Minimum Norm Regularization of Descriptor Systems by Output Feedback

Delin Chu * and Volker Mehrmann †
Fakultät für Mathematik
TU Chemnitz-Zwickau
D-09107 Chemnitz, FRG

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Abstract

We study the regularization problem for linear, constant coefficient descriptor systems $E\dot{x} = Ax + Bu$, $y_1 = Cx$, $y_2 = \Gamma\dot{x}$ by proportional and derivative mixed output feedback. Necessary and sufficient conditions are given, which guarantee that there exist output feedbacks such that the closed-loop system is regular, has index at most one and $E + BG\Gamma$ has a desired rank, i.e. there is a desired number of differential and algebraic equations. To resolve the freedom in the choice of the feedback matrices we then discuss how to obtain the desired regularizing feedback of minimum norm and show that this approach leads to useful results in the sense of robustness only if the rank of E is decreased. Numerical procedures are derived to construct the desired feedback gains. These numerical procedures are based on orthogonal matrix transformations which can be implemented in a numerically stable way.

Keywords: Descriptor system, output feedback, mixed feedback, regularization, condensed form, index, orthogonal transformations.

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* Supported by Alexander von Humboldt Foundation. email: chu.delin@mathematik.tu-chemnitz.de

† Supported by *Deutsche Forschungsgemeinschaft*, Research Grant Me 790/7-2. email: mehrmann@mathematik.tu-chemnitz.de

1 Introduction

We consider linear, time-invariant descriptor systems of the form

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t)+Bu(t),\ x(t_0)=x^0, \\ y_1(t) & = & Cx(t), \\ y_2(t) & = & \Gamma\dot{x}, \end{array} \tag{1}$$

where $E, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, \Gamma \in R^{q\times n}$. In this paper we always assume without loss of generality that B is full column rank, and that C and Γ have full row rank. If this is not the case then the system can be easily transformed so that this is the case.

Descriptor systems of the form (1) have been studied extensively in the last 20 years, since they allow automatic model generation, see [3, 7, 14, 15] and the references therein.

Existence and uniqueness of (classical) solutions to (1) are guaranteed if (E, A) is regular, i.e., if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in C^2$. The *index* of system (1) denoted by $\mathrm{ind}(E, A)$, is the dimension of the largest block associated with an infinite eigenvalue in the Kronecker canonical form of (E, A) [9].

Systems that are regular and of index at most one can be separated into purely dynamical and algebraic parts (fast and slow modes), and in theory the algebraic part can be eliminated to give a reduced-order standard system. The reduction process, however, may be ill-conditioned for numerical computation and lead to large errors in the reduced order system. If the system is not regular or if $\mathrm{ind}(E, A) > 1$, then impulses can arise in the response of the system if the control is not sufficiently smooth [10, 18]. Since the linear constant coefficient system is usually only a model that approximates a nonlinear model, disturbances in the real application will in general lead to impulsive solutions if the system is of index higher than one. Therefore, an appropriate feedback control should be chosen to ensure that the closed-loop system is regular and of index at most one and if there are several such feedbacks then we need to specify how the non-uniqueness can be resolved.

We therefore study the following problem:

Problem 1 The output feedback regularization problem OFR. *For a system of the form (1) and an integer $r \leq n$, give necessary and sufficient conditions to ensure the existence of feedback matrices $F \in R^{m\times p}, G \in$*

$R^{m\times q}$, *such that the closed-loop system pencil* $(E+BG\Gamma, A+BFC)$ *is regular, has index at most one and* $\mathrm{rank}(E+BG\Gamma)=r$.

This problem has already been discussed in a series of papers, see [2, 3, 6, 15, 16, 17] and the references therein. All the previous characterizations, however, are not complete, since the set of possible ranks for $E+BG\Gamma$ has not been completely characterized. We will give the complete characterization.

For certain choices of ranks, reliable numerical methods for constructing the regularizing feedbacks are presented in [1, 2, 3, 6]. There is, however, usually still freedom in the choice of the feedbacks which can be resolved in many different ways. A natural choice would be to use minimum norm feedbacks, which would be a least squares approach. This approach has not been investigated previously. For this reason we also study the following problem.

Problem 2 The minimum norm output feedback regularization problem MNOFR. *For a system of the form (1) and an integer* $r \le n$, *characterize* $\inf \| \begin{bmatrix} F & G \end{bmatrix} \|$ *for all matrices* F, G *such that* $(E+BG\Gamma, A+BFC)$ *is regular, has index at most one and* $\mathrm{rank}(E+BG\Gamma)=r$. *(Here* $\| \; \|$ *can be any consistent matrix norm.)*

We will show that it is in general not a good idea to use the minumum norm feedback, since it does not lead to a robust regularization, in particular, when the chosen integer r is larger or equal to $\mathrm{rank}(E)$.

The main ingredients for the analysis of the system properties as well as for the construction of the appropriate feedbacks are condensed forms under orthogonal equivalence transformations which can be implemented via numerically stable methods. We will describe such forms in Section 2. In Section 3 we then discuss the complete solution of Problem OFR and we discuss the minimum norm problem MNOFR in Section 4.

We use the following notation:

We denote a full column rank matrix with its columns spanning the right nullspace of the matrix M by $S_\infty(M)$ and with its columns spanning the left nullspace of M by $T_\infty(M)$, respectively. Furthermore we use the following abbreviations:

$$\begin{array}{rcl} r_e & := & \mathrm{rank}(E),\ r_a := \mathrm{rank}(A),\ r_b := \mathrm{rank}(B),\ r_\gamma := \mathrm{rank}(\Gamma), \\ r_{eb} & := & \mathrm{rank}\begin{bmatrix} E & B \end{bmatrix},\ r_{e\gamma} := \mathrm{rank}\begin{bmatrix} E \\ \Gamma \end{bmatrix},\ r_{eb\gamma} := \mathrm{rank}\begin{bmatrix} E & B \\ \Gamma & 0 \end{bmatrix}. \end{array} \tag{2}$$

In all the following discussions the rank of the closed loop matrix $E + BG\Gamma$ will be allowed to vary in order to have more freedom in the choice of feedbacks. The set of all possible ranks for this matrix is easily characterized, e.g. [6], as

$$\begin{aligned} \Pi_{eb\gamma} &:= \{\mathrm{rank}(E+BG\Gamma) | G \in R^{m\times q}\} \\ &= \{r | r \text{ is an integer and } r_{eb} + r_{e\gamma} - r_{eb\gamma} \leq r \leq \min(r_{eb}, r_{e\gamma})\}. \end{aligned} \tag{3}$$

2 Condensed Forms

In this section for systems of the form (1) three useful condensed forms are discussed. The forms we describe here are slight modifications of the condensed forms in [2, 3, 6, 15].

Theorem 1 *Let $E \in R^{n\times n}$, $B \in R^{n\times m}$, $\Gamma \in R^{q\times n}$. Then there exist orthogonal matrices U, V, P and W such that*

$$UEV = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & \tilde{s}_4 & \tilde{s}_5 \end{array} \\ \left[\begin{array}{ccccc} E_{11} & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ E_{31} & E_{32} & E_{33} & E_{34} & 0 \\ E_{41} & E_{42} & 0 & E_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array},$$

$$UBP = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{cc} t_3 & t_4 \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ B_{31} & B_{32} \\ 0 & B_{42} \\ 0 & 0 \end{array}\right] \end{array}, \tag{4}$$

$$W\Gamma V = \begin{array}{c} \\ \tilde{s}_4 \\ t_1 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & \tilde{s}_4 & \tilde{s}_5 \end{array} \\ \left[\begin{array}{ccccc} \Gamma_{11} & \Gamma_{12} & 0 & \Gamma_{14} & 0 \\ \Gamma_{21} & 0 & 0 & 0 & 0 \end{array}\right] \end{array},$$

with nonsingular blocks E_{11}, Γ_{21}, E_{22}, E_{33}, B_{31}, B_{42}, Γ_{14}.

