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Abstract

We study the disturbance decoupling problem for linear time invariant descriptor systems. We give necessary and sufficient conditions for the existence of a solution to the disturbance decoupling problem via a proportional and/or derivative feedback that also makes the resulting closed-loop system regular and/or of index at most one. All results are proved constructively based on condensed forms that can be computed using orthogonal matrix transformations, i.e., transformations that can be implemented in a numerically stable way.

Keywords: Descriptor system, state feedback, disturbance decoupling, orthogonal matrix transformation.

AMS subject classification: 93B05, 93B40, 93B52, 65F35

1 Introduction

We consider linear and time-invariant continuous descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Gq(t);\ x(0-) = x_0,\ t \geq 0 \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where E, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $G \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{q\times n}$, and $\dot{x} = dx/dt$. The term $q(t), t \geq 0$ represents a disturbance, which may represent modelling or measuring errors, noise or higher order terms in linearization. We study the problem of constructing feedbacks that surpress this disturbance in the sense that $q(t)$ does not affect the input-output behaviour of the system. In this paper, we always assume without loss of generality that B, G are full column rank, and C is full row rank, i.e., $\text{rank}\,(B) = m$, $\text{rank}\,(G) = p$, $\text{rank}\,(C) = q$. If this is not the case then this can be easily achieved by considering appropriate submatrices after a change of basis. In the following we denote a matrix with orthogonal columns spanning the right nullspace of a matrix M by $S_\infty(M)$ and a matrix with orthogonal columns spanning the left nullspace of M by $T_\infty(M)$. Moreover, we denote the polynomial degree of a polynomial $f(s)$ by $\deg(f(s))$ and by $\text{rank}\,[\cdot](s)$ the rank relative to the field of rational functions.

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When $E = I$, (1) is a standard linear time-invariant system. Our attention, however, will focus on the case that E is singular. In this case existence and uniqueness of (classical) solutions to (1) is guaranteed if (E, A) is regular, i.e. if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in \mathbf{C}^2$.

The system (1) is said to have index at most one if the dimension of the largest nilpotent block in the Kronecker canonical form of (E, A) is at most one [9].

It is well-known that systems that are regular and of index at most one, can be separated into purely dynamical and purely algebraic parts (fast and slow modes), and in theory the algebraic part can be elminated to give a reduced-order standard system. The reduction process, however, may be ill-conditioned with respect to numerical computation. If the index is larger than 1, then impulses can arise in the response of the system if the control is not sufficiently smooth [3, 5, 10, 15]. Therefore, an appropriate feedback control should be chosen to ensure that the closed-loop system is regular and of index at most one. The disturbance decoupling problem for descriptor systems has been studied in [1, 14, 13, 12]. Fletcher and Aasaraai [13] were the first to formulate and to solve the problem with respect to continuous descriptor systems. However, as the problem has been formulated there, disturbance decoupling is achieved if, among other conditions, the output is independent of the input disturbance in the sense that there is a set of admissible initial conditions such that the response of the system is zero. But, since the disturbance input is usually unknown, it is not clear how, and if at all, a given initial state x_0 can be qualified as an admissible initial condition. Banaszuk et al. [14] solve the problem using the concepts of sliding and coasting subspaces by means of a set of necessary and sufficient conditions for obtaining disturbance decoupling in implicit discrete systems. Lebret [12] presents structurally equivalent characterizations of the solutions of the disturbance decoupling problems for implicit discrete systems. Recently, Ailon [1] considered the standard disturbance decoupling problem for continuous-time descriptor systems as formulated in the standard state-space system theory [17], i.e., given the system (1), find (if possible) a proportional state feedback such that, regardless of the initial value of x_0, the disturbance input has no influence on the output of the systems for $t \geq 0$, and yet the uniqueness of solutions for the closed-loop system is ensured. Necessary and sufficient conditions for the cases rank $\begin{bmatrix} E & G \end{bmatrix} = n$ and rank $\begin{bmatrix} E & B & G \end{bmatrix} = n$ are obtained in [1] via analogy to standard state-space systems. But the obtained conditions are rather cumbersome and are only partly given in terms of the original data (E, A, B, C, G). Moreover, the derivative and combined derivative and proportional state feedback, the index and numerical aspects of the algorithms have not been considered in [1].

According to [1], a proportional feedback $u = Fx$ solves the disturbance decoupling problem in system (1) if the matrix pencil $(E, A + BF)$ is regular, and

$$C(sE - (A + BF))^{-1}G = 0$$

The above discussion leads us to study the following problem:

The disturbance decoupling problem *For a system of the form (1) find necessary and sufficient conditions under which there exists a proportional and derivative feedback of the form* $u(t) = Fx(t) - K\dot{x}(t)$*, such that matrix pencil* $(E + BK, A + BF)$ *is regular and*

$$C(s(E + BK) - (A + BF))^{-1}G = 0,$$

where $C(s(E + BK) - (A + BF))^{-1}G$ *is the transfer-function matrix of the closed-loop system*

$$\begin{aligned} (E + BK)\dot{x}(t) &= (A + BF)x(t) + Gq(t) \\ y(t) &= Cx(t). \end{aligned} \tag{2}$$

In addition, if possible, it is required that $(E + BK, A + BF)$ *is of index at most one.*

This paper is strongly inspired by the work in [3, 4, 8]. We give necessary and sufficient conditions for solving the disturbance decoupling problem. All our results are proven constructively, based on condensed forms under orthogonal matrix transformations which can be implemented as numerically stable algorithms.

2 Preliminaries

Given an arbitrary matrix pencil (E, A), it is well-known [9, 8, 16] that there exist nonsingular matrices X and Y transforming the pencil (E, A) to Kronecker canonical form (KCF)

$$X(sE - A)Y = \text{diag}\{sI - J_f, L_{\epsilon_1}, \ldots, L_{\epsilon_s}, sJ_\infty - I, L^T_{\eta_1}, \ldots, L^T_{\eta_t}\}, \tag{3}$$

where $J_f \in R^{n_f \times n_f}$ and $J_\infty \in R^{n_\infty \times n_\infty}$ are in Jordan canonical form. Here J_∞ is nilpotent and associated with the infinite eigenvalues of the pencil. The matrix L_k is a bidiagonal matrix of size $k \times (k+1)$

$$L_k := \begin{bmatrix} s & -1 & & & \\ & s & -1 & & \\ & & \ddots & \ddots & \\ & & & s & -1 \end{bmatrix}$$

and the index sets $\{\epsilon_i, i = 1, \ldots, s\}$ and $\{\eta_j, j = 1, \ldots, t\}$ are the left and right Kronecker indexes of (E, A), (see [16, 8]). If we define

$$\begin{aligned} sE_1 - A_1 &:= \text{diag}\{sI - J_f, L_{\epsilon_1}, \ldots, L_{\epsilon_s}\}, \\ sE_2 - A_2 &:= \text{diag}\{sJ_\infty - I, L^T_{\eta_1}, \ldots, L^T_{\eta_t}\}, \end{aligned} \tag{4}$$

then E_1 is of full row rank, and $sE_2 - A_2$ is of full column rank for any finite $s \in \mathbf{C}$.

It is in general impossible to compute the Kronecker canonical form with a finite precision algorithm, since this is an ill conditioned problem, small changes in the data can drastically change the canonical form. Instead one can obtain a condensed from under orthogonal equivalence transformations. This form, the generalized upper triangular (GUPTRI) form is well studied [6, 7, 8] and has been implemented in LAPACK [2]. It displays all the invariants, in particular the left and right Kronecker indices, but it is not the complete canonical form.

Lemma 1 *[6, 7] Given a matrix pencil* (E, A), $E, A \in \mathbf{R}^{n \times l}$ *there exist orthogonal matrices* $P \in \mathbf{R}^{n \times n}$, $Q \in \mathbf{R}^{l \times l}$ *such that* (PEQ, PAQ) *are in the following GUPTRI form:*

$$P(sE - A)Q = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad l_1 \quad & \quad l_2 \quad & \quad l_3 \quad & \quad l_4 \quad \end{array} \\ \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} \\ 0 & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} \\ 0 & 0 & sE_{33} - A_{33} & sE_{34} - A_{34} \\ 0 & 0 & 0 & sE_{44} - A_{44} \end{bmatrix} \end{array}, \tag{5}$$

where $sE_{11} - A_{11}$ *contains all the* L_j *blocks of* $sE - A$, $sE_{22} - A_{22}$ *and* $sE_{33} - A_{33}$ *are upper triangular and regular, and contain the regular finite and infinite structure of* $sE - A$, *and* $sE_{44} - A_{44}$ *contains all the* L^T_j *blocks of* $sE - A$.

Using the GUPTRI form, we are able to determine the indices and spaces that we introduce in the following definition.

Definition 2 *Given a matrix pencil (E, A), $E, A \in \mathbf{R}^{n\times l}$ in Kronecker canonical form (3). Then*

$$\begin{aligned} r_i(E,A) &:= n_f + \epsilon_1 + \ldots + \epsilon_s, \\ c_i(E,A) &:= n_\infty + \eta_1 + \ldots + \eta_t, \end{aligned}$$

define the row and column index of (E, A), respectively.

Furthermore we define the row-subspace $V_r(E, A)$ and the column-subspace $V_c(E, A)$ of (E, A) by

$$\begin{aligned} V_r(E,A) &:= \text{span}(X^T \begin{bmatrix} 0 \\ I_{n-r_i(E,A)} \end{bmatrix}) \\ V_c(E,A) &:= \text{span}(Y \begin{bmatrix} I_{l-c_i(E,A)} \\ 0 \end{bmatrix}). \end{aligned}$$

From Lemma 1 we obtain the indexs $r_i(E, A), c_i(E, A)$ and the spaces $V_r(E, A), V_c(E, A)$ directly as

$$r_i(E,A) = n_1 + n_2, \quad c_i(E,A) = l_3 + l_4$$

and

$$V_r(E,A) = \text{span}(P^T \begin{bmatrix} 0 \\ I_{n_3+n_4} \end{bmatrix}), \quad V_c(E,A) = \text{span}(Q \begin{bmatrix} I_{l_1+l_2} \\ 0 \end{bmatrix}).$$

It is obvious that

$$\text{rank}\,(sE - A) = r_i(E,A) + c_i(E,A),$$

and if $E_i, A_i, (i = 1, 2)$ are defined by (4), then

$$\begin{aligned} \text{range}(E_1) &= \text{range}(T_\infty^T(V_r(E,A))EV_c(E,A)), \\ \text{range}(E_2) &= \text{range}(V_r^T(E,A)ES_\infty(V_c^T(E,A))), \\ \text{range}(A_1) &= \text{range}(T_\infty^T(V_r(E,A))AV_c(E,A)), \\ \text{range}(A_2) &= \text{range}(V_r^T(E,A)AS_\infty(V_c^T(E,A))). \end{aligned} \tag{6}$$

The following lemma gives a useful characterization of regular, index at most one pencils.

Lemma 3 *[3] Given $E, A \in \mathbf{R}^{n\times n}$. Then the following are equivalent:*

1. *The pencil (E, A) is regular and of index at most one;*
2. $\text{rank}\left(\begin{bmatrix} E & AS_\infty(E) \end{bmatrix}\right) = n$;
3. $\text{rank}\left(\begin{bmatrix} E \\ T_\infty^T(E)A \end{bmatrix}\right) = n$;
4. *$T_\infty^T(E)AS_\infty(E)$ is nonsingular;*
5. $\deg(\det(sE - A)) = \text{rank}(E)$.

