, we introduce the operators

$$(P^{\mathbb{T}}_n f)(t) := \sum_{k=-n-1}^{n} f_k t^k,$$

$$(W^{\mathbb{T}}_n f)(t) := f_{-1} t^{-n-1} + \cdots + f_{-n-1} t^{-1} + f_n + \cdots + f_0 t^n$$

and consider the algebra $\mathcal{A}^{\mathbb{T}}$ and the ideal (in $\mathcal{A}^{\mathbb{T}}$) $\mathcal{I}^{\mathbb{T}}$ related to these operator sequences and defined analogously to $\mathcal{A}$ and $\mathcal{I}$.

Let M_n be the Multhopp interpolation operator that assigns to every Riemann integrable function f on $\mathbb{T}$ the polynomial $(M_n f)(t) = \sum_{k=-n-1}^{n} \alpha_k t^k$ coinciding with f in the nodes $e^{(ik\pi)/(n+1)}$ $(k = -n-1, \ldots, n)$. We remind that $\|M_n f - f\|_{L^2(\mathbb{T})} \to 0$ for all Riemann integrable f. We further introduce the projections $(Pf)(t) = \sum_{k=0}^{\infty} f_k t^k$, $Q = I - P$ and $(Tf)(t) = \frac{1}{2}(f(t) - f(t^{-1}))$. Note that T is the orthogonal projection onto the subspace of all odd functions (that means, the space of all $f \in L^2(\mathbb{T})$ for which $f(t) = -f(\bar{t})$ for all $t \in \mathbb{T}$). If a is a complex-valued function on $[-1, 1]$, we define a function $\hat{a}$ on $\mathbb{T}$ by $\hat{a}(e^{i\varphi}) := a(\cos\varphi)$.

Finally, we use the following mapping F from L^2_σ onto the subspace of all odd functions:

$$(Fu)(t) := \begin{cases} \sqrt{\pi} u(\Re t) w_\sigma(\Re t), & \Im t > 0 \\ -(Fu)(\bar{t}), & \Im t < 0 \\ 0, & t = \pm 1, \end{cases}$$

where $\Re t$ and $\Im t$ denote the real and the imaginary part of a complex number t, respectively. Using the formula

$$U_n(x) = \frac{(x + i\sqrt{1-x^2})^{n+1} - (x - i\sqrt{1-x^2})^{n+1}}{\sqrt{2\pi} i \sqrt{1-x^2}},$$

we obtain

$$(F\tilde{u}_n)(t) = \frac{t^{n+1} - t^{-n-1}}{\sqrt{2} i},$$

which shows that F is an isometric isomorphism between L^2_σ and the space of all odd functions in $L^2(\mathbb{T})$. The following lemma summarizes the transformation of some operators we are interested in.

Lemma 5.2 *The following identities hold:*
$\widetilde{L^\sigma_n} = F^{-1} M_n F$,
$bI = F^{-1} \hat{b} F$,
$P^\sigma_n = F^{-1} P^{\mathbb{T}}_n F P^\sigma_n$,
$V^* = F^{-1}(t^{-1}P + tQ)F$,
$P^\sigma_n F^{-1} T P^{\mathbb{T}}_n = P^\sigma_n F^{-1} T$,
$P^{\mathbb{T}}_n F P^\sigma_n = F P^\sigma_n$,
$P^\sigma_n F^{-1} T W^{\mathbb{T}}_n = W^\sigma_n F^{-1} T(tP + QtI)$,
$W^{\mathbb{T}}_n F P^\sigma_n = t^{-1} F W^\sigma_n$.

Lemma 5.3 ([JS], Lemma 2.5) *If $f \in C(\mathbb{T})$, then the sequences $\{QM_n f P P^{\mathbb{T}}_n\}$ and $\{PM_n f Q P^{\mathbb{T}}_n\}$ belong to $\mathcal{I}^{\mathbb{T}}$.*

Lemma 5.4 *Let $f \in C[-1,1]$. Then the sequence $\{\widetilde{L^\sigma_n} f V^* P^\sigma_n - V^* \widetilde{L^\sigma_n} f P^\sigma_n\}$ belongs to $\mathcal{I}$.*

Proof. Using Lemma 5.2, we can transform the sequence under consideration as follows:

$$\{\widetilde{L^\sigma_n} f V^* P^\sigma_n - V^* \widetilde{L^\sigma_n} f P^\sigma_n\} \tag{5.7}$$

$$= \{F^{-1}\left[M_n \hat{f}(t^{-1}P + tQ)P^{\mathbb{T}}_n - (t^{-1}P + tQ) M_n \hat{f} P^{\mathbb{T}}_n\right] F P^\sigma_n\}$$