Proof. See [3, 6]. □

From this condensed form we immediately obtain a relationship between the ranks introduced in (2) and the block sizes $t_i, \tilde{s}_i$.

Corollary 2 *Let $E \in R^{n\times n}$, $B \in R^{n\times m}$, $\Gamma \in R^{q\times n}$ be in the condensed form (4). Then*

$$\begin{array}{rcl}
t_1 & = & r_\gamma + r_{eb} - r_{eb\gamma}, \\
t_2 & = & r_{eb\gamma} - r_b - r_\gamma, \\
t_3 & = & r_b + r_{e\gamma} - r_{eb\gamma}, \\
t_4 & = & r_{eb\gamma} - r_{e\gamma}, \\
t_5 & = & n - r_{eb} \\
\tilde{s}_4 & = & r_{eb\gamma} - r_{eb}, \\
\tilde{s}_5 & = & n - r_{e\gamma}
\end{array} \tag{5}$$

and

$$\begin{array}{rcl}
r_{eb\gamma} & = & t_1 + t_2 + t_3 + t_4 + \tilde{s}_4, \\
r_{e\gamma} & = & t_1 + t_2 + t_3 + \tilde{s}_4, \\
r_{eb} & = & t_1 + t_2 + t_3 + t_4, \\
r_b & = & m = t_3 + t_4, \\
r_\gamma & = & q = t_1 + \tilde{s}_4.
\end{array} \tag{6}$$

Using partly nonorthogonal equivalence transformations, the condensed from (4) can be significantly simplified.

Corollary 3 *Let $E \in R^{n\times n}$, $B \in R^{n\times m}$, $\Gamma \in R^{q\times n}$. Then there exist nonsingular matrices X, Y and orthogonal matrices P, W such that*

$$\begin{array}{rcl}
XEY & = & \begin{bmatrix} I_{t_1} & 0 & 0 & 0 & 0 \\ 0 & I_{t_2} & 0 & 0 & 0 \\ 0 & 0 & I_{t_3} & 0 & 0 \\ 0 & 0 & 0 & \tilde{E}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
W\Gamma Y & = & \begin{bmatrix} 0 & 0 & 0 & I_{\tilde{s}_4} & 0 \\ I_{t_1} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad XBP = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{t_3} & 0 \\ 0 & I_{t_4} \\ 0 & 0 \end{bmatrix}.
\end{array} \tag{7}$$

Moreover, the transformations can be chosen such that $\tilde{E}_{44} \in R^{t_4 \times \tilde{s}_4}$ *and*

$$\tilde{E}_{44} = \begin{cases} \begin{bmatrix} \sigma_1 & \ldots & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \ldots & \sigma_{t_4} & 0 \end{bmatrix} & \text{if } \tilde{s}_4 \geq t_4 \\ \begin{bmatrix} \sigma_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & \sigma_{\tilde{s}_4} \\ 0 & 0 & 0 \end{bmatrix} & \text{if } t_4 \geq \tilde{s}_4 \end{cases} \tag{8}$$

with $\sigma_1 \geq \ldots \geq \sigma_t \geq 0$, *where* $t = \min(t_4, \tilde{s}_4)$.

Proof. The proof is a direct consequence of Theorem 1. After performing a restricted singular value decomposition, see [19] of $B_{42}^{-1} E_{44} \Gamma_{14}^{-1}$ one uses the nonsingular blocks to eliminate all other blocks. □

Note that (7) is actually itself a restricted singular value decomposition of the triple (E, B, Γ).

If E, B, Γ are in the condensed form (7), then we immediately have a characterization of the following spaces, which are needed to describe the complete solution of problem OFR.

$$\begin{array}{rcl} S_{e\gamma} & := & S_\infty(\begin{bmatrix} E \\ \Gamma \end{bmatrix}), \quad T_{eb} := T_\infty(\begin{bmatrix} E & B \end{bmatrix}), \\ \tilde{S}_{eb} & := & S_\infty(T_\infty^T(B)E), \quad \tilde{T}_{e\gamma} := T_\infty(ES_\infty(\Gamma)), \\ \tilde{T}_{e\gamma}^0 & := & T_\infty(\begin{bmatrix} E \\ 0_{p\times n} \end{bmatrix} S_\infty(\Gamma)). \end{array} \tag{9}$$

Although these spaces look complicated they can be easily characterized in term of the condensed form (7), see Appendix B. Now we further split the fourth block column of E, Γ and the first block row of Γ.

Lemma 4 *Let* E, A, B, C, Γ *be as in (1). Then there exist nonsingular matrices* X, Y *and orthogonal matrices* P, W *such that*

$$XEY \; = \; \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccccc} t_1 & t_2 & t_3 & s_4 & s_5 & s_6 & s_7 \end{array} \\ \begin{bmatrix} I_{t_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{t_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{t_3} & E_{34} & E_{35} & E_{36} & 0 \\ 0 & 0 & 0 & E_{44} & E_{45} & E_{46} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$
\begin{array}{rcl}
WTY & = & \begin{array}{c} \\ s_4 \\ s_5 \\ s_6 \\ t_1 \end{array}\begin{array}{c} \begin{array}{ccccccc} t_1 & t_2 & t_3 & s_4 & s_5 & s_6 & s_7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 0 & 0 & I_{s_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{s_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{s_6} & 0 \\ I_{t_1} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}, \\
XBP & = & \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array}\begin{array}{c} \begin{array}{cc} t_3 & t_4 \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ I_{t_3} & 0 \\ 0 & I_{t_4} \\ 0 & 0 \end{array}\right] \end{array}, \\
XAY & = & \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array}\begin{array}{c} \begin{array}{ccccccc} t_1 & t_2 & t_3 & s_4 & s_5 & s_6 & s_7 \end{array} \\ \left[\begin{array}{ccccccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} \end{array}\right] \end{array}, \\
CY & = & \begin{array}{c} \\ p \end{array}\begin{array}{c} \begin{array}{ccccccc} t_1 & t_2 & t_3 & s_4 & s_5 & s_6 & s_7 \end{array} \\ \left[\begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{array}\right] \end{array},
\end{array} \tag{10}
$$

where

$$
\begin{array}{rcl}
s_6 & := & \min(r_{eb\gamma} - r_{eb}, \quad \operatorname{rank}(T_{eb}^T A \tilde{S}_{eb}) - \operatorname{rank}(T_{eb}^T A S_{e\gamma})), \\
s_5 & := & \min(r_{eb\gamma} - r_{eb}, \quad \operatorname{rank}((\tilde{T}_{e\gamma}^0)^T \begin{bmatrix} A \\ C \end{bmatrix} \tilde{S}_{eb}) \\
& & \qquad - \operatorname{rank}((\tilde{T}_{e\gamma}^0)^T \begin{bmatrix} A \\ C \end{bmatrix} S_{e\gamma})) - s_6, \\
s_4 & := & r_{eb\gamma} - r_{eb} - s_5 - s_6,
\end{array} \tag{11}
$$

and furthermore

$$
\begin{array}{rcl}
\operatorname{rank}\begin{bmatrix} A_{56} & A_{57} \end{bmatrix} & = & s_6 + \operatorname{rank}(A_{57}) = s_6 + \operatorname{rank}(T_{eb}^T A S_{e\gamma}), \\
\operatorname{rank}\begin{bmatrix} A_{15} & A_{16} & A_{17} \\ A_{45} & A_{46} & A_{47} \\ A_{55} & A_{56} & A_{57} \\ C_5 & C_6 & C_7 \end{bmatrix} & = & s_5 + s_6 + \operatorname{rank}\begin{bmatrix} A_{17} \\ A_{47} \\ A_{57} \\ C_7 \end{bmatrix} \\
& = & s_5 + s_6 + \operatorname{rank}((\tilde{T}_{e\gamma}^0)^T \begin{bmatrix} A \\ C \end{bmatrix} S_{e\gamma}).
\end{array} \tag{12}
$$