If the system is not regular and of index at most one this can often be achieved by feedback. A characterization, when this is possible is the following:

Lemma 4 *[3] Given $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$.*
a) There exists $F \in \mathbf{R}^{m\times n}$ such that $(E, A+BF)$ is regular and of index at most one if and only if

$$\operatorname{rank}\left[\begin{array}{cc} T_\infty^T(E)AS_\infty(E) & T_\infty^T(E)B \end{array}\right] = n - \operatorname{rank}(E). \tag{7}$$

b) There exist $F, K \in \mathbf{R}^{m\times n}$ such that $(E+BK, A+BF)$ is regular and of index at most one if and only if

$$\operatorname{rank}(T_\infty^T(\left[\begin{array}{cc} E & B \end{array}\right])AS_\infty(T_\infty^T(B)E) = n - \operatorname{rank}\left[\begin{array}{cc} E & B \end{array}\right]. \tag{8}$$

c) If (8) holds, then there exists $K \in \mathbf{R}^{m\times n}$ such that $(E+BK, A)$ is regular and of index at most one.

After having introduced some preliminaries, in the next section we now discuss some suitable condensed forms for triples and quintuples of matrices under orthogonal equivalence transformations.

3 Condensed Forms

In this section we introduce condensed forms under orthogonal equivalence transformations. The key Lemma that we will use frequently is the following:

Lemma 5 *Given $\hat{E}, \hat{A} \in \mathbf{R}^{t\times l}$, $\hat{B} \in \mathbf{R}^{t\times s}$, with B of full column rank. Then there exist orthogonal matrices $U \in \mathbf{R}^{t\times t}, V \in \mathbf{R}^{l\times l}$, such that*

$$\begin{array}{rcl} U\hat{E}V & = & \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} l-l_1 & l_1 \end{array} \\ \left[\begin{array}{cc} E_{11} & E_{12} \\ 0 & E_{22} \\ 0 & E_{32} \end{array}\right] \end{array}, \quad U\hat{B} = \begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array}\!\!\left[\begin{array}{c} B_1 \\ B_2 \\ 0 \end{array}\right], \\ U\hat{A}V & = & \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} l-l_1 & l_1 \end{array} \\ \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \\ 0 & A_{32} \end{array}\right] \end{array}, \end{array} \tag{9}$$

where E_{11} and B_2 are of full row rank, and

$$\operatorname{rank}(sE_{32} - A_{32}) = l_1, \quad \forall s \in \mathbf{C}$$

Proof. The proof is given constructively via Algorithm 1 in Appendix A. □

Using Lemma 5 and the GUPTRI form (5) we obtain the following condensed form for quintuples (E, A, B, C, G).

Theorem 6 *Given a system of the form (1), there exist orthogonal matrices $U, V \in \mathbf{R}^{n\times n}$ such that*

$$UEV = \begin{array}{c} \\ p \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \\ \tilde{n}_5 \end{array}\begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \\ 0 & E_{42} & E_{43} \\ 0 & 0 & E_{53} \end{bmatrix}\end{array}, \quad UAV = \begin{array}{c} \\ p \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \\ \tilde{n}_5 \end{array}\begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ 0 & A_{42} & A_{43} \\ 0 & 0 & A_{53} \end{bmatrix}\end{array}$$

$$UB = \begin{array}{c} p \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \\ \tilde{n}_5 \end{array}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \\ 0 \end{bmatrix}, \quad UG = \begin{array}{c} p \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \\ \tilde{n}_5 \end{array}\begin{bmatrix} G_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (10)$$

$$CV = \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{bmatrix} 0 & C_2 & C_3 \end{bmatrix}\end{array},$$

where G_1, E_{21}, B_3 and E_{42} are of full row rank and furthermore

$$\begin{aligned} \operatorname{rank}(sE_{53} - A_{53}) &= n_3, \quad \forall s \in \mathbf{C}, \\ \operatorname{rank}\begin{bmatrix} sE_{42} - A_{42} & sE_{43} - A_{43} \\ 0 & sE_{53} - A_{53} \\ C_2 & C_3 \end{bmatrix} &= n_2 + n_3, \quad \forall s \in \mathbf{C}. \end{aligned}$$

Proof. The proof is given constructively via Algorithm 2 in Appendix A. □

Using these two condensed forms we can characterize the following indices which are needed for the solution of the disturbance decoupling problem.

Corollary 7

a) Let $\hat{E}$, $\hat{A}$, $\hat{B}$ be in the condensed form (9). Then

$$\begin{aligned} \rho &:= r_i(\begin{bmatrix} \hat{E} & 0 \end{bmatrix}, \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix}) = t_1, \\ \gamma &:= c_i(T_\infty^T(\hat{B})\hat{E}, T_\infty^T(\hat{B})\hat{A}) = l_1. \end{aligned}$$

b) Let E, A, B, C, G be in the condensed form (10), then

$$\begin{aligned} \tau &:= r_i(\begin{bmatrix} T_\infty^T(G)E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(G)A & T_\infty^T(G)B \\ C & 0 \end{bmatrix}) = \tilde{n}_2, \\ \mu &:= c_i(\begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})E \\ 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})A \\ C \end{bmatrix}) = n_2 + n_3, \\ \eta &:= \operatorname{rank}(T_\infty^T(G)B) - \tau \\ &\quad + r_i(\begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})E \\ 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})A \\ C \end{bmatrix}) = \tilde{n}_3. \end{aligned}$$

Corollary 8 *Given a system of the form (1) in condensed form (10). Then we have the following implications.*
a) If condition (7) holds, then $E_{53} = 0$ and A_{53} is nonsingular.
b) If condition (8) holds, then $E_{53} = 0$, A_{53} is nonsingular and

$$\operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & E_{13} & B_1 \\ E_{21} & E_{22} & E_{23} & B_2 \\ 0 & E_{32} & E_{33} & B_3 \\ 0 & E_{42} & E_{43} & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_1 \\ E_{21} & E_{22} & B_2 \\ 0 & E_{32} & B_3 \\ 0 & E_{42} & 0 \end{bmatrix}. \tag{11}$$

Proof. a) According to Lemma 4, there exists $F := \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ such that $(E, A+BF)$ is regular and of index at most one. Hence,

$$\operatorname{rank}(sE_{53} - A_{53}) = \tilde{n}_5.$$

But $sE_{53} - A_{53}$ is full column rank for any finite $s \in C$ and thus, $n_3 = \tilde{n}_5$ and $\det(sE_{53} - A_{53}) = \det(-A_{53})$. Therefore, the nonsingularity of A_{53} follows directly from the regularity of $(E, A + BF)$. Moreover

$$\begin{aligned}
\operatorname{rank}(E) &= \deg(\det(sE - A - BF)) \\
&= \deg(\det(\begin{bmatrix} sE_{11} - A_{11} - B_1F_1 & sE_{12} - A_{12} - B_1F_2 \\ sE_{21} - A_{21} - B_2F_1 & sE_{22} - A_{22} - B_2F_2 \\ -A_{31} - B_3F_1 & sE_{32} - A_{32} - B_3F_2 \\ 0 & sE_{42} - A_{42} \end{bmatrix})) + \deg(\det(sE_{53} - A_{53})) \\
&= \deg(\det(\begin{bmatrix} sE_{11} - A_{11} - B_1F_1 & sE_{12} - A_{12} - B_1F_2 \\ sE_{21} - A_{21} - B_2F_1 & sE_{22} - A_{22} - B_2F_2 \\ -A_{31} - B_3F_1 & sE_{32} - A_{32} - B_3F_2 \\ 0 & sE_{42} - A_{42} \end{bmatrix})) \\
&\leq \operatorname{rank}(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ 0 & E_{32} \\ 0 & E_{42} \end{bmatrix})
\end{aligned}$$

But, we have

$$\operatorname{rank}(E) \geq \operatorname{rank}(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ 0 & E_{32} \\ 0 & E_{42} \end{bmatrix}) + \operatorname{rank}(E_{53})$$

and hence $E_{53} = 0$.

b) Using Lemma 4 there exist $F := \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ and $K := \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$ such that $(E + BK, A + BF)$ is regular and of index at most one. Since E_{53}, A_{53} are not affected by feedback it follows from part a) that $E_{53} = 0$ and that A_{53} is nonsingular.

Similar as in a), we can prove that

$$\operatorname{rank}(E + BK) = \operatorname{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 \\ B_3K_1 & E_{32} + B_3K_2 \\ 0 & E_{42} \end{bmatrix}).$$

However, since $E_{53} = 0$, we also have

$$\begin{aligned}
\operatorname{rank}(E + BK) &= \operatorname{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 & E_{13} + B_1K_3 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 & E_{23} + B_2K_3 \\ B_3K_1 & E_{32} + B_3K_2 & E_{33} + B_3K_3 \\ 0 & E_{42} & E_{43} \end{bmatrix}) \\
&\geq \operatorname{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 \\ B_3K_1 & E_{32} + B_3K_2 \\ 0 & E_{42} \end{bmatrix})
\end{aligned}$$

and hence

$$\operatorname{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 & E_{13} + B_1K_3 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 & E_{23} + B_2K_3 \\ B_3K_1 & E_{32} + B_3K_2 & E_{33} + B_3K_3 \\ 0 & E_{42} & E_{43} \end{bmatrix}) = \operatorname{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 \\ B_3K_1 & E_{32} + B_3K_2 \\ 0 & E_{42} \end{bmatrix})$$

which implies (11). □

In this section we have discussed condensed froms. Both forms will prove very useful in studying the disturbance decoupling of system (1) by proportional and derivative state feedback, respectively.

4 The range of ranks for $s(E + BK) - (A + BF)$

Since the existence of solutions of the disturbance decoupling problem (without index requirement) is equivalent to the existence of matrices F, K such that

$$\begin{aligned}
\operatorname{rank}(s(E + BK) - (A + BF)) &= n, \\
\operatorname{rank}(C(s(E + BK) - (A + BF))^{-1}G) &= 0,
\end{aligned}$$

we discuss in this section the possible values of $\operatorname{rank}(s(E + BK) - (A + BF))$ that can be achieved. The following theorem gives the key tool for the solution of the disturbance decoupling problem.