We will show that the term in brackets belongs to $\mathcal{I}^{\mathbb{T}}$. Note that we can insert the operator M_n before the expression $(t^{-1}P + tQ)M_n \hat{f} P^{\mathbb{T}}_n$ since the space $\operatorname{im} M_n$ is left invariant by $t^{-1}P + tQ$. We have

$$\{M_n \hat{f} t Q P^{\mathbb{T}}_n - M_n t Q M_n \hat{f} P^{\mathbb{T}}_n\}$$

$$= \{M_n t P^{\mathbb{T}}_n (M_n \hat{f} Q P^{\mathbb{T}}_n - Q M_n \hat{f} P^{\mathbb{T}}_n)\}$$

$$= \{M_n t P^{\mathbb{T}}_n\}\{P^{\mathbb{T}}_n M_n \hat{f} Q P^{\mathbb{T}}_n - Q M_n \hat{f} P P^{\mathbb{T}}_n\} \in \mathcal{I}^{\mathbb{T}}$$

according to Lemma 5.3. (Note that $\{M_n t P_n^{\mathbb{T}}\} \in \mathcal{A}^{\mathbb{T}}$ ([JS]).) Analogously,

$$\{M_n t^{-1} P M_n \hat{f} P_n^{\mathbb{T}} - M_n \hat{f} t^{-1} P P_n^{\mathbb{T}}\} = \{M_n t^{-1} P_n^{\mathbb{T}}\}\{P M_n \hat{f} Q P_n^{\mathbb{T}} - Q M_n \hat{f} P P_n^{\mathbb{T}}\} \in \mathcal{I}^{\mathbb{T}}.$$

Hence, there are operators $K_1,\ K_2 \in \mathcal{K}(L^2(\mathbb{T}))$ such that (5.7) equals

$$\{P_n^\sigma F^{-1} T (P_n^{\mathbb{T}} K_1 P_n^{\mathbb{T}} + W_n^{\mathbb{T}} K_2 W_n^{\mathbb{T}} + C_n) F P_n^\sigma\},$$

where $\|C_n\| \to 0$. (We insert the projection T to be able to consider the three summands individually, since it is not guaranteed that each of them maps into im F.) If we use the relations given in Lemma 5.2, we see that the latter expression equals

$$\{P_n^\sigma F^{-1} T K_1 F P_n^\sigma + W_n^\sigma F^{-1} T (tP + QtI) K_2 t^{-1} F W_n^\sigma + P_n^\sigma F^{-1} T C_n F P_n^\sigma\},$$

which is obviously an element of $\mathcal{I}$. ■

Proposition 5.1 *Let $a \in PC$, $b \in PC_0$. The coset $\{\widetilde{L_n^\sigma}(aI + bS) P_n^\sigma\} + \mathcal{I}$ commutes with all elements of $\bigcup\limits_{t\in[-1,1]} M_t$.*

Proof. In the case of the multiplication operator $A = aI$, the assertion is a consequence of

$$\widetilde{L_n^\sigma} a P_n^\sigma \widetilde{L_n^\sigma} f P_n^\sigma = \widetilde{L_n^\sigma} f P_n^\sigma \widetilde{L_n^\sigma} a P_n^\sigma = \widetilde{L_n^\sigma} a f P_n^\sigma.$$

As for the singular integral operator, we start with considering coefficients of the form χb with a Riemann integrable function χ and $b \in C_{0,0}^{1,\eta}$, where $\eta > \max\{\frac{1}{4} + \frac{\alpha}{2}, \frac{1}{4} + \frac{\beta}{2}, 0\}$. Equation (4.5) and Lemma 4.11 give

$$\chi b S = \chi K + i \chi b \varphi^{-1}(xI - V^*)$$

with $\chi K \in \mathcal{K}(L^2_\sigma, R)$, where R denotes the Banach space of all Riemann integrable functions on $[-1,1]$, endowed with the supremum norm. Consequently, the commutator $\widetilde{L_n^\sigma} f P_n^\sigma\ \widetilde{L_n^\sigma} \chi K P_n^\sigma - \widetilde{L_n^\sigma} \chi K P_n^\sigma\ \widetilde{L_n^\sigma} f P_n^\sigma$ converges uniformly to $f\chi K - \chi K f I \in \mathcal{K}(L^2_\sigma)$ and is therefore of the form

$$P_n^\sigma (f\chi K - \chi K f I) P_n^\sigma + C_n, \qquad \|C_n\| \to 0,$$

which is contained in $\mathcal{I}$.