Proof. By Corollary 3 there exist matrices X, Y, P, W such that XEY, XBP, $W\Gamma Y$ are in the condensed form (7). Partition

$$XAY = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{A}_{15} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{A}_{25} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} & \tilde{A}_{35} \\ \tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44} & \tilde{A}_{45} \\ \tilde{A}_{51} & \tilde{A}_{52} & \tilde{A}_{53} & \tilde{A}_{54} & \tilde{A}_{55} \end{bmatrix}, \quad CY = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 & \tilde{C}_5 \end{bmatrix} \tag{13}$$

accordingly. Consider the submatrix $\begin{bmatrix} \tilde{A}_{53} & \tilde{A}_{54} & \tilde{A}_{55} \end{bmatrix}$ of XAY. It is easy to see that there exists a matrix K_0 of appropriate dimensions such that

$$\begin{aligned} \operatorname{rank}(\begin{bmatrix} \tilde{A}_{54} + \tilde{A}_{53}K_0 & \tilde{A}_{55} \end{bmatrix}) &= \max_K(\operatorname{rank} \begin{bmatrix} \tilde{A}_{54} + \tilde{A}_{53}K & \tilde{A}_{55} \end{bmatrix}) \\ &= s_6 + \operatorname{rank}(\tilde{A}_{55}) \\ &= s_6 + \operatorname{rank}(T_{eb}^T A S_{e\gamma}) \end{aligned}$$

and

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} \tilde{A}_{14} + \tilde{A}_{13}K_0 & \tilde{A}_{15} \\ \tilde{A}_{44} + \tilde{A}_{43}K_0 & \tilde{A}_{45} \\ \tilde{A}_{54} + \tilde{A}_{53}K_0 & \tilde{A}_{55} \\ \tilde{C}_4 + \tilde{C}_3K_0 & \tilde{C}_5 \end{bmatrix} &= \max_K(\operatorname{rank} \begin{bmatrix} \tilde{A}_{14} + \tilde{A}_{13}K & \tilde{A}_{15} \\ \tilde{A}_{44} + \tilde{A}_{43}K & \tilde{A}_{45} \\ \tilde{A}_{54} + \tilde{A}_{53}K & \tilde{A}_{55} \\ \tilde{C}_4 + \tilde{C}_3K & \tilde{C}_5 \end{bmatrix}) \\ = s_5 + s_6 + \begin{bmatrix} \tilde{A}_{15} \\ \tilde{A}_{45} \\ \tilde{A}_{55} \\ \tilde{C}_5 \end{bmatrix} &= s_5 + s_6 + \operatorname{rank}((\tilde{T}_{e\gamma}^0)^T \begin{bmatrix} A \\ C \end{bmatrix} S_{e\gamma}). \end{aligned}$$

Finally there exists an orthogonal matrix V such that

$$\begin{bmatrix} A_{14} & A_{15} & A_{16} \\ A_{24} & A_{25} & A_{26} \\ A_{34} & A_{35} & A_{36} \\ A_{44} & A_{45} & A_{46} \\ A_{54} & A_{55} & A_{56} \\ C_4 & C_5 & C_6 \end{bmatrix} := \begin{bmatrix} \tilde{A}_{14} + \tilde{A}_{13}K_0 \\ \tilde{A}_{24} + \tilde{A}_{23}K_0 \\ \tilde{A}_{34} + \tilde{A}_{33}K_0 \\ \tilde{A}_{44} + \tilde{A}_{43}K_0 \\ \tilde{A}_{54} + \tilde{A}_{53}K_0 \\ \tilde{C}_4 + \tilde{C}_3K_0 \end{bmatrix} V$$

is partitioned so that the following conditions are satisfied.

$$\operatorname{rank} \begin{bmatrix} A_{56} & \tilde{A}_{55} \end{bmatrix} = s_6 + \operatorname{rank}(\tilde{A}_{55}),$$

$$\operatorname{rank}\begin{bmatrix} A_{15} & A_{16} & \tilde{A}_{15} \\ A_{45} & A_{46} & \tilde{A}_{45} \\ A_{55} & A_{56} & \tilde{A}_{55} \\ C_5 & C_6 & \tilde{C}_5 \end{bmatrix} = s_5 + s_6 + \operatorname{rank}\begin{bmatrix} \tilde{A}_{15} \\ \tilde{A}_{45} \\ \tilde{A}_{55} \\ \tilde{C}_5 \end{bmatrix}.$$

Leaving of the ˜ in the first three block columns and setting

$$\begin{bmatrix} A_{17} \\ A_{27} \\ A_{37} \\ A_{47} \\ A_{57} \\ C_7 \end{bmatrix} := \begin{bmatrix} \tilde{A}_{15} \\ \tilde{A}_{25} \\ \tilde{A}_{35} \\ \tilde{A}_{45} \\ \tilde{A}_{55} \\ \tilde{C}_5 \end{bmatrix}, \quad W := \begin{bmatrix} V^T & \\ & I_{t_1} \end{bmatrix} W,$$

$$Y := Y \begin{bmatrix} I_{t_1+t_2} & & & \\ & I_{t_3} & K_0 & \\ & & I_{\tilde{s}_4} & \\ & & & I_{\tilde{s}_5} \end{bmatrix} \begin{bmatrix} I_{t_1+t_2+t_3} & & \\ & V & \\ & & I_{\tilde{s}_5} \end{bmatrix},$$

we have obtained the desired condensed from. □

Corollary 5 *Given a system (E, A, B, C, Γ) in condensed form (10). Then for any K_1, K_2, K_3 of appropriate dimensions, the following two inequalities hold:*

$$\begin{aligned} \operatorname{rank}(\begin{bmatrix} A_{54} & A_{55} & A_{56} & A_{57} \end{bmatrix} &+ A_{53}\begin{bmatrix} K_1 & K_2 & K_3 & 0 \end{bmatrix}) \\ &\leq \operatorname{rank}\begin{bmatrix} A_{56} & A_{57} \end{bmatrix}, \end{aligned} \tag{14}$$

$$\begin{aligned} \operatorname{rank}(\begin{bmatrix} A_{14} & A_{15} & A_{16} & A_{17} \\ A_{44} & A_{45} & A_{46} & A_{47} \\ A_{54} & A_{55} & A_{56} & A_{57} \\ C_4 & C_5 & C_6 & C_7 \end{bmatrix} + \begin{bmatrix} A_{13} \\ A_{43} \\ A_{53} \\ C_3 \end{bmatrix}\begin{bmatrix} K_1 & K_2 & K_3 & 0 \end{bmatrix}) \\ \leq \operatorname{rank}\begin{bmatrix} A_{15} & A_{16} & A_{17} \\ A_{45} & A_{46} & A_{47} \\ A_{55} & A_{56} & A_{57} \\ C_5 & C_6 & C_7 \end{bmatrix}. \end{aligned} \tag{15}$$

Furthermore we have the following equivalences

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} A_{56} & A_{57} \end{bmatrix} &= n - r_{eb} \ \text{ iff} \\ \operatorname{rank}(T_{eb}^T A \tilde{S}_{eb}) &= n - r_{eb} \ \text{ and} \\ n - r_{eb\gamma} &\leq \operatorname{rank}(T_{eb}^T A S_{e\gamma}). \end{aligned} \tag{16}$$

$$\operatorname{rank}\begin{bmatrix} A_{17} \\ A_{47} \\ A_{57} \\ C_7 \end{bmatrix} = n - r_{e\gamma} \quad \text{iff} \quad \operatorname{rank}((\tilde{T}^0_{e\gamma})^T \begin{bmatrix} A \\ C \end{bmatrix} S_{e\gamma}) = n - r_{e\gamma} \tag{17}$$

Proof. The proof is an immediate consequence of Lemma 4. □

In this section we have introduced three condensed forms. Condensed form (4) uses only orthogonal tranformations and thus can be implemented as a numerically stable procedure. We will use this form to design a numerical algorithm to compute regularizing feedbacks for the solution of Problem OFR.