Theorem 9 *Let matrices $\hat{E}, \hat{A} \in \mathbf{R}^{t\times l}, \hat{B} \in \mathbf{R}^{t\times s}$ be given.*
a) There exists $F \in \mathbf{R}^{s\times l}$, such that $\operatorname{rank}(s\hat{E} - (\hat{A} + \hat{B}F)) = r$ if and only if

$$\rho + \gamma \leq r \leq \min(l, \operatorname{rank}\begin{bmatrix} s\hat{E} - \hat{A} & \hat{B} \end{bmatrix}), \qquad (12)$$

where ρ and γ are defined as in Corollary 7a).
b) There exist $F, K \in \mathbf{R}^{s\times l}$, such that

$$\operatorname{rank}(s(\hat{E} + \hat{B}K) - (\hat{A} + \hat{B}F)) = r$$

if and only

$$\operatorname{rank}\begin{bmatrix} s\hat{E} - \hat{A} & \hat{B} \end{bmatrix} - \operatorname{rank}(\hat{B}) \leq r \leq \min(l, \operatorname{rank}\begin{bmatrix} s\hat{E} - \hat{A} & \hat{B} \end{bmatrix}). \qquad (13)$$

Proof. For orthogonal matrices U, V such that $(U\hat{E}V, U\hat{A}V, U\hat{B})$ are in condensed form (9), and for $F, K \in \mathbf{R}^{s\times l}$, partition

$$FV =: \left[\begin{array}{cc} F_1 & F_2 \end{array} \right], \quad KV =: \left[\begin{array}{cc} K_1 & K_2 \end{array} \right].$$

Let also t_2 be as in (9). a) A simple calculation yields that for any F,

$$\mathrm{rank}\,(s\hat{E} - (\hat{A} + \hat{B}F)) = \gamma + \mathrm{rank}\left[\begin{array}{c} sE_{11} - A_{11} - B_1F_1 \\ -A_{21} - B_2F_2 \end{array} \right].$$

Since E_{11}, B_2 are of full row rank this implies that

$$\begin{array}{rcl} \rho + \gamma & = & \mathrm{rank}\,(sE_{11} - A_{11} - B_1F_1) + \gamma \\ & \leq & \mathrm{rank}\,(sE - (A + BF)) \\ & \leq & \gamma + \min(\rho + t_2, l - \gamma) \\ & = & \min(l, \mathrm{rank}\left[\begin{array}{cc} s\hat{E} - \hat{A} & \hat{B} \end{array} \right]), \end{array}$$

which implies (12). Conversely let r be any integer satisfying (12). Let Q, Z be orthogonal matrices such that

$$E_{11}Z = \left[\begin{array}{cc} \Sigma_{11} & 0 \end{array} \right], \quad \left[\begin{array}{c} A_{11} \\ A_{21} \end{array} \right] Z = \left[\begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right], \quad \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right] Q = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & 0 \end{array} \right]$$

with nonsingular Σ_{11}, B_{21}. Choose

$$F = Q \left[\begin{array}{cc} -B_{21}^{-1}\Phi_{21} & -B_{21}^{-1}(\Phi_{22} - \Delta) \\ 0 & 0 \end{array} \right] Z^T,$$

where Δ is any matrix of suitable size satisfying

$$\mathrm{rank}\,(\Delta) = r - \rho - \gamma.$$

Then we have

$$\mathrm{rank}\,(s\hat{E} - (\hat{A} + \hat{B}F)) = \rho + \gamma + \mathrm{rank}\,(\Delta) = r.$$

b) For $(\hat{E}, \hat{A}, \hat{B})$ in condensed form (9) choose P, Q orthogonal such that

$$P \left[\begin{array}{c} E_{11} \\ 0 \end{array} \right] = \left[\begin{array}{c} \Theta_1 \\ \Theta_2 \end{array} \right], \quad P \left[\begin{array}{c} A_{11} \\ A_{21} \end{array} \right] = \left[\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right], \quad P \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right] Q = \left[\begin{array}{cc} 0 & 0 \\ \Psi_2 & 0 \end{array} \right],$$

with Ψ_2 nonsingular. Note that $\left[\begin{array}{cc} E_{11} & B_1 \\ 0 & B_2 \end{array} \right]$ is full row rank, hence also Θ_1 is of full row rank. Therefore, there exists an orthogonal matrix Z such that

$$\left[\begin{array}{c} \Theta_1 \\ \Theta_2 \end{array} \right] Z = \left[\begin{array}{cc} \Theta_{11} & 0 \\ \Theta_{21} & \Theta_{22} \end{array} \right], \quad \left[\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right] Z = \left[\begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right],$$

with Θ_{11} nonsingular. Then for any $F, K \in \mathbf{R}^{s\times l}$, we partition

$$F_1Z = \left[\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \end{array} \right], \quad K_1Z = \left[\begin{array}{cc} K_{11} & K_{12} \\ K_{21} & K_{22} \end{array} \right]$$

accordingly and obtain

$$\begin{aligned}
&\operatorname{rank}(s(E+BK)-(A+BF))=\gamma \\
+\ &\operatorname{rank}\begin{bmatrix} s\Theta_{11}-\Phi_{11} & -\Phi_{12} \\ s(\Theta_{21}+\Psi_2 K_{11})-(\Phi_{21}+\Psi_2 F_{11}) & s(\Theta_{22}+\Psi_2 K_{12})-(\Phi_{22}+\Psi_2 F_{12}) \end{bmatrix}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\gamma+\operatorname{rank}(s\Theta_{11}-\Phi_{11}) \\
\leq\ &\operatorname{rank}(s(E+BK)-(A+BF)) \\
\leq\ &\gamma+\min(l-\gamma, t_2+\rho).
\end{aligned}$$

Thus, (13) follows directly from Lemma 5.

Conversely let r satisfy (13). Set

$$\begin{bmatrix} F_{11} & F_{12} \end{bmatrix} = -\Psi_2^{-1}\begin{bmatrix} \Phi_{21} & \Phi_{22} \end{bmatrix}, \quad \begin{bmatrix} K_{11} & K_{12} \end{bmatrix} = -\Psi_2^{-1}\begin{bmatrix} \Theta_{21} & \Theta_{22}-\Delta \end{bmatrix},$$

where Δ is any real matrix of suitable size satisfying

$$\operatorname{rank}(\Delta) = r-\gamma-\operatorname{rank}(\Theta_{11}).$$

With these F and K, we obtain

$$\operatorname{rank}(s(\hat{E}+\hat{B}K)-(\hat{A}+\hat{B}F)) = \gamma+\rho+\operatorname{rank}(\Delta) = r.$$

□

Remark 1 In Part a) of Theorem 9 if $r = \min(l, \operatorname{rank}\begin{bmatrix} s\hat{E}-\hat{A} & \hat{B} \end{bmatrix})$, then we can also choose F in the following way.

Let orthogonal matrices P, W, Q be chosen such that

$$P\Phi_{22}W = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad PB_{21}Q = \begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \end{bmatrix},$$

where $\Sigma, \Psi_{11}, \Psi_{22}$ are nonsingular. Then choose

$$F := Q\begin{bmatrix} 0 & F_{12} \\ 0 & 0 \end{bmatrix} Z^T \tag{14}$$

with $F_{12} = Q\begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} W^T$, where Δ is any matrix of suitable size such that $\begin{bmatrix} \Sigma & 0 \\ 0 & \Delta \end{bmatrix}$ has full rank.

In Part b) of Theorem 9 if $r = \min(l, \operatorname{rank}\begin{bmatrix} s\hat{E}-\hat{A} & \hat{B} \end{bmatrix})$, then we can choose $K = 0$ and F as in (14). We can also choose $F = 0$ and construct an appropriate K similar to that in the construction of F in (14).

Theorem 9 will be applied in many different ways to prove the existence and also to construct appropriate feedbacks for the disturbance decoupling problem.

5 Disturbance Decoupling Without Index Requirement

In this section we derive necessary and sufficient conditions for the solvability of the disturbance decoupling problem without imposing the requirement that the index of the closed-loop system is at most one. We need the following lemma, see also [1].

Lemma 10 *Consider a system of the form (1). If (E, A) is regular, then the following are equivalent:*
a) $C(sE - A)^{-1}G = 0$;
b) $\operatorname{rank} \begin{bmatrix} sE - A & G \\ C & 0 \end{bmatrix} = n$;
c) $\operatorname{rank} \begin{bmatrix} T_\infty^T(G)(sE - A) \\ C \end{bmatrix} = n - \operatorname{rank}(G)$.

Proof. The proof follows directly from the fact that if $\det(sE - A) \neq 0$ then

$$\operatorname{rank}(C(sE - A)^{-1}G) = \operatorname{rank} \begin{bmatrix} sE - A & G \\ C & 0 \end{bmatrix} - \operatorname{rank}(sE - A).$$

□

Our next theorem yields the solution for the disturbance decoupling problem in the case that no index requirement is given, i.e. we require regularity and disturbance decoupling.

Theorem 11 *Consider a system of the form (1).*
a) There exists a feedback matrix $F \in \mathbf{R}^{m \times n}$ such that the pencil $(E, A + BF)$ is regular and $C(sE - (A + BF))^{-1}G = 0$ if and only if the following three conditions hold:

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} sE - A & B \end{bmatrix} &= n; & (15) \\ \operatorname{rank} \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} sE - A & B & G \\ C & 0 & 0 \end{bmatrix}; & (16) \\ \tau + \mu &\leq n - p, & (17) \end{aligned}$$

where τ, μ are as in Corollary 7 b).
b) There exist feedback matrices $F, K \in \mathbf{R}^{m \times n}$, such that the pencil $(E + BK, A + BF)$ is regular and $C(s(E + BK) - (A + BF))^{-1}G = 0$ if and only if conditions (15) and (16) hold and furthermore

$$\operatorname{rank} \begin{bmatrix} T_\infty^T(G)(sE - A) & T_\infty^T(G)B \\ C & 0 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} T_\infty^T(G)B \\ 0 \end{bmatrix} \leq n - p. \qquad (18)$$

Proof. By Theorem 6 there exist orthogonal matrices U, V such that UEV, UAV, UB, CV, UG are of the form (10). Thus for the proof we may assume w.l.o.g. that the system is already in form (10). Since the proof is constructive, a numerical method can be based on this proof after first transforming to condensed form.

a) **Necessity**: Let $F \in \mathbf{R}^{m \times n}$ be such that $(E, A + BF)$ is regular and $C(sE - (A + BF))^{-1}G = 0$. Partition $F =: \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ compatibly with (E, A, B). From Lemma 10 we have that

$$\begin{aligned} \operatorname{rank}(sE-(A+BF)) &= n, \\ \operatorname{rank}\begin{bmatrix} sE_{21}-(A_{21}+B_2F_1) \\ -A_{32}-B_3F_1 \end{bmatrix} &= n-p-n_2-n_3. \end{aligned}$$

Hence, conditions (15) and (17) follow directly from Theorem 9. From Lemma 10 we also have that

$$\operatorname{rank}\begin{bmatrix} sE-A-BF & G \\ C & 0 \end{bmatrix} = n = \operatorname{rank}\begin{bmatrix} sE-A-BF \\ C \end{bmatrix}.$$

Hence we have

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} sE-A & B & G \\ C & 0 & 0 \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} sE-A-BF & B & G \\ C & 0 & 0 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} sE-A-BF & B \\ C & 0 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} sE-A & B \\ C & 0 \end{bmatrix}, \end{aligned}$$

which implies (16).

Sufficiency: From (15), (16) and (17) we deduce the following three conditions

$$\operatorname{rank}\begin{bmatrix} sE_{42}-A_{42} & sE_{43}-A_{43} \\ 0 & sE_{53}-A_{53} \end{bmatrix} = \tilde{n}_4+\tilde{n}_5; \tag{19}$$

$$p+\tau \le n_1 \le p+\tilde{n}_2+\tilde{n}_3 = p+\tau+\eta; \tag{20}$$

$$\operatorname{rank}\begin{bmatrix} sE_{11}-A_{11} & B_1 \\ sE_{21}-A_{21} & B_2 \\ -A_{31} & B_3 \end{bmatrix} = p+\tilde{n}_2+\tilde{n}_3 = p+\tau+\eta. \tag{21}$$

By partitioning B_3 into two row blocks and using orthogonal column compression (see [11]), we may assume w.l.o.g. that

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{matrix} p \\ \tilde{n}_2 \\ n_1-p-\tilde{n}_2 \\ p+\tilde{n}_2+\tilde{n}_3-n_1 \end{matrix} \overset{\begin{matrix} m+n_1-p-\tilde{n}_2-\tilde{n}_3 & p+\tilde{n}_2+\tilde{n}_3-n_1 \end{matrix}}{\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \\ 0 & \Psi_{42} \end{bmatrix}},$$

where Ψ_{31} is full of row rank and Ψ_{42} is nonsingular. Anologously then

$$\begin{bmatrix} -A_{31} & sE_{32}-A_{32} & sE_{33}-A_{33} \end{bmatrix} = \begin{bmatrix} -\Phi_{31} & s\Theta_{32}-\Phi_{32} & s\Theta_{33}-\Phi_{33} \\ -\Phi_{41} & s\Theta_{42}-\Phi_{42} & s\Theta_{43}-\Phi_{43} \end{bmatrix}.$$