If we abbreviate $c := i\chi b \varphi^{-1}$, it remains to consider

$$\widetilde{L_n^\sigma} f c V^* P_n^\sigma - \widetilde{L_n^\sigma} c V^* \widetilde{L_n^\sigma} f P_n^\sigma = \widetilde{L_n^\sigma} c P_n^\sigma (\widetilde{L_n^\sigma} f V^* P_n^\sigma - V^* \widetilde{L_n^\sigma} f P_n^\sigma),$$

and Lemma 5.4 shows that this expression is in $\mathcal{I}$. Hence, the assertion is true for coefficients of the form χb, where $b \in C_{0,0}^{1,\eta}$, and χ is a characteristic function of a subinterval. Arbitrary PC_0-coefficients can be approximated in the supremum norm by sums of such functions (compare the proof of Lemma 4.5). Finally, we take into account that $\|\{\widetilde{L_n^\sigma} b S P_n^\sigma\} + \mathcal{I}\| \leq$ $const\, \|b\|_\infty$ and that the ideal $\mathcal{I}$ is closed, which completes the proof of the proposition. ■

Now we are able to give local representatives for $\{A_n\} + \mathcal{I}$.

Lemma 5.5 *Let $\tau \in [-1,1]$, $a, a_\tau \in PC$, $b, b_\tau \in PC_0$ such that*

$$a_\tau(\tau \pm 0) = a(\tau \pm 0), \qquad b_\tau(\pm 0) = b(\tau \pm 0). \tag{5.8}$$

Then $\{\widetilde{L_n^\sigma}(aI + bS)P_n^\sigma\} + \mathcal{I}$ and $\{\widetilde{L_n^\sigma}(a_\tau I + b_\tau S)P_n^\sigma\} + \mathcal{I}$ are M_τ-equivalent.
If further $b_\tau = \chi_{[-1,\tau]} b_1 + \chi_{[\tau,1]} b_2$ with $b_1, b_2 \in C_{0,0}^{1,\eta}$, $\eta > \max\{\frac{1}{4} + \frac{\alpha}{2}, \frac{1}{4} + \frac{\beta}{2}, 0\}$, then $\{\widetilde{L_n^\sigma}((a_\tau + ib_\tau \varphi^{-1} x) I - ib_\tau \varphi^{-1} V^*) P_n^\sigma\} + \mathcal{I}$ is M_τ- equivalent to both cosets.

Proof. Let $f \in m_\tau$. We have

$$\left\| \{\widetilde{L_n^\sigma} f P_n^\sigma\}\{\widetilde{L_n^\sigma}((a-a_\tau)I + (b-b_\tau)S)P_n^\sigma\}\right\|_{\mathcal{A}/\mathcal{I}}$$

$$\leq \|\widetilde{L_n^\sigma} f(a-a_\tau)P_n^\sigma + \widetilde{L_n^\sigma} f(b-b_\tau)P_n^\sigma \, \widetilde{L_n^\sigma} S P_n^\sigma\|_{\mathcal{L}(L^2_\sigma)}$$

$$\leq \|f(a-a_\tau)\|_\infty + const\, \|f(b-b_\tau)\|_\infty,$$

which can be made arbitrarily small by a suitable choice of f. Thus, we have proved the first assertion (note Proposition 5.1). For the second, formula (4.5) shows

$$bS = ib_\tau\varphi^{-1}(xI - V^*) + \chi_{[-1,t]}K_1 + \chi_{[t,1]}K_2$$

with $K_1, K_2 \in \mathcal{K}(L^2_\sigma, C^{0,\lambda})$ and therefore $\widetilde{L_n^\sigma}(\chi_{[-1,t]}K_1 + \chi_{[t,1]}K_2)P_n^\sigma \in \mathcal{I}$. ■

Remark 5.6 *If we have one of the special cases with respect to the exponents of the weight σ considered in Subsection 4.2, we slightly modify the preceding lemma and its proof. If for instance $\alpha = \frac{1}{2}$, we require that $b(x) = o((1-x)^\xi)$ for $x \to 1$, $\xi > \frac{1+\max\{\alpha,\beta,0\}}{2}$, which ensures $\|\widetilde{L_n^\sigma}(1-x)^\xi S P_n^\sigma\| \leq const$. Then we estimate as follows: $\|\widetilde{L_n^\sigma} f(b-b_\tau)P_n^\sigma \, \widetilde{L_n^\sigma} S P_n^\sigma\| \leq \|f(1-x)^{-\xi}(b-b_\tau)\|_\infty \, \|\widetilde{L_n^\sigma}(1-x)^\xi S P_n^\sigma\|$.*

In the following we are going to give stability conditions. If the coefficients of the singular integral operator are continuous, we can obtain a very general result.