Condensed forms (7) and (10) follow directly from (4), but since they are obtained via non-orthogonal transformations usually there do not exist backward stable procedures to compute these forms. However, condensed forms (7), (10) are important, since they provide the basis for the solvability theory for problem OFR. The form (10) is not unique, but (7) is essentially unique (except for permutations). This form will be used to establish algebraic characterizations for Problem MNOFR.

3 Output Feedback Regularization

In this section we give necessary and sufficient conditions for problem OFR.

Theorem 6 *Given a system of the form (1) with $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $\Gamma \in R^{q\times n}$. Let r be any integer satisfying*

$$r_{eb} + r_{e\gamma} - r_{eb\gamma} + s_4 \le r \le r_{eb} + r_{e\gamma} - r_{eb\gamma} + s_4 + s_5 \tag{18}$$

with s_4, s_5 given by (11). There exist feedback matrices $F \in R^{m\times p}$ and $G \in R^{m\times q}$, such that $(E + BG\Gamma, A + BFC)$ is regular, has index at most one and $\operatorname{rank}(E + BG\Gamma) = r$ *if and only if the following three conditions hold*

$$\begin{aligned} \operatorname{rank}(T_{eb}^T A \tilde{S}_{eb}) &= n - r_{eb}, & (19)\\ \operatorname{rank}((\tilde{T}^0_{e\gamma})^T \begin{bmatrix} A \\ C \end{bmatrix} S_{e\gamma}) &= n - r_{e\gamma}, & (20)\\ \operatorname{rank}(T_{eb}^T A S_{e\gamma}) &\ge n - r_{eb\gamma}. & (21) \end{aligned}$$

Proof. Without loss of generality, we may assume that the system matrices (E, A, B, C, Γ) are in the condensed form (10). We partition any $F \in R^{m\times p}$, and $G \in R^{m\times q}$ as

$$F = \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \end{bmatrix}$$

compatible with the partitioning in B, C, and Γ.

Necessity. Let F, G be such that $(E + BG\Gamma, A + BFC)$ is regular, has index at most one and $\operatorname{rank}(E+BG\Gamma) = r$. Then a simple calculation yields that

$$\operatorname{rank}\begin{bmatrix} E_{44}+G_{21} & E_{45}+G_{22} & E_{46}+G_{23} \end{bmatrix} = r - (t_1+t_2+t_3).$$

Hence, there exists an orthogonal matrix $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$, such that

$$\begin{bmatrix} E_{44}+G_{21} & E_{45}+G_{22} & E_{46}+G_{23} \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} \hat{E}_{44} & 0 \end{bmatrix},$$

where $\hat{E}_{44}$ has full column rank $r - (t_1+t_2+t_3)$ and Q is partititioned compatibly. Let $K_1 := E_{34} + G_{11}$, $K_2 := E_{35} + G_{12}$ and $K_3 := E_{36} + G_{13}$ and set

$$\begin{bmatrix} A_{14}-A_{13}K_1 & A_{15}-A_{13}K_2 & A_{16}-A_{13}K_3 \\ A_{24}-A_{23}K_1 & A_{25}-A_{23}K_2 & A_{26}-A_{23}K_3 \\ A_{34}-A_{33}K_1 & A_{35}-A_{33}K_2 & A_{36}-A_{33}K_3 \\ A_{44}-A_{43}K_1 & A_{45}-A_{43}K_2 & A_{46}-A_{43}K_3 \\ A_{54}-A_{53}K_1 & A_{55}-A_{53}K_2 & A_{56}-A_{53}K_3 \\ C_4-C_3K_1 & C_5-C_3K_2 & C_6-C_3K_3 \end{bmatrix} Q_2 = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Psi \end{bmatrix}.$$

Then, we have that

$$\begin{aligned} n &= \operatorname{rank}\begin{bmatrix} E+BG\Gamma & (A+BFC)S_\infty(E+BG\Gamma) \end{bmatrix} \\ &= \operatorname{rank} \begin{array}{c} \begin{array}{cccccc} t_1 & t_2 & t_3 & r-t_1-t_2-t_3 & n-r-s_7 & s_7 \end{array} \\ \begin{bmatrix} I_{t_1} & 0 & 0 & 0 & \Phi_1 & A_{17} \\ 0 & I_{t_2} & 0 & 0 & \Phi_2 & A_{27} \\ G_{14} & 0 & I_{t_3} & 0 & \Phi_3+F_{11}\Psi & A_{37}+F_{11}C_7 \\ G_{24} & 0 & 0 & \hat{E}_{44} & \Phi_4+F_{21}\Psi & A_{47}+F_{21}C_7 \\ 0 & 0 & 0 & 0 & \Phi_5 & A_{57} \end{bmatrix} \end{array}. \end{aligned}$$

This implies that the submatrix

$$\begin{bmatrix} I_{t_1} & 0 & \Phi_1 & A_{17} \\ G_{24} & \hat{E}_{44} & \Phi_4 + F_{21}\Psi & A_{47} + F_{21}C_7 \\ 0 & 0 & \Phi_5 & A_{57} \end{bmatrix}$$

is nonsingular. Therefore, Lemma 4 implies that

$$\operatorname{rank} \begin{bmatrix} \Phi_5 & A_{57} \end{bmatrix} = n - r_{eb} \tag{22}$$

and

$$\operatorname{rank} \begin{bmatrix} \Phi_1 & A_{17} \\ \Phi_4 & A_{47} \\ \Phi_5 & A_{57} \\ \Psi & C_7 \end{bmatrix} = n - r. \tag{23}$$

Then (14) implies that

$$n - r_{eb} = \operatorname{rank} \begin{bmatrix} \Phi_5 & A_{57} \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} A_{56} & A_{57} \end{bmatrix} \leq n - r_{eb}. \tag{24}$$

This together with (16) yields conditions (19) and (21). Condition (23) implies that

$$\operatorname{rank} \begin{bmatrix} A_{17} \\ A_{47} \\ A_{57} \\ C_7 \end{bmatrix} = n - r_{e\gamma}. \tag{25}$$

This together with (17) yields (20).

Moreover, conditions (15) and (23) together yield

$$n - r = \operatorname{rank} \begin{bmatrix} \Phi_1 & A_{17} \\ \Phi_4 & A_{47} \\ \Phi_5 & A_{57} \\ \Psi & C_7 \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} A_{15} & A_{16} & A_{17} \\ A_{45} & A_{46} & A_{47} \\ A_{55} & A_{56} & A_{57} \\ C_5 & C_6 & C_7 \end{bmatrix}. \tag{26}$$

Then (24) implies that the number of columns of Φ_5 is at least $n - r_{eb} - \operatorname{rank}(A_{57}) = s_6$. With $r_{e\gamma} = t_1 + t_2 + t_3 + s_4 + s_5 + s_6$ therefore, (25) and (26) and Lemma 4 yield that

$$t_1 + t_2 + t_3 + s_4 \leq r \leq t_1 + t_2 + t_3 + s_4 + s_5$$

which is (18).