Using $F_{21} := -\Psi_{42}^{-1}\Phi_{41}$ we then obtain from (21) that

$$\operatorname{rank}\begin{bmatrix} sE_{11}-A_{11}-\Psi_{12}F_{21} & \Psi_{11} \\ sE_{21}-A_{21}-\Psi_{22}F_{21} & \Psi_{21} \\ -\Phi_{31}-\Psi_{32}F_{21} & \Psi_{31} \end{bmatrix} = n_1. \tag{22}$$

Thus, from Theorem 9 we obtain that there exists F_{11} such that

$$\text{rank}\begin{bmatrix} sE_{11} - A_{11} - \Psi_{12}F_{21} - \Psi_{11}F_{11} \\ sE_{21} - A_{21} - \Psi_{22}F_{21} - \Psi_{21}F_{11} \\ -\Phi_{31} - \Psi_{32}F_{21} - \Psi_{31}F_{11} \end{bmatrix} = n_1. \tag{23}$$

Since $sE_{53} - A_{53}$ is of full column rank for any finite $s \in \mathbf{C}$ and since (19) holds, it follows that

$$\text{rank}\,(sE_{42} - A_{42}) = \tilde{n}_4 \tag{24}$$

and

$$\text{rank}\,(sE_{53} - A_{53}) = \tilde{n}_5 = n_3, \quad \forall s \in \mathbf{C}. \tag{25}$$

Therefore, Theorem 9 implies that there exists F_{22} such that

$$\text{rank}\begin{bmatrix} s\Theta_{42} - \Phi_{42} - \Psi_{42}F_{22} & s\Theta_{43} - \Phi_{43} \\ sE_{42} - A_{42} & sE_{43} - A_{43} \\ 0 & sE_{53} - A_{53} \end{bmatrix} = n_2 + n_3.$$

Now, set

$$F = \begin{bmatrix} F_{11} & 0 & 0 \\ F_{21} & F_{22} & 0 \end{bmatrix}.$$

Then we have that

$$\begin{aligned} & sE - A - BF \\ = & \begin{bmatrix} sE_{11} - A_{11} - \Psi_{11}F_{11} - \Psi_{12}F_{21} & sE_{12} - A_{12} - \Psi_{12}F_{22} & sE_{13} - A_{13} \\ sE_{21} - A_{21} - \Psi_{21}F_{11} - \Psi_{22}F_{21} & sE_{22} - A_{22} - \Psi_{22}F_{22} & sE_{23} - A_{23} \\ -\Phi_{31} - \Psi_{31}F_{11} - \Psi_{32}F_{21} & s\Theta_{32} - \Phi_{32} - \Psi_{32}F_{22} & s\Theta_{33} - \Phi_{33} \\ & s\Theta_{42} - \Phi_{42} - \Psi_{42}F_{22} & s\Theta_{43} - \Phi_{43} \\ & sE_{42} - A_{42} & sE_{43} - A_{43} \\ & & sE_{53} - A_{53} \end{bmatrix}. \end{aligned}$$

Clearly, for this F, the pencil $(E, A + BF)$ is regular, and Lemma 10 implies that $C(sE - (A + BF))^{-1}G = 0$.

b) **Necessity:** Let $F, K \in \mathbf{R}^{m \times n}$ be such that $(E + BK, A + BF)$ is regular and $C(s(E + BK) - (A + BF))^{-1}G = 0$. Then as in a), we have that

$$\text{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = \text{rank}\begin{bmatrix} s(E + BK) - (A + BF) & B \end{bmatrix} = n,$$

and

$$\begin{aligned} \text{rank}\begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} &= \text{rank}\begin{bmatrix} s(E + BK) - (A + BF) & B \\ C & 0 \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} s(E + BK) - (A + BF) & B & G \\ C & 0 & 0 \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} sE - A & B & G \\ C & 0 & 0 \end{bmatrix}, \end{aligned}$$

which implies that (15) and (16) hold.

On the other hand, from Lemma 10, we have that

$$\operatorname{rank}\begin{bmatrix} s(E_{21}+B_2K_1)-(A_{21}+B_2F_1) \\ -(A_{31}+B_3F_1) \end{bmatrix} = n-p-n_2-n_3,$$

and thus (18) follows directly from Part 2 in Theorem 9 b).

Sufficiency: Because of (18), we may assume w.l.o.g. that

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{matrix} p \\ n_1-p \\ p+\tilde{n}_2+\tilde{n}_3-n_1 \end{matrix} \overset{\begin{matrix} m+n_1-p-\tilde{n}_2-\tilde{n}_3 & p+\tilde{n}_2+\tilde{n}_3-n_1 \end{matrix}}{\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \\ 0 & \Psi_{32} \end{bmatrix}}$$

with nonsingular Ψ_{32}. If this were not the case, then, by performing a singular value decomposition, (see [11]),

$$P\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} Q = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma \end{bmatrix}$$

with Σ nonsingular and diagonal, we can partition

$$\begin{bmatrix} 0 & \\ & \Sigma \end{bmatrix} =: \begin{matrix} n_1-p \\ p+\tilde{n}_2+\tilde{n}_3-n_1 \end{matrix} \overset{\begin{matrix} m+n_1-p-\tilde{n}_2-\tilde{n}_3 & p+\tilde{n}_2+\tilde{n}_3-n_1 \end{matrix}}{\begin{bmatrix} \Psi_{21} & \Psi_{22} \\ 0 & \Psi_{32} \end{bmatrix}}.$$

Denote the associated transformed parts of E and A similarly as

$$\begin{bmatrix} sE_{21}-A_{21} & sE_{22}-A_{22} & sE_{23}-A_{23} \\ -A_{31} & sE_{32}-A_{32} & sE_{33}-A_{33} \end{bmatrix}$$
$$=: \begin{bmatrix} s\Theta_{21}-\Phi_{21} & s\Theta_{22}-\Phi_{22} & s\Theta_{23}-\Phi_{23} \\ s\Theta_{31}-\Phi_{31} & s\Theta_{32}-\Phi_{32} & s\Theta_{33}-\Phi_{33} \end{bmatrix}$$

and set

$$K_{21} := -\Psi_{32}^{-1}\Theta_{31}, \quad F_{21} := -\Psi_{32}^{-1}\Phi_{31}.$$

Using (16), similar to the proof of sufficiency in a), we have

$$\operatorname{rank}\begin{bmatrix} s(E_{11}+\Psi_{12}K_{21})-(A_{11}+\Psi_{12}F_{21}) & \Psi_{11} \\ s(\Theta_{21}+\Psi_{22}K_{21})-(\Phi_{21}+\Psi_{22}F_{21}) & \Psi_{21} \end{bmatrix} = n_1. \tag{26}$$

Thus, from Theorem 9 we get that there exists a matrix F_{11} such that

$$\operatorname{rank}\begin{bmatrix} s(E_{11}+\Psi_{12}K_{21})-(A_{11}+\Psi_{12}F_{21})-\Psi_{11}F_{11} \\ s(\Theta_{21}+\Psi_{22}K_{21})-(\Phi_{21}+\Psi_{22}F_{21})-\Psi_{21}F_{11} \end{bmatrix} = n_1. \tag{27}$$

Using (15), we obtain conditions (24) and (25), hence Theorem 9 yields that there exists F_{22} such that

$$\begin{bmatrix} s\Theta_{32}-\Phi_{32}-\Psi_{32}F_{22} & s\Theta_{33}-\Phi_{33} \\ sE_{42}-A_{42} & sE_{43}-A_{43} \\ & sE_{53}-A_{53} \end{bmatrix} = n_2+n_3.$$

Now, with

$$K = \begin{bmatrix} 0 & 0 & 0 \\ K_{21} & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} F_{11} & 0 & 0 \\ F_{21} & F_{22} & 0 \end{bmatrix}$$

we obtain that

$$\begin{aligned} & s(E + BK) - (A + BF) \\ = & \begin{bmatrix} s(E_{11} + \Psi_{12}K_{21}) - (A_{11} + \Psi_{11}F_{11} + \Psi_{12}F_{21}) & sE_{12} - (A_{12} + \Psi_{12}F_{22}) & sE_{13} - A_{13} \\ s(\Theta_{21} + \Psi_{22}K_{21}) - (\Phi_{21} + \Psi_{21}F_{11} + \Psi_{22}F_{21}) & s\Theta_{22} - (\Phi_{22} + \Psi_{22}F_{22}) & s\Theta_{23} - \Phi_{23} \\ 0 & s\Theta_{32} - (\Phi_{32} + \Psi_{32}F_{22}) & s\Theta_{33} - \Phi_{33} \\ 0 & sE_{42} - A_{42} & sE_{43} - A_{43} \\ 0 & 0 & sE_{53} - A_{53} \end{bmatrix}. \end{aligned}$$

Clearly, for these F, K, we have that $(E + BK, A + BF)$ is regular, and Lemma 10 implies that $C(s(E + BK) - (A + BF))^{-1}G = 0$. □

Remark 2 For the construction of the matrices F_{11}, F_{22} in the proof of Theorem 9 b), we have used in (23) and (27) the term Ψ_{32}^{-1}. An alternative construction to compute an appropriate F_{11} without the explicit inversion of Ψ_{32} is as follows:

Since (23) holds if and only if

$$\operatorname{rank}\left(\begin{bmatrix} sE_{11} - A_{11} & \Psi_{11} \\ sE_{21} - A_{21} & \Psi_{22} \\ -\Phi_{31} & \Psi_{32} \\ -\Phi_{41} & \Psi_{42} \end{bmatrix} + \begin{bmatrix} \Psi_{11} \\ \Psi_{21} \\ \Psi_{31} \\ 0 \end{bmatrix} F_{11} \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} \right) = n_1 + \operatorname{rank}(\Psi_{42}),$$

we can first perform an orthogonal transformation such that

$$\begin{bmatrix} sE_{11} - A_{11} & \Psi_{12} \\ sE_{21} - A_{21} & \Psi_{22} \\ -\Phi_{31} & \Psi_{32} \\ -\Phi_{41} & \Psi_{42} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} & \star \\ s\tilde{E}_{21} - \tilde{A}_{21} & \star \\ -\tilde{\Phi}_{31} & \star \\ 0 & \tilde{\Psi}_{42} \end{bmatrix},$$

where $Z_{11} \in \mathbf{R}^{n_1 \times n_1}$, $\tilde{\Psi}_{42}$ is nonsingular and $\tilde{\Psi}_{42}$ has the same size as Ψ_{42}. Because $\begin{bmatrix} E_{21} & 0 \\ -\Phi_{41} & \Psi_{42} \end{bmatrix}$ is of full row rank, also $\tilde{E}_{21}$ is of full row rank. Moreover, we know that Ψ_{31} is of full row rank and, in addition, since Ψ_{42} is nonsingular, Z_{11} is also nonsingular, so we can use the numerical procedure given in Remark 1 to obtain F_{11} satisfying

$$\operatorname{rank} \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} - \Psi_{11}F_{11} \\ s\tilde{E}_{21} - \tilde{A}_{21} - \Psi_{21}F_{11} \\ -\tilde{\Phi}_{31} - \Psi_{31}F_{11} \end{bmatrix} = n_1.$$

The technique above can be easily generalized to compute F_{11} such that (27) holds without explicit inversion of Ψ_{42}.

Remark 3 Condition (16) has also been obtained in [1] under the additional assumptions that $\operatorname{rank}\begin{bmatrix} E & G \end{bmatrix} = n$ and $\operatorname{rank}\begin{bmatrix} E & B & G \end{bmatrix} = n$.

By exchanging the roles of E and A, we can obtain analogous results for the case that only derivative feedback is used. The results are listed in Appendix B.