Theorem 5.1 *Let $a, b \in C[-1,1]$ with $b(\pm 1) = 0$, and let $A = aI + bS$. Then the sequence $\{A_n\} = \{\widetilde{L_n^\sigma} A P_n^\sigma\}$ is stable if and only if A is invertible in L^2_σ.*

Proof. Due to Theorem 3.4 and Remark 4.19, we only have to consider the invertibility of the coset $\{A_n\}+\mathcal{I}$. Let $\tau \in [-1,1]$. Note that the invertibility of A implies $a^2(\tau) - b^2(\tau) \neq 0$ ([GK, Th. IX.4.1]). We choose $a_\tau \in C$, $b_\tau \in C^{1,\eta}_{0,0}$ such that (5.8) is fulfilled, and additionally $a_\tau^2(x) - b_\tau^2(x) \neq 0$ for all $x \in [-1,1]$. Then, because of Lemma 5.5, an M_τ-equivalent local representative of $\{A_n\} + \mathcal{I}$ is given by $\{A_n^t\} + \mathcal{I}$, where (cf. Lemma 5.2)

$$A_n^\tau = \widetilde{L_n^\sigma}\left((a_\tau + ib_\tau\varphi^{-1}x)I - ib_\tau\varphi^{-1}V^*\right)P_n^\sigma$$

$$= F^{-1}M_n\left((\hat{a}_\tau + i\hat{b}_\tau\hat{\varphi}^{-1}\hat{x})I - i\hat{b}_\tau\hat{\varphi}^{-1}(t^{-1}P + tQ)\right)P_n^{\mathbb{T}} F P_n^\sigma$$

$$= F^{-1}\left[M_n(c_\tau P + \tilde{c_\tau} Q)P_n^{\mathbb{T}}\right] F P_n^\sigma$$

with $c_\tau(s) = \hat{a}_\tau + i\hat{b}_\tau\hat{\varphi}^{-1}(\hat{x} - s^{-1})$ and $\tilde{c_\tau}(s) = c_\tau(s^{-1})$, and hence

$$c_\tau(x+iy) = \begin{cases} a_\tau(x) - b_\tau(x), & y \geq 0 \\ a_\tau(x) + b_\tau(x), & y < 0. \end{cases}$$

Let

$$\begin{aligned} B_n^\tau &= \widetilde{L_n^\sigma}\left(\left(\frac{a_\tau}{a_\tau^2 - b_\tau^2} - i\frac{b_\tau}{a_\tau^2 - b_\tau^2}\varphi^{-1}x\right)I + i\frac{b_\tau}{a_\tau^2 - b_\tau^2}\varphi^{-1}V^*\right)P_n^\sigma \\ &= F^{-1}\left[M_n(c_\tau^{-1}P + \tilde{c_\tau}^{-1}Q)P_n^{\mathbb{T}}\right] F P_n^\sigma. \end{aligned}$$

Evidently, $B_n^\tau \in \mathcal{A}$. The proof of [JS, Theorem 2.1] now yields

$$A_n^\tau B_n^\tau = F^{-1}\left[M_n(c_\tau P + \tilde{c_\tau} Q)P_n^{\mathbb{T}}\right]\left[M_n(c_\tau^{-1}P + \tilde{c_\tau}^{-1}Q)P_n^{\mathbb{T}}\right] F P_n^\sigma$$

$$= F^{-1}[P_n^{\mathbb{T}} + C_n^{\mathbb{T}}]F P_n^\sigma = P_n^\sigma + C_n,$$

where $\{C_n^{\mathbb{T}}\} \in \mathcal{I}^{\mathbb{T}}$, $\{C_n\} \in \mathcal{I}$ (cf. the proof of Lemma 5.4). Analogously, $B_n^\tau A_n^\tau = P_n^\sigma + C_n'$, $\{C_n'\} \in \mathcal{I}$, and Theorem 3.2 yields the assertion. ■

As for arbitrary piecewise continuous coefficients, we restrict ourselves to giving a sufficient stability condition in a special case.