Sufficiency. Let $\tilde{s} := r - t_1 - t_2 - t_3 - s_4$ and partition the fifth block column of A and C as

$$\begin{bmatrix} A_{15} \\ A_{25} \\ A_{35} \\ A_{45} \\ A_{55} \\ C_5 \end{bmatrix} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ p \end{array} \begin{array}{c} \begin{array}{cc} \tilde{s} & s_5 - \tilde{s} \end{array} \\ \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \\ \Phi_{31} & \Phi_{32} \\ \Phi_{41} & \Phi_{42} \\ \Phi_{51} & \Phi_{52} \\ \Psi_1 & \Psi_2 \end{bmatrix} \end{array}$$

Then, (16) implies that $\begin{bmatrix} \Phi_{52} & A_{56} & A_{57} \end{bmatrix}$ has full row rank $n - r_{eb}$, and (17) and (12) imply that

$$\begin{bmatrix} \Phi_{12} & A_{16} & A_{17} \\ \Phi_{42} & A_{46} & A_{47} \\ \Phi_{52} & A_{56} & A_{57} \\ \Psi_2 & C_6 & C_7 \end{bmatrix}$$

has full column rank $n - r$. But then there exist $\Delta_1, \tilde{\Delta}_2$, Δ_4 and F_{21} such that the matrix

$$\begin{array}{c} \begin{array}{cccccc} t_1 & s_4 & \tilde{s} & s_5 - \tilde{s} & s_6 & s_7 \end{array} \\ \begin{bmatrix} I_{t_1} & 0 & 0 & \Phi_{12} & A_{16} & A_{17} \\ \Delta_4 & \Delta_1 & \tilde{\Delta}_2 & \Phi_{42} + F_{21}\Psi_2 & A_{46} + F_{21}C_6 & A_{47} + F_{21}C_7 \\ 0 & 0 & 0 & \Phi_{52} & A_{56} & A_{57} \end{bmatrix} \end{array}$$

is nonsingular. Set

$$\begin{aligned} G &:= \begin{array}{c} \begin{array}{cccc} s_4 & s_5 & s_6 & t_1 \end{array} \\ \begin{bmatrix} -E_{34} & -E_{35} & -E_{36} & 0 \\ -E_{44} & -E_{45} & -E_{46} & 0 \end{bmatrix} \end{array} + \begin{array}{c} \begin{array}{ccccc} s_4 & \tilde{s} & s_5 - \tilde{s} & s_6 & t_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \Delta_1 & \tilde{\Delta}_2 & 0 & 0 & \Delta_4 \end{bmatrix} \end{array}, \\ F &:= \begin{bmatrix} 0 \\ F_{21} \end{bmatrix}. \end{aligned}$$

Then it follows that $(E + BG\Gamma, A + BFC)$ is regular, has index at most one, and $\text{rank}(E + BG\Gamma) = r$. $\square$

As direct corollaries of Theorem 6 we obtain the characterizations in the special cases $C = \Gamma$, $C = 0$, respectively and even the state feedback case $\Gamma = 0, C = I$.

Corollary 7 *[6] Given a system of the form (1) with $\Gamma = C$ and let r be an integer satisfying*

$$r_{eb} + r_{e\gamma} - r_{eb\gamma} \leq r \leq r_{e\gamma} - s_6 \tag{27}$$

with s_6 as in (11). There exist feedback matrices F, G such that $(E + BG\Gamma, A + BF\Gamma)$ is regular, has index at most one, and $\operatorname{rank}(E + BG\Gamma) = r$ *if and only if the three conditions (19), (21) and*

$$\operatorname{rank}(\tilde{T}_{e\gamma}^T A S_{e\gamma}) = n - r_{e\gamma} \tag{28}$$

hold.

Proof. Since $\Gamma = C$, by the condensed form (10) it is easy to verify that

$$\operatorname{rank}(\tilde{T}_{e\gamma}^T A S_{e\gamma}) = \operatorname{rank}((\tilde{T}_{e\gamma}^0)^T \begin{bmatrix} A \\ \Gamma \end{bmatrix} S_{e\gamma}),\ s_4 = 0,\ s_5 = r_{eb\gamma} - r_{eb} - s_6.$$

Hence, Corollary 7 follows directly from Theorem 6. $\square$

Corollary 8 *Given a system of the form (1) with $C = 0$ and let r be an integer satisfying*

$$\begin{aligned} r_{e\gamma} \quad - \quad & \min(r_{eb\gamma} - r_{eb}, \operatorname{rank}(\tilde{T}_{e\gamma}^T A \tilde{S}_{eb}) - \operatorname{rank}(\tilde{T}_{e\gamma}^T A S_{e\gamma})) \\ \leq r \quad & \leq r_{e\gamma} - s_6 \end{aligned} \tag{29}$$

with s_6 as in (11). There exists a feedback G such that $(E + BG\Gamma, A)$ is regular, has index at most one and $\operatorname{rank}(E + BG\Gamma) = r$ *if and only if the three conditions (19), (21) and (28) hold.*

Proof. Since $C = 0$, we have from the condensed form (10) that

$$\operatorname{rank}(\tilde{T}_{e\gamma}^T A S_{e\gamma}) = \operatorname{rank}((\tilde{T}_{e\gamma}^0)^T \begin{bmatrix} A \\ 0 \end{bmatrix} S_{e\gamma}),$$

$$s_4 = r_{eb\gamma} - r_{eb} - \min(r_{eb\gamma} - r_{eb}, \operatorname{rank}(\tilde{T}_{e\gamma}^T A \tilde{S}_{eb}) - \operatorname{rank}(\tilde{T}_{e\gamma}^T A S_{e\gamma}))$$

and

$$r_{eb} + r_{e\gamma} - r_{eb\gamma} + s_4 + s_5 = r_{e\gamma} - s_6$$

Thus, Corollary 8 follows directly from Theorem 6. $\square$

Corollary 9 *[2] Given a system of the form (1) with $\Gamma = 0, C = I$. There exists a feedback F such that $(E, A + BFC)$ is regular and has index at most one if and only if (21) holds.*

Proof. Since $\Gamma = 0, C = I$, we have from the condensed form (4) that $t_1 = t_4 = \tilde{s}_4 = 0$. Then (21) holds if and only if rank $\left[\begin{array}{ccc} E & AS_\infty(E) & B \end{array} \right] = n$, which is equivalent to the existence of the state feedback F such that $(E, A + BFC)$ is regular and has index at most one, see [2]. □

A numerical procedure to compute feedbacks with the properties described in Theorem 6, Corollaries 7 and 8 is described in Algorithm 2 in Appendix A.

Note that the possible ranks rank$(E + BG\Gamma)$ depend on the relationship between C and Γ as the previous characterizations demonstrate. But as will show in the next section it does not make much sense to increase or keep the rank of E, if we require a robust regularization.

4 Minimum Norm Feedback

In this section, algebraic characterizations for minimum norm feedback matrices F, G which solve problem OFR are given. We introduce the following notation. Let $\Omega(E, A, B, C, \Gamma, r) := \{(F, G) \,|\, F \in R^{m\times p}, G \in R^{m\times q}, \text{rank}(E + BG\Gamma) = r, (E + BG\Gamma, A + BFC) \text{ is regular and of index at most one }\}$. Furthermore define the distances

$$\begin{array}{rcl} \delta(E, A, B, C, \Gamma, r) & := & \inf\{\| \left[\begin{array}{cc} F & G \end{array} \right] \| \,|(F, G) \in \Omega(E, A, B, C, \Gamma, r)\}, \\ \delta_1(E, A, B, \Gamma, r) & := & \delta(E, A, B, \Gamma, \Gamma, r), \\ \delta_2(E, A, B, \Gamma, r) & := & \delta(E, A, B, \Gamma, 0, r). \end{array}$$

In order to give the characterization, we need the following three Lemmas.