6 Disturbance decoupling with index requirement

In the previous section we have obtained necessary and sufficient conditions which ensure that there exist feedback matrices F, K such that $(E+BK, A+BF)$ is regular and $C(s(E+BK)-(A+BF))^{-1}G = 0$. However, as pointed out in Section 1, it is very important to take the index of the systems into account whenever one designs a controller for a descriptor system. Motivatived by this, we discuss in this section the disturbance decoupling problem with the additional requirement that the index of the closed loop system is at most one. In order to do this in a coordinate free way, we need to introduce some further spaces.

First we consider the smallest (with respect to dimension) row space and the largest column space that can be achieved in the closed loop system via state feedback and characterize these spaces by Theorem 6. Set

$$\begin{aligned} \tilde{V}_r &:= \min_F V_r(\begin{bmatrix} T_\infty^T(G)E \\ 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(G)(A+BF) \\ C \end{bmatrix}) \\ \tilde{V}_c &= \max_F V_c(\begin{bmatrix} T_\infty^T(G)E \\ 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(G)(A+BF) \\ C \end{bmatrix}) \end{aligned}$$

and introduce

$$\begin{aligned} \Lambda_1 &:= T_\infty(\begin{bmatrix} E\tilde{V}_c & B & G \end{bmatrix}), \\ \Lambda_2 &:= T_\infty^T(\Lambda_1)E\tilde{V}_c, \\ \Lambda_3 &:= T_\infty^T(\Lambda_1)A\tilde{V}_c, \\ \Lambda_4 &:= \tilde{V}_r^T T_\infty^T(G)ES_\infty(\tilde{V}_c^T)V_c(\Lambda_1^T ES_\infty(\tilde{V}_c^T), \Lambda_1^T AS_\infty(\tilde{V}_c^T)), \\ \Lambda_5 &:= T_\infty^T(\Lambda_1)B \end{aligned} \tag{28}$$

(29)

These spaces can be characterized we the condensed form of Theorem 6.

Corollary 12 *Given a system of the form (1) in the condensed form (10). Then*

$$\begin{aligned} \text{range}(\begin{bmatrix} E_{11} \\ E_{21} \\ 0_{\tilde{n}_3 \times n_1} \end{bmatrix}) &= \text{range}(\Lambda_2), \\ \text{range}(\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix}) &= \text{range}(\Lambda_3), \\ \text{range}(\begin{bmatrix} E_{32} \\ E_{42} \\ 0 \end{bmatrix}) &= \text{range}(\Lambda_4), \\ \text{range}(\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}) &= \text{range}(\Lambda_5). \end{aligned}$$

Using these spaces and their characterizations, we can now approach the solution to the disturbance decoupling problem with the requirement that the index is at most one.

Theorem 13 *Consider a system of the form (1). There exists a feedback matrix* $F \in \mathbf{R}^{m \times n}$ *such that the pencil* $(E, A+BF)$ *is regular, of index at most one, and*

$$C(sE-(A+BF))^{-1}G = 0$$

if and only if conditions (7), (17) and furthermore the following two conditions hold:

$$\begin{aligned} \operatorname{rank}(E) &= \operatorname{rank}(\Lambda_2) + \operatorname{rank}(\Lambda_4) & (30)\\ \operatorname{rank}\begin{bmatrix} \Lambda_2 & \Lambda_3 S_\infty(\Lambda_2) & \Lambda_5 \end{bmatrix} &= p + \tau + \eta, & (31) \end{aligned}$$

where the spaces Λ_i, *are as defined in (28) and* τ, η *are as in Corollary 7 b).*

Proof. We may assume w.l.o.g. that the system is in the form (10).

Necessity: Conditions (7) and (17) follow directly from Lemma 4 and Theorem 11, respectively. Hence we only need to prove conditions (30)–(31).

Let $F \in \mathbf{R}^{m \times n}$ be such that $(E, A+BF)$ is regular, of index at most one, and $C(sE-(A+BF))^{-1}G = 0$. Then by Corollary 8, we have that $E_{53} = 0$ and A_{53} is nonsingular. Note that $(E, A+BF)$ is of index at most one, so

$$\begin{aligned} \operatorname{rank}(E) &= \deg(\det(sE-A-BF)) \\ &= \deg(\det\begin{bmatrix} sE_{11}-A_{11}-B_1F_1 & sE_{12}-A_{12}-B_1F_2 \\ sE_{21}-A_{21}-B_2F_1 & sE_{22}-A_{22}-B_2F_2 \\ -A_{31}-B_3F_1 & sE_{32}-A_{32}-B_3F_3 \\ 0 & sE_{42}-A_{42} \end{bmatrix}) \qquad (32)\\ &+ \deg(\det(sE_{53}-A_{53})). \end{aligned}$$

where $F = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ is partitioned compatibley. Hence, from (32) it follows that

$$\begin{aligned} \operatorname{rank}(E) &= \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ & E_{32} \\ & E_{42} \end{bmatrix} \\ &= \deg(\det\begin{bmatrix} sE_{11}-A_{11}-B_1F_1 & sE_{12}-A_{12}-B_1F_2 \\ sE_{21}-A_{21}-B_2F_1 & sE_{22}-A_{22}-B_2F_2 \\ -A_{31}-B_3F_1 & sE_{32}-A_{32}-B_3F_2 \\ 0 & sE_{42}-A_{42} \end{bmatrix}). \qquad (33) \end{aligned}$$

Using that E_{21}, E_{42} are of full row rank, we can assume w.l.o.g. (by performing appropriate equivalence transformations) that

$$\begin{aligned} E_{11} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_{11} & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} \Sigma_{21} & 0 & 0 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Theta_{32} & 0 \end{bmatrix}, \\ E_{42} &= \begin{bmatrix} 0 & 0 & \Sigma_{42} \end{bmatrix}, \quad E_{12} = \begin{bmatrix} \tilde{E}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{22} = 0, \end{aligned}$$

where $\Sigma_{11} \in \mathbf{R}^{p_1 \times p_1}, \Sigma_{21} \in \mathbf{R}^{\tilde{n}_2 \times \tilde{n}_2}, \Sigma_{42} \in \mathbf{R}^{\tilde{n}_4 \times \tilde{n}_4}$ are nonsingular and $\Theta_{32} \in \mathbf{R}^{(n_2 - \tilde{n}_4) \times t}$ is of full column rank. Partition comptabily

$$\begin{aligned} A_{11} + B_1 F_1 &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \end{bmatrix}, \quad A_{12} + B_1 F_2 = \begin{bmatrix} \Phi_{14} & \Phi_{15} & \Phi_{16} \\ \Phi_{24} & \Phi_{25} & \Phi_{26} \end{bmatrix}, \\ A_{21} + B_2 F_1 &= \begin{bmatrix} \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix}, \quad A_{22} + B_2 F_2 = \begin{bmatrix} \Phi_{34} & \Phi_{35} & \Phi_{36} \end{bmatrix}, \\ A_{31} + B_3 F_1 &= \begin{bmatrix} \Phi_{41} & \Phi_{42} & \Phi_{43} \\ \Phi_{51} & \Phi_{52} & \Phi_{53} \end{bmatrix}, \quad A_{32} + B_3 F_2 = \begin{bmatrix} \Phi_{44} & \Phi_{45} & \Phi_{46} \\ \Phi_{54} & \Phi_{55} & \Phi_{56} \end{bmatrix}, \\ A_{42} &= \begin{bmatrix} \Phi_{64} & \Phi_{65} & \Phi_{66} \end{bmatrix}. \end{aligned}$$

Then (33) yields that

$$\begin{bmatrix} T_\infty^T(\tilde{E}_{12})\Phi_{13} & T_\infty^T(\tilde{E}_{12})\Phi_{14}S_\infty(\tilde{E}_{12}) \\ \Phi_{43} & \Phi_{44}S_\infty(\tilde{E}_{12}) \\ T_\infty^T(\Theta_{32})\Phi_{53} & T_\infty^T(\Theta_{32})\Phi_{54}S_\infty(\tilde{E}_{12}) \end{bmatrix} \text{ is nonsingular}$$

Hence, we know that

$$\operatorname{rank} \begin{bmatrix} T_\infty^T(\tilde{E}_{12})\Phi_{13} \\ \Phi_{43} \\ T_\infty^T(\Theta_{32})\Phi_{53} \end{bmatrix} = n_1 - p_1 - \tilde{n}_2. \tag{34}$$

But, from Lemma 10 we have

$$\operatorname{rank} \begin{bmatrix} s\Sigma_{21} - \Phi_{31} & -\Phi_{32} & -\Phi_{33} \\ -\Phi_{41} & -\Phi_{42} & -\Phi_{43} \\ -\Phi_{51} & -\Phi_{52} & -\Phi_{53} \end{bmatrix} = n - p - n_2 - n_3 = n_1 - p$$

and hence

$$\operatorname{rank} \begin{bmatrix} \Phi_{43} \\ T_\infty^T(\Theta_{32})\Phi_{53} \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} \Phi_{43} \\ \Phi_{53} \end{bmatrix} \leq n_1 - p - \tilde{n}_2. \tag{35}$$

Thus, by (34), we have

$$\operatorname{rank} (T_\infty^T(\tilde{E}_{12})\Phi_{13}) \geq p - p_1, \tag{36}$$

where $p - p_1$ is the row size of Φ_{13}. This implies that $\tilde{E}_{12} = 0$ and hence (33) implies that

$$\operatorname{rank} (E) = \operatorname{rank} \begin{bmatrix} E_{11} \\ E_{21} \\ 0_{\tilde{n}_3 \times n_1} \end{bmatrix} + \operatorname{rank} \begin{bmatrix} E_{32} \\ E_{42} \end{bmatrix} \tag{37}$$

Thus, Corollary 12 gives (30).

Furthermore, using (36) we have that

$$\operatorname{rank} (\Phi_{13}) = p - p_1, \qquad \operatorname{rank} \begin{bmatrix} \Phi_{43} \\ \Phi_{53} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \Phi_{43} \\ T_\infty^T(\Theta_{32})\Phi_{53} \end{bmatrix} = n_1 - p - \tilde{n}_2$$

and thus

$$\operatorname{rank} \begin{bmatrix} E_{11} & A_{11}S & B_1 \\ E_{21} & A_{21}S & B_2 \\ 0 & A_{31}S & B_3 \end{bmatrix} = p + \tilde{n}_2 + \tilde{n}_3, \tag{38}$$

where

$$S := S_\infty\left(\begin{bmatrix} E_{11} \\ E_{21} \\ 0_{\tilde{n}_3 \times n_1} \end{bmatrix}\right).$$

Then (31) follows directly from Corollary 12.