Proposition 5.2 *Let $a \in PC, b \in PC_0$ such that $A = aI + bS$ is invertible in L_σ^2 and that moreover for the one-sided limits*

$$\left|a(\tau \pm 0) + ib(\tau \pm 0)t\varphi^{-1}(\tau)\right| > \left|b(\tau \pm 0)\varphi^{-1}(\tau)\right|$$

holds for all $\tau \in [-1,1]$. Then $\{A_n\} = \{\widetilde{L_n^\sigma} A P_n^\sigma\}$ is stable.

Proof. Again we only have to show the invertibility of the coset $\{A_n\} + \mathcal{I}$. For $t \in [-1,1]$ choose $a_\tau \in PC$, $b_1, b_2 \in C_{0,0}^{1,\eta}$ ($\eta > \max\{\frac{1}{4} + \frac{\alpha}{2}, \frac{1}{4} + \frac{\beta}{2}, 0\}$) such that a_τ and $b_\tau := \chi_{[-1,t]}b_1 + \chi_{[t,1]}b_2$ satisfy (5.8) and

$$\left|a_\tau(s \pm 0) + ib_\tau(s \pm 0)s\varphi^{-1}(s)\right| > \left|b_\tau(s \pm 0)\varphi^{-1}(s)\right| \tag{5.9}$$

for all $s \in [-1,1]$. Then $\{A_n^\tau\} + \mathcal{I}$, where A_n^τ is defined as in the proof of Theorem 5.1, is an M_τ-equivalent local representative of $\{A_n\}$. We have

$$A_n^\tau = \left(\widetilde{L_n^\sigma}(a_\tau + ib_\tau\varphi^{-1}x)P_n^\sigma\right)\left(P_n^\sigma - i\widetilde{L_n^\sigma}\underbrace{b_\tau\varphi^{-1}(a_\tau + ib_\tau\varphi^{-1}x)^{-1}}_{=:c_\tau} P_n^\sigma V^* P_n^\sigma\right)$$

(for the invertibility of $a_\tau + ib_\tau\varphi^{-1}x$ in L^∞ note (5.9)). Since $\|\{\widetilde{L_n^\sigma} c_\tau P_n^\sigma\}\|_{\mathcal{A}} \leq \|c_\tau\|_\infty < 1$ and $\|V^* P_n^\sigma\| = 1$, the sequence $\{P_n^\sigma - i\widetilde{L_n^\sigma} c_\tau P_n^\sigma V^* P_n^\sigma\}$ is invertible in $\mathcal{A}$, which of course also implies the invertibility of $\{A_n^\tau\} + \mathcal{I}$ in $\mathcal{A}/\mathcal{I}$. ■

6 Implementation and numerical results

6.1 Some remarks on the implementation

For practical computations, we write the weighted polynomial v_n that solves (2.5) in the form

$$v_n(x) = w_{\sigma^{-1}}(x)\sum_{k=0}^{n-1}\xi_k P_k^{(\sigma)}(x),$$

where $P_k^{(\sigma)}$ denotes the monic orthogonal polynomial of degree k with respect to the weight $w_{\sigma^{-1}}$, and we solve the system of linear equations

$$\sum_{k=0}^{n-1}\underbrace{\left[\left(aw_{\sigma^{-1}}P_k^{(\sigma)}\right)(x_{jn}^\varphi) + b(x_{jn}^\varphi)\left(Sw_{\sigma^{-1}}P_k^{(\sigma)}\right)(x_{jn}^\varphi)\right]}_{=:\,a_{jk}}\xi_k = f(x_{jn}^\varphi), \qquad j = 1,\dots,n \tag{6.10}$$

to determine the ξ_k, $k = 0,\dots,n-1$.