Lemma 10 *Let $E, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, \Gamma \in R^{m\times q}$ and let $r \leq n$ be an integer satisfying (18). Assume that (E, B, Γ) are given in the condensed form (7) and that A, C are partitioned accordingly as in (13). Denote by U_r the set of matrices $Q \in R^{(r_{eb\gamma} - r_{eb})\times(r_{e\gamma} - r)}$ satisfying $Q^TQ = I_{r_{e\gamma} - r}$,*

$$\text{rank} \left[\begin{array}{ccc} \tilde{A}_{53} & \tilde{A}_{54}Q & \tilde{A}_{55} \end{array} \right] = n - r_{eb} \qquad (30)$$

and

$$\operatorname{rank}\begin{bmatrix} \tilde{A}_{13} & \tilde{A}_{14}Q & \tilde{A}_{15} \\ \tilde{A}_{43} & \tilde{A}_{44}Q & \tilde{A}_{45} \\ \tilde{A}_{53} & \tilde{A}_{54}Q & \tilde{A}_{55} \\ \tilde{C}_3 & \tilde{C}_4Q & \tilde{C}_5 \end{bmatrix} \geq n - r. \tag{31}$$

Then

$$\delta(E, A, B, C, \Gamma, r) = \inf\{\|\tilde{E}_{44}Q\| \; Q \in U_r\}. \tag{32}$$

Proof. Note first that $r_{eb\gamma} - r_{eb} \geq r_{e\gamma} - r$. Let $\hat{s}_4 := r - t_1 - t_2 - t_3 = r_{eb\gamma} + r - r_{eb} - r_{e\gamma}$. Then for any $Q_2 \in U_r$ let $Q_1 \in R^{(r_{eb\gamma}-r_{eb})\times \hat{s}_4}$ be such that $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ is orthogonal.

For any $\epsilon > 0$, there exists $\tilde{G}_{11}$ (depending on ϵ), such that $\|\tilde{G}_{11}\| < \epsilon$,

$$\begin{bmatrix} \tilde{A}_{54}Q_2 - \tilde{A}_{53}\tilde{G}_{11} & \tilde{A}_{55} \end{bmatrix} \text{has full row rank} \quad n - r_{eb} \tag{33}$$

and

$$\begin{bmatrix} \tilde{A}_{14}Q_2 - \tilde{A}_{13}\tilde{G}_{11} & \tilde{A}_{15} \\ \tilde{A}_{44}Q_2 - \tilde{A}_{43}\tilde{G}_{11} & \tilde{A}_{45} \\ \tilde{A}_{54}Q_2 - \tilde{A}_{53}\tilde{G}_{11} & \tilde{A}_{55} \\ \tilde{C}_4Q_2 - \tilde{C}_3\tilde{G}_{11} & \tilde{C}_5 \end{bmatrix} \text{ has full column rank.} \tag{34}$$

Set

$$Z_Q := \begin{bmatrix} I_{t_1} & 0 & \tilde{A}_{14}Q_2 - \tilde{A}_{13}\tilde{G}_{11} & \tilde{A}_{15} \\ 0 & \tilde{E}_{44}Q_1 & \tilde{A}_{44}Q_2 - \tilde{A}_{43}\tilde{G}_{11} & \tilde{A}_{45} \\ 0 & 0 & \tilde{A}_{54}Q_2 - \tilde{A}_{53}\tilde{G}_{11} & \tilde{A}_{55} \end{bmatrix}.$$

Then there exist matrices $\tilde{G}_{21}$, $\tilde{F}_{21}$, $\tilde{G}_{22}$ (all depending on ϵ), such that $\|\begin{bmatrix} \tilde{G}_{21} & \tilde{F}_{21} & \tilde{G}_{22} \end{bmatrix}\| < \epsilon$ and

$$\operatorname{rank}(Z_Q + \begin{bmatrix} 0 \\ I_{t_4} \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{G}_{21} & \tilde{F}_{21} & \tilde{G}_{22} \end{bmatrix} \begin{bmatrix} 0 & I_{\hat{s}_4} & 0 & 0 \\ 0 & 0 & \tilde{C}_4Q_2 - \tilde{C}_3\tilde{G}_{11} & \tilde{C}_5 \\ I_{t_1} & 0 & 0 & 0 \end{bmatrix})$$

$$= n - t_2 - t_3. \tag{35}$$

Let $G_{11} := \begin{bmatrix} 0 & \tilde{G}_{11} \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$, $G_{21} := \begin{bmatrix} \tilde{G}_{21} & -\tilde{E}_{44} \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$, and $G_{22} = \tilde{G}_{22}$. Then with $G_Q := \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix}$, and $F_Q := \begin{bmatrix} 0 \\ F_{21} \end{bmatrix}$, we obtain that $(E + BG_Q\Gamma, A + BF_QC)$ is regular, has index at most one and $\mathrm{rank}(E + BG_Q\Gamma) = r$. This implies that

$$\delta(E, A, B, C, \Gamma, r) \le \lim_{\epsilon\to 0} \| \begin{bmatrix} G_Q & F_Q \end{bmatrix} \| = \|\tilde{E}_{44}Q_2\|, \qquad (36)$$

and hence

$$\delta(E, A, B, C, \Gamma, r) \le \inf\{\|\tilde{E}_{44}Q\| \quad |Q \in U_r\}.$$

To prove (32) we show that the opposite inequality in (36) also holds. For any $\epsilon > 0$, there exist matrices F, G (depending on ϵ), such that $(E + BG\Gamma, A + BFC)$ is regular, has index at most one, $\mathrm{rank}(E + BG\Gamma) = r$, and

$$\| \begin{bmatrix} F & G \end{bmatrix} \| < \delta(E, A, B, C, \Gamma, r) + \epsilon.$$

Then there exists an orthogonal matrix $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ with $Q_2 \in R^{(r_{eb\gamma}-r_{eb})\times(r_{e\gamma}-r)}$, such that

$$G_{21}Q_2 = -\tilde{E}_{44}Q_2$$

and (35) holds with $\tilde{G}_{21} = G_{21}Q_2$ and $\tilde{G}_{11} = G_{11}Q_2$. Hence (33) and (34) hold and thus $Q_2 \in U_r$. This implies that

$$\begin{aligned} \inf\{\|\tilde{E}_{44}Q\| \quad Q \in U_r\} &\le \lim_{\epsilon\to 0} \|\tilde{E}_{44}Q_2\| \\ &\le \lim_{\epsilon\to 0} \| \begin{bmatrix} F & G \end{bmatrix} \| = \delta(E, A, B, C, \Gamma, r). \end{aligned}$$

This finishes the proof. $\square$

Lemma 11 *Given a real matrix Φ of the form*

$$\Phi := \begin{array}{c} \\ \tau_1 \\ \tau_2 \end{array} \begin{array}{c} \begin{array}{cc} \mu_1 & \mu_2 \end{array} \\ \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Psi_1 & \Psi_2 \end{bmatrix} \end{array},$$

where $\mathrm{rank}(\begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}) = \tau_1$. *Let*

$$k_1 := \tau_1 - \mathrm{rank}(\Phi_2), \quad k_2 := \mathrm{rank}(\begin{bmatrix} \Phi_2 \\ \Psi_2 \end{bmatrix}), \quad k_3 := \mathrm{rank}(\Phi) - k_1 - k_2.$$