Sufficiency By Corollary 12 and Corollary 8 then it follows that $E_{53} = 0$, A_{53} is nonsingular, and furthermore (31) and (38) hold. Using conditions (17) and (38) and considering that E_{21}, B_3 and E_{42} are of full row rank, we can compute orthogonal matrices P_1, P_2, Z_1, Z_2, W, such that

$$P_1 \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} Z_1 = \begin{array}{c} \\ p-p_1 \\ p_1 \\ \tilde{n}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \tilde{n}_2 & p_1 & n_1-p_1-\tilde{n}_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ \Theta_{21} & \Sigma_{11} & 0 \\ \Sigma_{21} & 0 & 0 \end{bmatrix} \end{array},$$

$$E_{21} Z_1 = \begin{array}{c} \\ \tilde{n}_2 \end{array} \begin{array}{c} \begin{array}{cc} \tilde{n}_2 & n_1-\tilde{n}_2 \end{array} \\ \begin{bmatrix} \tilde{\Sigma}_{21} & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} P_1 & & \\ & P_2 & \\ & & I_{\tilde{n}_4} \end{bmatrix} \begin{bmatrix} E_{12} & E_{13} \\ E_{22} & E_{23} \\ E_{32} & E_{33} \\ E_{42} & E_{43} \end{bmatrix} \begin{bmatrix} Z_2 & \\ & I_{n_3} \end{bmatrix} = \begin{array}{c} \\ p-p_1 \\ p_1 \\ \tilde{n}_2 \\ n_1-p-\tilde{n}_2 \\ p+\tilde{n}_2+\tilde{n}_3-n_1 \\ \tilde{n}_4 \end{array} \begin{array}{c} \begin{array}{ccc} n_2-\tilde{n}_4 & \tilde{n}_4 & n_3 \end{array} \\ \begin{bmatrix} \Theta_{14} & \Theta_{15} & \Theta_{16} \\ \Theta_{24} & \Theta_{25} & \Theta_{26} \\ \Theta_{34} & \Theta_{35} & \Theta_{36} \\ 0 & \Theta_{45} & \Theta_{46} \\ \Theta_{54} & \Theta_{55} & \Theta_{56} \\ 0 & \Sigma_{42} & \Theta_{66} \end{bmatrix} \end{array}$$

and

$$\begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} W = \begin{array}{c} \\ p-p_1 \\ p_1 \\ \tilde{n}_2 \\ t_1 \\ \tilde{n}_3-t_1 \end{array} \begin{array}{c} \begin{array}{cc} \tilde{n}_3-t_1 & m+t_1-\tilde{n}_3 \end{array} \\ \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \\ \Psi_{41} & \Psi_{42} \\ \Psi_{51} & 0 \end{bmatrix} \end{array},$$

where $t_1 := n_1 - p - \tilde{n}_2$ and $\Sigma_{21}, \tilde{\Sigma}_{21}, \Psi_{51}, \Sigma_{11}, \Sigma_{42}$ are nonsingular. Moreover, Θ_{32} is square because $n_2 - \tilde{n}_4 = p + \tilde{n}_2 + \tilde{n}_3 - n_1$. Denote compatibly

$$\begin{bmatrix} P_1 & & \\ & P_2 & \\ & & I_{\tilde{n}_4} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ 0 & A_{42} & A_{43} \end{bmatrix} \begin{bmatrix} Z_1 & & \\ & Z_2 & \\ & & I_{n_3} \end{bmatrix}$$

$$=: \begin{array}{c} \\ p-p_1 \\ p_1 \\ \tilde{n}_2 \\ t_1 \\ \tilde{n}_3-t_1 \\ \tilde{n}_4 \end{array} \begin{array}{c} \begin{array}{cccccc} \tilde{n}_2 & p_1 & t_2 & n_2-\tilde{n}_4 & \tilde{n}_4 & n_3 \end{array} \\ \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35} & \Phi_{36} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \Phi_{45} & \Phi_{46} \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} & \Phi_{56} \\ 0 & 0 & 0 & \Phi_{64} & \Phi_{65} & \Phi_{66} \end{bmatrix} \end{array},$$

where $t_2 := n_1 - p_1 - \tilde{n}_2$. From conditions (17) and (38) we have that $\begin{bmatrix} \Phi_{13} & \Psi_{11} & \Psi_{12} \\ \Phi_{43} & \Psi_{41} & \Psi_{42} \\ \Phi_{53} & \Psi_{51} & 0 \end{bmatrix}$ is of full row rank and $\begin{bmatrix} \Phi_{13} \\ \Phi_{43} \end{bmatrix}$ is square. Thus, we can compute F_{23} such that

$$\begin{bmatrix} \Phi_{13} + \Psi_{12} F_{23} & \Psi_{11} \\ \Phi_{43} + \Psi_{42} F_{23} & \Psi_{41} \\ \Phi_{53} & \Psi_{51} \end{bmatrix}$$

is nonsingular. *It should be pointed out that in this step, no matrix inversion is necessary.*

The second step is to use the method given in the Appendix of [3] to compute matrices F_{14}, F_{15} such that the pencil

$$\left(\begin{bmatrix} \Theta_{54} & \Theta_{55} \\ 0 & \Sigma_{42} \end{bmatrix}, \begin{bmatrix} \Phi_{54} + \Psi_{51} F_{14} & \Phi_{55} + \Psi_{51} F_{15} \\ \Phi_{64} & \Phi_{65} \end{bmatrix} \right) \tag{39}$$

is regular and of index at most one.

Let

$$\begin{aligned} F_{11} &:= -\Psi_{51}^{-1}\Phi_{51}, \quad F_{12} := -\Psi_{51}^{-1}\Phi_{52}, \quad F_{13} := -\Psi_{51}^{-1}\Phi_{53} \\ F &:= W \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & 0 \\ 0 & 0 & F_{23} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1^T & & \\ & Z_2^T & \\ & & I_{n_3} \end{bmatrix} \in \mathbf{R}^{m \times n}. \end{aligned}$$

and

$$\begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} & \tilde{\Phi}_{14} & \tilde{\Phi}_{15} \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} & \tilde{\Phi}_{23} & \tilde{\Phi}_{24} & \tilde{\Phi}_{25} \\ \tilde{\Phi}_{31} & \tilde{\Phi}_{32} & \tilde{\Phi}_{33} & \tilde{\Phi}_{34} & \tilde{\Phi}_{35} \\ \tilde{\Phi}_{41} & \tilde{\Phi}_{42} & \tilde{\Phi}_{43} & \tilde{\Phi}_{44} & \tilde{\Phi}_{45} \\ \tilde{\Phi}_{51} & \tilde{\Phi}_{52} & \tilde{\Phi}_{53} & \tilde{\Phi}_{54} & \tilde{\Phi}_{55} \end{bmatrix} := \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \\ \Psi_{41} & \Psi_{42} \\ \Psi_{51} & 0 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} \\ 0 & 0 & F_{23} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \Phi_{45} \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} \end{bmatrix}.$$

Obviously, we have

$$\tilde{\Phi}_{5i} = 0, \quad i = 1, 2, 3$$

and $\begin{bmatrix} \tilde{\Phi}_{13} \\ \tilde{\Phi}_{43} \end{bmatrix}$ is nonsingular.

Note that $E_{53} = 0$, and hence we have

$$\begin{bmatrix} P_1 & & \\ & P_2 & \\ & & I_{\tilde{n}_4 + \tilde{n}_5} \end{bmatrix} (sE - A - BF) \begin{bmatrix} Z_1 & & \\ & Z_2 & \\ & & I_{n_3} \end{bmatrix}$$

$$= \begin{bmatrix} -\tilde{\Phi}_{11} & -\tilde{\Phi}_{12} & -\tilde{\Phi}_{13} & s\Theta_{14}-\tilde{\Phi}_{14} & s\Theta_{15}-\tilde{\Phi}_{15} & s\Theta_{16}-\Phi_{16} \\ s\Theta_{21}-\tilde{\Phi}_{21} & s\Sigma_{11}-\tilde{\Phi}_{22} & -\tilde{\Phi}_{23} & s\Theta_{24}-\tilde{\Phi}_{24} & s\Theta_{25}-\tilde{\Phi}_{25} & s\Theta_{26}-\Phi_{26} \\ s\Sigma_{21}-\tilde{\Phi}_{31} & -\tilde{\Phi}_{32} & -\tilde{\Phi}_{33} & s\Theta_{34}-\tilde{\Phi}_{34} & s\Theta_{35}-\tilde{\Phi}_{35} & s\Theta_{36}-\Phi_{36} \\ -\tilde{\Phi}_{41} & -\tilde{\Phi}_{42} & -\tilde{\Phi}_{43} & -\tilde{\Phi}_{44} & s\Theta_{45}-\tilde{\Phi}_{45} & s\Theta_{46}-\Phi_{46} \\ 0 & 0 & 0 & s\Theta_{54}-\tilde{\Phi}_{54} & s\Theta_{55}-\tilde{\Phi}_{55} & s\Theta_{56}-\Phi_{56} \\ 0 & 0 & 0 & -\Phi_{64} & s\Sigma_{42}-\Phi_{65} & s\Theta_{66}-\Phi_{66} \\ 0 & 0 & 0 & 0 & 0 & -A_{53} \end{bmatrix}.$$

Then by Lemma 10 it is easy to see that the pencil $(E, A+BF)$ is regular, of index at most one, and $C(sE-(A+BF))^{-1}G = 0$. □

Theorem 14 *Given a system of the form (1). There exist feedback matrices $F, K \in \mathbf{R}^{m\times n}$ such that the pencil $(E+BK, A+BF)$ is regular, of index at most one, and $C(s(E+BK)-(A+BF))^{-1}G = 0$ if and only if the conditions (8), (18) hold and*

$$T_\infty^T(\begin{bmatrix} \Lambda_2 & \Lambda_5 \end{bmatrix})\Lambda_3 S_\infty(T_\infty^T(\Lambda_5)\Lambda_2) \tag{40}$$

has full row rank, where again the Λ_i are deined as in (28).

Proof. We may assume w.l.o.g. that the system in the form (10).

Necessity: Conditions (8) and (18) follow directly from Lemma 4 and Theorem 11, respectively. Thus it suffices to prove that the matrix in (40) has full row rank.

Let $F, K \in \mathbf{R}^{m\times n}$ be such that $(E+BK, A+BF)$ is regular, of index at most one, and $C(s(E+BK)-(A+BF))^{-1}G = 0$. Then, considering that $\begin{bmatrix} E_{21}+B_2K_1 & B_2 \\ B_3K_1 & B_3 \end{bmatrix}$ is of full row rank, from (38) we have that

$$\begin{bmatrix} E_{11}+B_1K_1 & (A_{11}+B_1F_1)\tilde{S} & B_1 \\ E_{21}+B_2K_1 & (A_{21}+B_2F_1)\tilde{S} & B_2 \\ B_3K_1 & (A_{31}+B_3F_1)\tilde{S} & B_3 \end{bmatrix}$$

is of full row rank, where $\tilde{S} := S_\infty(\begin{bmatrix} E_{11}+B_1K_1 \\ E_{21}+B_2K_1 \\ B_3K_1 \end{bmatrix})$. Equivalently, we obtain that

$$\begin{bmatrix} E_{11} & A_{11}\tilde{S}_\infty & B_1 \\ E_{21} & A_{21}\tilde{S}_\infty & B_2 \\ 0 & A_{31}\tilde{S}_\infty & B_3 \end{bmatrix}$$

is of full row rank. Thus, using the relation between $S_\infty(T_\infty^T(\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix})\begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix})$ and $\tilde{S}_\infty$, we have that

$$T_\infty^T(\begin{bmatrix} E_{11} & B_1 \\ E_{21} & B_2 \\ 0 & B_3 \end{bmatrix})\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} S_\infty(T_\infty^T(\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix})\begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix})$$

is of full row rank and condition (40) follows directly from Corollary 12.