It is well-known that the orthogonal polynomials satisfy a three-term recurrence formula of the form

$$P_{k+1}^{(\sigma)}(x) = (x - \alpha_k)P_k^{(\sigma)}(x) - \beta_k P_{k-1}^{(\sigma)}(x), \qquad j = 0, 1, 2, \ldots, \tag{6.11}$$

where $P_{-1}^{(\sigma)} \equiv 0$ and $P_1^{(\sigma)} \equiv 1$. In the case of Jacobi weights, there are explicit formulas for the α_k and β_k. This allows us to compute the matrix coefficients a_{jk} recursively. We have

$$a_{j,k+1} = (x_{jn}^{\varphi} - \alpha_k)a_{jk} - \beta_k a_{j,k-1}, \qquad k = 1, \ldots, n-1,$$

with the initial values

$$a_{j0} = (aw_{\sigma^{-1}})\,(x_{jn}^{\varphi}) + b(x_{jn}^{\varphi})\varrho_0(x_{jn}^{\varphi})$$

and

$$\begin{aligned} a_{j1} &= (x_{jn}^{\varphi} - \alpha_0)\,(aw_{\sigma^{-1}})\,(x_{jn}^{\varphi}) + b(x_{jn}^{\varphi})\left(\beta_0 + (x_{jn}^{\varphi} - \alpha_0)\varrho_0(x_{jn}^{\varphi})\right) \\ &= (x_{jn}^{\varphi} - \alpha_0)a_{j0} + b(x_{jn}^{\varphi})\beta_0, \end{aligned}$$

where $\beta_0 := \frac{1}{\pi i}\int_{-1}^{1} w_{\sigma^{-1}}(t)\,dt$ and $\varrho_0(x) := (Sw_{\sigma^{-1}})\,(x)$. The computation of ϱ_0 is based upon formula (2.2) from [GW]. If (6.10) is solved, one can efficiently compute the values of v_n from the coefficients ξ_j using (6.11).

6.2 Numerical examples

In the following examples, we approximated the error $\|u - v_n\|_\sigma$ by the quadrature rule $\sqrt{Q_m^{\varphi}(|u - v_n|^2\sigma\varphi^{-1})}$ with $m = 256$. We always chose $a \equiv 1$ and $\alpha = \beta = 0$.

Example 1 $b(x) = i\sqrt{1-x^2}$, $f(x) = 1 + \dfrac{\sqrt{1-x^2}}{\pi}\ln\dfrac{1-x}{1+x}$, $u \equiv 1$.

Example 2 $b(x) = i\sqrt{1-x^2}$, $f(x) = |x| + x\dfrac{\sqrt{1-x^2}}{\pi}\ln\dfrac{(1+x)(1-x)}{x^2}$, $u(x) = |x|$.

Example 3 $b(x) = i\sqrt{1-x^2}$, $f(x) = \operatorname{sgn} x + \dfrac{\sqrt{1-x^2}}{\pi}\ln\dfrac{(1+x)(1-x)}{x^2}$, $u(x) = \operatorname{sgn} x$.

	$\sqrt{Q_m^{\varphi}(\lvert u - v_n\rvert^2\sigma\varphi^{-1})}$		
n	Ex. 1	Ex. 2	Ex. 3
8	7.99 E-3	8.04 E-3	4.98 E-2
16	4.04 E-3	4.12 E-3	3.56 E-2
32	2.01 E-3	2.04 E-3	2.52 E-2
64	9.49 E-4	9.59 E-4	1.77 E-2
128	3.65 E-4	3.68 E-4	1.18 E-2
256	2.65 E-6	2.07 E-5	5.78 E-3
512	1.02 E-7	1.78 E-6	4.28 E-4

We also considered some examples with $b(\pm 1) \neq 0$, which are not covered by the theoretical results of this paper. One is inclined to conjecture that in this case the sequence A_n is stable if and only if the operators $A = aI + bS$ and $\widetilde{A} = aI - bw_\sigma^{-1} S w_\sigma I$ are invertible in L^2_σ. In Example 4 this is the case, whereas in Example 5 the operator A is invertible but $\widetilde{A}$ is not. In both cases the approximate solutions seem to converge, but in Example 5 the convergence is somewhat slower in despite of the same smoothness of the input data.

Example 4 $b = -\frac{i}{10}$, $f(x) = |x| - \dfrac{x}{10\pi} \ln \dfrac{(1+x)(1-x)}{x^2}$, $u(x) = |x|$.

Example 5 $b = -i$, $f(x) = |x| - \dfrac{x}{\pi} \ln \dfrac{(1+x)(1-x)}{x^2}$, $u(x) = |x|$.

	$\sqrt{Q_m^\varphi(\|u - v_n\|^2 \sigma \varphi^{-1})}$	
n	Ex. 4	Ex. 5
8	8.33 E-3	8.36 E-3
16	4.21 E-3	4.46 E-3
32	2.07 E-3	2.27 E-3
64	9.67 E-4	1.09 E-3
128	3.70 E-4	4.61 E-4
256	1.63 E-5	1.45 E-4
512	1.26 E-6	8.80 E-6

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