Assume that l, t are given positive integers such that

$$l \le \mu_1, \quad k_1 + k_2 \le t \le \mathrm{rank}(\Phi).$$

Then there exist orthogonal matrices $U_1 \in R^{\tau_1\times\tau_1}$, $U_2 \in R^{(\tau_1+\tau_2-k_1)\times(\tau_1+\tau_2-k_1)}$, $V_1 \in R^{\mu_1\times\mu_1}$ *and* $V_2 \in R^{(\mu_1-k_1)\times(\mu_1-k_1)}$, *such that*

$$\begin{bmatrix} I_{k_1} & \\ & U_2 \end{bmatrix}\begin{bmatrix} U_1 & \\ & I_{\tau_2} \end{bmatrix}\begin{bmatrix} \Phi_1 & \Phi_2 \\ \Psi_1 & \Psi_2 \end{bmatrix}\begin{bmatrix} V_1 & \\ & I_{\mu_2} \end{bmatrix}\begin{bmatrix} I_{k_1} & & \\ & V_2 & \\ & & I_{\mu_2} \end{bmatrix}$$

$$= \begin{array}{c} \\ k_1 \\ k_2 \\ \tau_1+\tau_2-k_1-k_2 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_3 & \mu_1-k_1-k_3 & \mu_2 \end{array} \\ \begin{bmatrix} \Sigma_1 & 0 & 0 & 0 \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & 0 & 0 \end{bmatrix} \end{array}, \tag{37}$$

where Σ_1 *is nonsingular,* Ψ_{24} *has full row rank and* Ψ_{32} *has full column rank.*

Furthermore, for any orthogonal matrix $Q \in R^{\mu_1\times l}$, *partition*

$$\begin{bmatrix} I_{k_1} & \\ & V_2^T \end{bmatrix} V_1^T Q := \begin{array}{c} k_1 \\ k_3 \\ \mu_1-k_1-k_3 \end{array} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}.$$

Then

$$\operatorname{rank}\begin{bmatrix} \Phi_1 Q & \Phi_2 \end{bmatrix} = \tau_1, \quad \operatorname{rank}\begin{bmatrix} \Phi_1 Q & \Phi_2 \\ \Psi_1 Q & \Psi_2 \end{bmatrix} \geq t$$

if and only if

$$\operatorname{rank}(Q_1) = k_1, \quad \operatorname{rank}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \geq t - k_2.$$

Proof. The proof for the first part is given constructively via Algorithm 3 in Appendix A. The second part follows trivially from the first. □

The next lemma is needed to determine the infimum in (32).

Lemma 12 *Given a real matrix* M *of the form*

$$M := \begin{array}{c} \\ v \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix} \end{array}.$$

Let l, t *be given positive integers such that*

$$l \leq \min(v, z_1+z_2+z_3), \quad z_1 \leq t \leq \min(z_1+z_2, l),$$

and let

$$U_{t,l} := \{Q = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array}\!\!\begin{array}{c} l \\ \left[\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \end{array}\right] \end{array} \| \quad Q^TQ = I_l \quad \mathrm{rank}(Q_1) = z_1, \quad \mathrm{rank}\left[\begin{array}{c} Q_1 \\ Q_2 \end{array}\right] \geq t\}.$$

Then

$$\inf\{\|MQ\| \quad | \; Q \in U_{t,l}\} = \sigma_{z_1+z_2+z_3-l+1}(M).$$

Proof.

It is well known [11] that

$$\inf\{\|MQ\| \quad |Q \in R^{(z_1+z_2+z_3)\times l}, Q^TQ = I_l\} = \sigma_{z_1+z_2+z_3-l+1}(M)$$

and thus $\inf\{\|MQ\| \quad |Q \in U_{t,l}\} \geq \sigma_{z_1+z_2+z_3-l+1}(M)$. Hence it suffices to show that

$$\inf\{\|MQ\| \quad | \; Q \in U_{t,l}\} \leq \sigma_{z_1+z_2+z_3-l+1}(M). \tag{38}$$

To see this, let

$$\tilde{Q} = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \end{array}\!\!\begin{array}{c} l \\ \left[\begin{array}{c} \tilde{Q}_1 \\ \tilde{Q}_2 \\ \tilde{Q}_3 \end{array}\right] \end{array}, \quad \tilde{Q}^T\tilde{Q} = I_l$$

be such that $\|M\tilde{Q}\| = \sigma_{z_1+z_2+z_3-t+1}(M)$.

If $\mathrm{rank}(\tilde{Q}_1) = z_1$ and $\mathrm{rank}\left[\begin{array}{c} \tilde{Q}_1 \\ \tilde{Q}_2 \end{array}\right] \geq t$, then (38) follows. Otherwise, if $\mathrm{rank}(\tilde{Q}_1) < z_1$ or $\left[\begin{array}{c} \tilde{Q}_1 \\ \tilde{Q}_2 \end{array}\right] < t$, then we can assume w.l.o.g. that

$$\tilde{Q}_1 = \tilde{P}_1 \left[\begin{array}{ccc} \Pi_1 & 0 & 0 \\ 0 & 0_{z_1-\tau_1} & 0 \end{array}\right],$$

and

$$\tilde{Q}_2 = \left[\begin{array}{ccc} Q_{21} & Q_{22} & Q_{23} \end{array}\right], \quad \tilde{Q}_3 = \left[\begin{array}{ccc} Q_{31} & Q_{32} & Q_{33} \end{array}\right],$$

where $\tilde{P}_1$ is orthogonal and $\Pi_1 \in R^{\tau_1\times\tau_1}$ is nonsingular.

For any ϵ with $0 < \epsilon < 1$, set

$$\begin{aligned}
\hat{Q}_1(\epsilon) &:= \tilde{P}_1 \begin{bmatrix} \Pi_1 & 0 & \\ 0 & \epsilon I_{z_1-\tau_1} & 0 \end{bmatrix}, \\
\hat{Q}_2(\epsilon) &:= \begin{bmatrix} Q_{21} & (1-\epsilon)Q_{22} & Q_{23} \end{bmatrix}, \\
\hat{Q}_3(\epsilon) &:= \begin{bmatrix} Q_{31} & (1-\epsilon)Q_{32} & Q_{33} \end{bmatrix}.
\end{aligned}$$

Then $\text{rank}(\hat{Q}_1(\epsilon)) = z_1$ and there exist orthogonal matrices $\hat{P}_2(\epsilon), \hat{V}_2(\epsilon)$ such that

$$\begin{aligned}
\begin{bmatrix} \hat{Q}_1(\epsilon) \\ \hat{Q}_2(\epsilon) \end{bmatrix} &= \hat{P}_2(\epsilon) \begin{bmatrix} \Pi_2(\epsilon) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{V}_2(\epsilon), \\
\hat{Q}_3(\epsilon) &= \begin{bmatrix} \hat{Q}_{31}(\epsilon) & \hat{Q}_{32}(\epsilon) & \hat{Q}_{33}(\epsilon) \end{bmatrix} \hat{V}_2(\epsilon),
\end{aligned}$$

where $\Pi_2(\epsilon) \in R^{\tau_2 \times \tau_2}$ is nonsingular. Let δ_ϵ satisfy that $0 < \delta_\epsilon < \epsilon$ and $\text{rank}(Q_1(\epsilon)) = z_1$, where $Q(\epsilon)$, $Q_1(\epsilon)$, $Q_2(\epsilon)$ and $Q_3(\epsilon)$ are defined as follows.

$$Q(\epsilon) := \begin{bmatrix} Q_1(\epsilon) \\ Q_2(\epsilon) \\ Q_3(\epsilon) \end{bmatrix}$$

with

$$\begin{aligned}
\begin{array}{cc} & l \\ \begin{array}{c} z_1 \\ z_2 \end{array} & \begin{bmatrix} Q_1(\epsilon) \\ Q_2(\epsilon) \end{bmatrix} \end{array} &:= \hat{P}_2(\epsilon) \begin{bmatrix} \Pi_2(\epsilon) & 0 & 0 \\ 0 & \delta_\epsilon I_{z_1+z_2-\tau_2} & 0 \end{bmatrix} \hat{V}_2(\epsilon) \\
Q_3(\epsilon) &:= \begin{bmatrix} \hat{Q}_{31}(\epsilon) & (1-\delta_\epsilon)\hat{Q}_{32}(\epsilon) & \hat{Q}_{33}(\epsilon) \end{bmatrix} \hat{V}_2(\epsilon).
\end{aligned}$$