Sufficiency: First we construct K_1 such that

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} E_{21}+B_2K_1 \\ B_3K_1\end{bmatrix} &= \operatorname{rank}\begin{bmatrix} E_{21} & B_2 \\ 0 & B_3\end{bmatrix} - \operatorname{rank}\begin{bmatrix} B_2 \\ B_3\end{bmatrix} \\ &= \min_X \operatorname{rank}\begin{bmatrix} E_{21}+B_2X \\ B_3X\end{bmatrix}, \\ \operatorname{rank}\begin{bmatrix} E_{11}+B_1K_1 \\ E_{21}+B_2K_1 \\ B_3K_1\end{bmatrix} &= \operatorname{rank}\begin{bmatrix} E_{11} & B_1 \\ E_{21} & B_2 \\ 0 & B_3\end{bmatrix} - \operatorname{rank}\begin{bmatrix} B_1 \\ B_2 \\ B_3\end{bmatrix} \\ &= \min_Y \operatorname{rank}\begin{bmatrix} E_{11}+B_1Y \\ E_{21}+B_2Y \\ B_3Y\end{bmatrix}.\end{aligned}$$

K_1 can be obtained as follows: Since E_{21}, B_3 are of full row rank, we can compute orthogonal matrices

$$P_1 \in \mathbf{R}^{p\times p}, \qquad P_2 \in \mathbf{R}^{(\tilde{n}_2+\tilde{n}_3)\times(\tilde{n}_2+\tilde{n}_3)}$$

and Z_1, W_1 such that

$$\begin{bmatrix} P_1 & \\ & P_2\end{bmatrix}\begin{bmatrix} E_{11} \\ E_{21} \\ 0\end{bmatrix} Z_1 = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \\ \Theta_{31} & 0 \\ \Theta_{41} & \Theta_{42}\end{bmatrix},$$

$$\begin{bmatrix} P_1 & \\ & P_2\end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3\end{bmatrix} W_1 = \begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \\ 0 & 0 \\ \Psi_{41} & 0\end{bmatrix},$$

where $\Theta_{31}, \Psi_{22}, \Psi_{41}$ are nonsingular. Then we obtain K_1 by solving the linear system

$$\begin{bmatrix} \Psi_{21} & \Psi_{22} \\ \Psi_{41} & 0\end{bmatrix} W_1^T K_1 Z_1 = -\begin{bmatrix} \Theta_{21} & \Theta_{22} \\ \Theta_{41} & \Theta_{42}\end{bmatrix}.$$

From the construction of K_1 and condition (40) it follows that we have

$$\operatorname{rank}\begin{bmatrix} E_{11}+B_1K_1 & A_{11}\tilde{S} & B_1 \\ E_{21}+B_2K_1 & A_{21}\tilde{S} & B_2 \\ B_3K_1 & A_{31}\tilde{S} & B_3\end{bmatrix} = p+\tilde{n}_2+\tilde{n}_3,$$

where $\tilde{S} := S_\infty(\begin{bmatrix} E_{11}+B_1K_1 \\ E_{21}+B_2K_1 \\ B_3K_1\end{bmatrix})$. Using this and condition (18), similarly to the proof of sufficiency in Theorem 13, we can compute a matrix F_1 and orthogonal matrices P_1, Z_1, Z_2 such that

$$\operatorname{rank}\begin{bmatrix} s(E_{21}+B_2K_1)-(A_{21}+B_2F_1) \\ sB_3K_1-(A_{31}+B_3F_1)\end{bmatrix} = n-p-n_2-n_3 \tag{41}$$

$$P_1\begin{bmatrix} s(E_{11}+B_1K_1)-(A_{11}+B_1F_1) \\ s(E_{21}+B_2K_1)-(A_{21}+B_2F_1) \\ sB_3K_1-(A_{31}+B_3F_1)\end{bmatrix} Z_1 = \begin{bmatrix} -\tilde{\Phi}_{11} & -\tilde{\Phi}_{12} & -\tilde{\Phi}_{13} \\ s\tilde{\Theta}_{21}-\tilde{\Phi}_{21} & s\Sigma_{11}-\tilde{\Phi}_{22} & -\tilde{\Phi}_{23} \\ s\Sigma_{21}-\tilde{\Phi}_{31} & -\tilde{\Phi}_{32} & -\tilde{\Phi}_{33} \\ -\tilde{\Phi}_{41} & -\tilde{\Phi}_{42} & -\tilde{\Phi}_{43} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P_1 & \\ & I_{\tilde{n}_4}\end{bmatrix}\begin{bmatrix} E_{12} \\ E_{22} \\ E_{32} \\ E_{42}\end{bmatrix} Z_2 = \begin{bmatrix} \Theta_{14} & \Theta_{15} \\ \Theta_{24} & \Theta_{25} \\ \Theta_{34} & \Theta_{35} \\ 0 & \Theta_{45} \\ \Theta_{54} & \Theta_{55} \\ 0 & \Sigma_{42}\end{bmatrix},$$

$$P_1\begin{bmatrix} B_1 \\ B_2 \\ B_3\end{bmatrix} = \begin{bmatrix} \Psi_{11} \\ \Psi_{21} \\ \Psi_{31} \\ \Psi_{41} \\ \Psi_{51}\end{bmatrix},$$

$$P_1\begin{bmatrix} G_1 \\ 0_{\tilde{n}_2\times p} \\ 0_{\tilde{n}_3\times p}\end{bmatrix} = \begin{bmatrix} G_{11} \\ G_{21} \\ G_{31} \\ 0 \\ 0\end{bmatrix}, \tag{42}$$

where $\Sigma_{11},\Sigma_{21},\Sigma_{42}$ and $\begin{bmatrix} \tilde{\Phi}_{13} \\ \tilde{\Phi}_{43}\end{bmatrix}$ are nonsingular, $\begin{bmatrix} \Psi_{41} \\ \Psi_{51}\end{bmatrix}$ is of full row rank, and Θ_{54} is square. Partition compatibly

$$P_1\begin{bmatrix} E_{13} \\ E_{23} \\ E_{33}\end{bmatrix} =: \begin{bmatrix} \Theta_{16} \\ \Theta_{26} \\ \Theta_{36} \\ \Theta_{46} \\ \Theta_{56}\end{bmatrix}$$

$$\begin{bmatrix} P_1 & \\ & I_{\tilde{n}_4}\end{bmatrix}\begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \\ A_{32} & A_{33} \\ A_{42} & A_{43}\end{bmatrix}\begin{bmatrix} Z_2 & \\ & I_{n_3}\end{bmatrix} =: \begin{bmatrix} \Phi_{14} & \Phi_{15} & \Phi_{16} \\ \Phi_{24} & \Phi_{25} & \Phi_{26} \\ \Phi_{34} & \Phi_{35} & \Phi_{36} \\ \Phi_{44} & \Phi_{45} & \Phi_{46} \\ \Phi_{54} & \Phi_{55} & \Phi_{56} \\ \Phi_{64} & \Phi_{65} & \Phi_{66}\end{bmatrix}.$$

By conditions (40) and (11) we have that

$$\operatorname{rank}\begin{bmatrix} \Theta_{14} & \Theta_{15} & \Theta_{16} & \Psi_{11} \\ 0 & \Theta_{45} & \Theta_{46} & \Psi_{41} \\ \Theta_{54} & \Theta_{55} & \Theta_{56} & \Psi_{51} \\ 0 & \Sigma_{42} & E_{43} & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix} \Theta_{14} & \Theta_{15} & \Psi_{11} \\ 0 & \Theta_{45} & \Psi_{41} \\ \Theta_{54} & \Theta_{55} & \Psi_{51} \\ 0 & \Sigma_{42} & 0\end{bmatrix} \tag{43}$$

Again, let an orthogonal matrix Z_3 be chosen such that

$$\begin{bmatrix} \Theta_{15} & \Theta_{16} \\ \Theta_{25} & \Theta_{26} \\ \Theta_{35} & \Theta_{36} \\ \Theta_{45} & \Theta_{46} \\ \Theta_{55} & \Theta_{56} \\ \Sigma_{42} & E_{43} \end{bmatrix} Z_3 = \begin{bmatrix} \tilde{\Theta}_{15} & \tilde{\Theta}_{16} \\ \tilde{\Theta}_{25} & \tilde{\Theta}_{26} \\ \tilde{\Theta}_{35} & \tilde{\Theta}_{36} \\ \tilde{\Theta}_{45} & \tilde{\Theta}_{46} \\ \tilde{\Theta}_{55} & \tilde{\Theta}_{56} \\ \tilde{\Sigma}_{42} & 0 \end{bmatrix}.$$

Then (43) yields that

$$\text{rank} \begin{bmatrix} \Theta_{14} & \tilde{\Theta}_{16} & \Psi_{11} \\ 0 & \tilde{\Theta}_{46} & \Psi_{41} \\ \Theta_{54} & \tilde{\Theta}_{56} & \Psi_{51} \end{bmatrix} = \text{rank} \begin{bmatrix} \Theta_{14} & \Psi_{11} \\ 0 & \Psi_{41} \\ \Theta_{54} & \Psi_{51} \end{bmatrix}$$

and in addition, we obtain that $\begin{bmatrix} \Psi_{41} \\ \Psi_{51} \end{bmatrix}$ is of full row rank, and since Θ_{54} is square, there exist matrices $\tilde{K}_2, \tilde{K}_3$ such that $\Theta_{54} + \Psi_{51}\tilde{K}_2$ is nonsingular and

$$\text{rank} \begin{bmatrix} \Theta_{14} + \Psi_{11}\tilde{K}_2 & \tilde{\Theta}_{16} + \Psi_{11}\tilde{K}_3 \\ \Psi_{41}\tilde{K}_2 & \tilde{\Theta}_{46} + \Psi_{41}\tilde{K}_3 \\ \Theta_{54} + \Psi_{51}\tilde{K}_2 & \tilde{\Theta}_{56} + \Psi_{51}\tilde{K}_3 \end{bmatrix} = \text{rank}\,(\Theta_{54} + \Psi_{51}\tilde{K}_2) \tag{44}$$

Let $\begin{bmatrix} \hat{K}_2 & K_3 \end{bmatrix} := \begin{bmatrix} 0 & \tilde{K}_3 \end{bmatrix} Z_3^T$ and $K := \begin{bmatrix} K_1 & \tilde{K}_2 & \hat{K}_2 & K_3 \end{bmatrix}$ then (41) and (44) imply that

$$\text{rank}\,(E + BK) = \text{rank}\,(\Sigma_{11}) + \text{rank}\,(\Sigma_{21}) + \text{rank} \begin{bmatrix} \Theta_{54} + \Psi_{51}\tilde{K}_2 & \Theta_{55} + \Psi_{51}\hat{K}_2 \\ 0 & \Sigma_{42} \end{bmatrix} \tag{45}$$

and that $\begin{bmatrix} \Theta_{54} + \Psi_{51}\tilde{K}_2 & \Theta_{55} + \Psi_{51}\hat{K}_2 \\ 0 & \Sigma_{42} \end{bmatrix}$ is nonsingular.

Taking $F := \begin{bmatrix} F_1 & 0 & 0 \end{bmatrix} V^T$, then using (45) and (41), it is easy to verify that $(E + BK, A + BF)$ is regular and of index at most one, because if set accordingly

$$\begin{bmatrix} \Phi_{15} & \Phi_{16} \\ \Phi_{25} & \Phi_{26} \\ \Phi_{35} & \Phi_{36} \\ \Phi_{45} & \Phi_{46} \\ \Phi_{55} & \Phi_{56} \\ A_{42} & A_{43} \\ 0 & A_{53} \end{bmatrix} Z_3 =: \begin{bmatrix} \tilde{\Phi}_{15} & \tilde{\Phi}_{16} \\ \tilde{\Phi}_{25} & \tilde{\Phi}_{26} \\ \tilde{\Phi}_{35} & \tilde{\Phi}_{36} \\ \tilde{\Phi}_{45} & \tilde{\Phi}_{46} \\ \tilde{\Phi}_{55} & \tilde{\Phi}_{56} \\ \tilde{\Phi}_{65} & \tilde{\Phi}_{66} \\ \tilde{\Phi}_{75} & \tilde{\Phi}_{76} \end{bmatrix}$$

then the nonsingularity of A_{53}, Σ_{42} and $\tilde{\Sigma}_{42}$ imply that $\tilde{\Phi}_{76}$ is nonsingular, and we also have

$$\begin{bmatrix} P_1 & \\ & I_{\tilde{n}_4+\tilde{n}_5} \end{bmatrix} (s(E + BK) - (A + BF)) \begin{bmatrix} Z_1 & & \\ & Z_2 & \\ & & I_{n_3} \end{bmatrix} \begin{bmatrix} I_{n_1} & \\ & Z_3 \end{bmatrix}$$

$$
= \begin{bmatrix}
-\tilde{\Phi}_{11} & -\tilde{\Phi}_{12} & -\tilde{\Phi}_{13} & s(\Theta_{14} + \Psi_{11}\tilde{K}_2) - \Phi_{14} & s\tilde{\Theta}_{15} - \tilde{\Phi}_{15} & s(\tilde{\Theta}_{16} + \Psi_{11}\tilde{K}_3) - \tilde{\Phi}_{16} \\
s\tilde{\Theta}_{21} - \tilde{\Phi}_{21} & s\Sigma_{11} - \tilde{\Phi}_{22} & -\tilde{\Phi}_{23} & s(\Theta_{24} + \Psi_{21}\tilde{K}_2) - \Phi_{24} & s\tilde{\Theta}_{25} - \tilde{\Phi}_{25} & s(\tilde{\Theta}_{26} + \Psi_{21}\tilde{K}_3) - \tilde{\Phi}_{26} \\
s\Sigma_{21} - \tilde{\Phi}_{31} & -\tilde{\Phi}_{32} & -\tilde{\Phi}_{33} & s(\Theta_{34} + \Psi_{31}\tilde{K}_2) - \Phi_{34} & s\tilde{\Theta}_{35} - \tilde{\Phi}_{35} & s(\tilde{\Theta}_{36} + \Psi_{31}\tilde{K}_3) - \tilde{\Phi}_{36} \\
-\tilde{\Phi}_{41} & -\tilde{\Phi}_{42} & -\tilde{\Phi}_{43} & s\Psi_{41}\tilde{K}_2 - \Phi_{44} & s\tilde{\Theta}_{45} - \tilde{\Phi}_{45} & s(\tilde{\Theta}_{46} + \Psi_{41}\tilde{K}_3) - \tilde{\Phi}_{46} \\
0 & 0 & 0 & s(\Theta_{54} + \Psi_{51}\tilde{K}_2) - \Phi_{54} & s\tilde{\Theta}_{55} - \tilde{\Phi}_{55} & s(\tilde{\Theta}_{56} + \Psi_{51}\tilde{K}_3) - \tilde{\Phi}_{56} \\
0 & 0 & 0 & -\Phi_{64} & s\tilde{\Sigma}_{42} - \tilde{\Phi}_{65} & -\tilde{\Phi}_{66} \\
0 & 0 & 0 & 0 & -\tilde{\Phi}_{75} & -\tilde{\Phi}_{76}
\end{bmatrix}
$$

Moreover, (42) gives that

$$
\operatorname{rank} \begin{bmatrix} s(E + BK) - (A + BF) & G \\ C & 0 \end{bmatrix} = n.
$$

Hence, we also have that $C(s(E + BK) - (A + BF))^{-1}G = 0$. □

In this section we have given necessary and sufficient conditions for the solution of the disturbance decoupling problem with the extra requirement that the index of the closed loop system is at most one. The results for the case that only derivative feedback is used are given in Appendix B.

7 Conclusions

In this paper we have studied the disturbance decoupling problem for descriptor systems. We have given necessary and sufficient conditions for solving this problem and at the same time ensuring that the resulting closed-loop system is regular and/or has index at most one. The proofs are constructive, based on condensed forms that can be computed via orthogonal matrix transformations.

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Appendix A–Numerical Algorithms

In the following algorithms we need row compressions, column compressions or simultaneous row and column compressions of matrices. Such compressions can be obtained in the usual way via QR-factorizations, rank revealing QR-factorizations, URV-decompositions or singular value decompositions, see [11].

Algorithm 1

Input: Matrices $\hat{E}, \hat{A} \in \mathbf{R}^{t\times l}$, $\hat{B} \in \mathbf{R}^{t\times s}$. with $\hat{B}$ of full column rank.

Output: Orthogonal matrices $U \in \mathbf{R}^{t\times t}, V \in \mathbf{R}^{l\times l}$ and the condensed form (9).

Step 1: Perform a row compression of $\hat{B}$:

$$U\hat{B} =: \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with B_1 of full row rank. Set

$$U\hat{E} =: \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad U\hat{A} =: \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Step 2: Compute the generalized upper triangular form of (E_2, A_2) using the LAPACK routine DGGBAK from LAPACK [2]:

$$QE_2V =: \begin{bmatrix} E_{21} & E_{22} \\ 0 & E_{32} \end{bmatrix}, \quad QA_2V =: \begin{bmatrix} A_{21} & A_{22} \\ 0 & A_{32} \end{bmatrix}$$

with E_{21} of full row rank and $sE_{32} - A_{32}$ of full column rank for any finite $s \in C$. Set

$$E_1V =: \begin{bmatrix} E_{11} & E_{12} \end{bmatrix}, \quad A_1V =: \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}.$$

Step 3: Perform a row compression:

$$Q_1 \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} =: \begin{bmatrix} E_{11} \\ 0 \end{bmatrix}$$

with E_{11} of full row rank. Set

$$\begin{aligned} Q_1 \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} &=: \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix}, \quad Q_1 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =: \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ Q_1 \begin{bmatrix} B_1 \\ 0 \end{bmatrix} &=: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad U := \begin{bmatrix} Q_1 & \\ & I \end{bmatrix} \begin{bmatrix} I & \\ & Q \end{bmatrix} U. \end{aligned}$$

Remark 4 In Step 2, $\begin{bmatrix} E_{11} & B_1 \\ E_{21} & 0 \end{bmatrix}$ is full row rank, hence, in Step 3, B_2 is full row rank.

Algorithm 2

Input: Matrices $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $G \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{q\times n}$.

Output: Orthogonal matrices U, V, P, W and the condensed form (10).

Step 1: Perform a row

$$UG =: \begin{bmatrix} G_1 \\ 0 \end{bmatrix}$$

and and a column compressions:

$$CV =: \begin{bmatrix} 0 & C_3 \end{bmatrix}$$

with G_1 and C_3 nonsingular. Set

$$UEV =: \begin{bmatrix} E_{11} & E_{13} \\ E_{21} & E_{23} \end{bmatrix}, \quad UAV =: \begin{bmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{bmatrix}, \quad UB =: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Step 2: Use Algorithm 1 to determine orthogonal transformations

$$U_2E_{21}V_1 =: \begin{bmatrix} E_{21} & E_{22} \\ 0 & E_{32} \\ 0 & E_{42} \end{bmatrix}, \quad U_2A_{21}V_1 =: \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \\ 0 & A_{42} \end{bmatrix}, \quad U_2B_2 =: \begin{bmatrix} B_2 \\ B_3 \\ 0 \end{bmatrix}$$

such that E_{21}, B_3 are of full column rank, and $sE_{42} - A_{42}$ is of full row rank for any finite $s \in \mathbf{C}$. Set

$$E_{11}V_1 =: \begin{bmatrix} E_{11} & E_{12} \end{bmatrix}, \quad U_2E_{23} =: \begin{bmatrix} E_{23} \\ E_{33} \\ E_{43} \end{bmatrix},$$

$$A_{11}V_1 =: \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}, \quad U_2A_{23} =: \begin{bmatrix} A_{23} \\ A_{33} \\ A_{43} \end{bmatrix}$$

Step 3: Compute the generalized upper triangular form of $\left(\begin{bmatrix} E_{42} & E_{43} \end{bmatrix}, \begin{bmatrix} A_{42} & A_{43} \end{bmatrix} \right)$ using the LAPACK routine DGGBAK from LAPACK [2].

$$Q \begin{bmatrix} E_{42} & E_{43} \end{bmatrix} Z =: \begin{bmatrix} E_{42} & E_{43} \\ & E_{53} \end{bmatrix}, \quad Q \begin{bmatrix} A_{42} & A_{43} \end{bmatrix} Z =: \begin{bmatrix} A_{42} & A_{43} \\ & A_{53} \end{bmatrix}$$

with E_{42} full row rank and $sE_{53} - A_{53}$ full column rank for any finite $s \in \mathbf{C}$. Set

$$\begin{bmatrix} E_{12} & E_{13} \\ E_{22} & E_{23} \\ E_{32} & E_{33} \\ \hline 0 & C_3 \end{bmatrix} Z =: \begin{bmatrix} E_{12} & E_{13} \\ E_{22} & E_{23} \\ E_{32} & E_{33} \\ \hline C_2 & C_3 \end{bmatrix}, \quad \begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} Z =: \begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$$

and

$$U := \begin{bmatrix} I & \\ & Q \end{bmatrix} \begin{bmatrix} I & \\ & U_2 \end{bmatrix}, \quad V := V \begin{bmatrix} V_1 & \\ & I \end{bmatrix} \begin{bmatrix} I & \\ & Z \end{bmatrix}.$$

Appendix B–Results for the case of derivative feedback only.

In this appendix, we list the results related to the disturbance decoupling of descriptor system (1) by derivative feedback only, i.e. for the case $F = 0$. Since these results are essentially dual results to the ones for state feedback by exchanging the roles of E and A, we omit the proofs.

We have the following analogous quantities. Let

$$\tilde{\rho} := r_i(\begin{bmatrix} \hat{A} & 0 \end{bmatrix}, \begin{bmatrix} \hat{E} & \hat{B} \end{bmatrix}), \quad \tilde{\gamma} := c_i(T_\infty^T(\hat{B})\hat{A}, T_\infty^T(\hat{B})\hat{E})$$

and

$$\begin{aligned}
\tilde{\tau} &:= r_i(\begin{bmatrix} T_\infty^T(G)A & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(G)E & T_\infty^T(G)B \\ C & 0 \end{bmatrix}), \\
\tilde{\mu} &:= c_i(\begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})A \\ 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})E \\ C \end{bmatrix}), \\
\tilde{\eta} &:= \mathrm{rank}(T_\infty^T(G)B) - \tilde{\tau} \\
&\quad + r_i(\begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})A \\ 0 \end{bmatrix}, \begin{bmatrix} T_\infty^T(\begin{bmatrix} B & G \end{bmatrix})E \\ C \end{bmatrix})
\end{aligned}$$

Using these quantities we have the following results.

Theorem 15 *Given $\hat{E}, \hat{A} \in \mathbf{R}^{t\times l}$, $\hat{B} \in \mathbf{R}^{t\times s}$. Then there exist a $K \in \mathbf{R}^{s\times l}$ such that*

$$\mathrm{rank}(s(\hat{E} + \hat{B}K) - \hat{A}) = r$$

if and only if

$$\tilde{\rho} + \tilde{\gamma} \leq r \leq \min(l, \mathrm{rank}\begin{bmatrix} s\hat{E} - \hat{A} & \hat{B} \end{bmatrix})$$

Theorem 16 *Given system of the from (1). There exists a feedback $K \in \mathbf{R}^{m\times n}$ such that $(E + BK, A)$ is regular and $C(s(E + BK) - A)^{-1}G = 0$ if and only if conditions (15) and (16) hold and furthermore*

$$\tilde{\tau} + \tilde{\mu} \leq n - p \tag{46}$$

Theorem 17 *Given a system of the form (1). There exists a feedback matrix $K \in \mathbf{R}^{m\times n}$ such that $(E + BK, A)$ is regular, of index at most one, and*

$$C(s(E + BK) - A)^{-1}G = 0$$

if and only if the conditions (8), (40) and (46) hold.