Since $\text{rank}(Q_1(\epsilon)) = z_1$, $\text{rank} \begin{bmatrix} Q_1(\epsilon) \\ Q_2(\epsilon) \end{bmatrix} \geq t$, $Q^T(\epsilon)Q(\epsilon) = I_l$, and

$$\lim_{\epsilon \to 0} \|Q(\epsilon) - \tilde{Q}\| = 0$$

it follows that

$$\inf\{\|MQ\| \quad | \; Q \in \tilde{U}_{t,l} \; \} \leq \lim_{\epsilon \to 0} \|MQ(\epsilon)\| = \|M\tilde{Q}\| = \sigma_{z_1+z_2+z_3-l+1}(M),$$

which implies (38). $\square$

Since for any nonsingular matrices X, Y and orthogonal matrices P, W as in (7), X, Y do not affect $\| \begin{bmatrix} F & G \end{bmatrix} \|$, where F, G are such that $(E + BG\Gamma, A + BFC)$ is regular, has index at most one and rank $(E + BG\Gamma) = r$, we can characterize the minimum norm regularization via the condensed form (7), in particular, via the restricted singular values of (E, B, Γ), i.e., the singular values of $\tilde{E}_{44}$ in (7).

Theorem 13 *Let $E, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, \Gamma \in R^{m\times q}$ and let $\tilde{E}_{44}$ be as in (7), (8).*

1. *If conditions (19)–(21) hold and r satisfies (18), then*

$$\delta(E, A, B, C, \Gamma, r) = \sigma_{r-t_1-t_2-t_3+1}.$$

2. *If $\Gamma = C$, conditions (19), (21) and (28) hold and r satisfies (27), then*

$$\delta_1(E, A, B, \Gamma, r) = \sigma_{r-t_1-t_2-t_3+1}.$$

3. *If $\Gamma = C$, conditions (19), (21) and (28) hold and r satisfies (29), then*

$$\delta_2(E, A, B, \Gamma, r) = \sigma_{r-t_1-t_2-t_3+1}.$$

In particular, we have in all three cases that if $r \geq r_e$ then

$$\begin{aligned} \delta(E, A, B, C, \Gamma, r) &= 0, \\ \delta_1(E, A, B, \Gamma, r) &= 0, \\ \delta_2(E, A, B, \Gamma, r) &= 0. \end{aligned}$$

Proof. The result follows directly by applying Lemma 10, Lemma 11 and Lemma 12. $\square$

Note that this result also covers the state feedback case $C = I$, $\Gamma = 0$.

Theorem 13 shows that it does not make sense to consider the minimum norm regularizing feedback if $r \geq r_e$, since we will be arbitrarily close to a system which is not regular and of index at most one. This certainly is not a robust regularization. Thus although minimum norm feedbacks are very important in other control problems, like pole placement or stabilization, since they remove ambiguity in the solution in a least squares sense, for the problem of regularization they do not lead to a useful solution.

If, however, we decrease the rank of E, i.e., we choose $r < r_e$, then the situation may be more satisfactory. In this case the minimal norm of the feedback is given by the appropriate restricted singular value of (E, B, Γ) and if this value is large enough, then there is some hope that the regularization will be robust, i.e., we are reasonably far away from a pencil that is non-regular or has an index larger than one. It is still an open problem how to obtain the exact distance of a closed loop system from the nearest non-regular pencil, see [4, 5]. This is the reason, why it is currently not clear how to choose the most robust regularization. This topic is currently under invertigation. Heuristic procedures to obtain a system that is robustly regular and of index at most one are discussed in [8, 12, 13].

5 Conclusions

We have given necessary and sufficient conditions for Problem OFR. The results characterize the relation among mixed feedback, combined derivative and proportional feedback and derivative feedback for the regularization of descriptor systems. Numerical procedures to compute the desired feedback gains are discussed in Appendix A.

Furthermore we have given algebraic characterizations of the minimum norm regularizing feedback and shown that this is not a good approach if the rank of E is increased.

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Appendix A–Numerical Algorithms

In this appendix, we describe a numerical procedure to construct feedback gains F, G such that $(E+BG\Gamma, A+BFC)$ is regular, has index at most one and $\operatorname{rank}(E+BG\Gamma) = r$. The procedure is based on the condensed form (4). For algorithms to compute the condensed form (4), see [3, 6].

In the following algorithms we need row compressions, column compressions or simultaneous row and column compressions of matrices. Such compressions can be obtained in the usual way via QR-factorizations, rank revealing QR-factorizations, URV-decompositions or singular value decompositions, see [11].

Algorithm 1
Input: Matrices

$$\begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} l_1 & l_2 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array}, \quad \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} m_1 \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{array},$$

satisfying

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} A_{21} & A_{22} & B_2 \end{bmatrix} &= n_2, \\ \operatorname{rank}(A_{21}) + l_2 &\geq n_2, \\ \operatorname{rank}\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} &= l_1 \end{aligned}$$

and an integer t satisfying

$$n_2 - \operatorname{rank}(A_{21}) \leq t \leq \min(\operatorname{rank}\begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix} - l_1, l_2).$$

Output: A matrix $K \in R^{m_1 \times l_2}$ and an orthogonal matrix $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \in R^{l_2 \times l_2}$ with $Q_2 \in R^{l_2 \times t}$, such that

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} A_{21} & (A_{22}+B_2K)Q_2 \end{bmatrix} &= n_2, \\ \operatorname{rank}\begin{bmatrix} A_{11} & (A_{12}+B_1K)Q_2 \\ A_{21} & (A_{22}+B_2K)Q_2 \end{bmatrix} &= l_1 + t. \end{aligned}$$

Remark 1 The main idea of this algorithm is to find K_0 such that

$$\operatorname{rank}\begin{bmatrix} A_{21} & A_{22}+B_2K_0 \end{bmatrix} = n_2$$

and

$$\operatorname{rank}\begin{bmatrix} A_{11} & A_{12}+B_1K_0 \\ A_{21} & A_{22}+B_2K_0 \end{bmatrix} = max_K \operatorname{rank}\begin{bmatrix} A_{11} & A_{12}+B_1K \\ A_{21} & A_{22}+B_2K \end{bmatrix}.$$

Step 1: Perform a simultaneous row and column compression of A_{21}:

$$\begin{bmatrix} I_{n_1} & \\ & U_1^T \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} V_1 & \\ & I_{l_2} \end{bmatrix} =: \begin{array}{c} \\ n_1 \\ \tau_1 \\ n_2-\tau_1 \end{array}\begin{array}{c} \begin{array}{ccc} \tau_1 & l_1-\tau_1 & l_2 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \tilde{\Sigma}_{31} & 0 & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \end{array}$$

with $\tilde{\Sigma}_{31}$ nonsingular. Partition

$$\begin{bmatrix} I_{n_1} & \\ & U_1^T \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} =: \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

analogously.
Step 2: Perform a row compression of A_{12} (note that by assumption it has full column rank):

$$\begin{bmatrix} U_2^T & \\ & I_{n_2} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \tilde{\Sigma}_{31} & 0 & A_{32} \\ 0 & 0 & A_{33} \end{bmatrix} =: \begin{array}{c} \\ l_1-\tau_1 \\ n_1+\tau_1-l_1 \\ \tau_1 \\ n_2-\tau_1 \end{array}\begin{array}{c} \begin{array}{ccc} \tau_1 & l_1-\tau_1 & l_2 \end{array} \\ \begin{bmatrix} A_{11} & \Sigma_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ \tilde{\Sigma}_{31} & 0 & A_{33} \\ 0 & 0 & A_{43} \end{bmatrix} \end{array}$$

with Σ_{12} nonsingular. Partition

$$\begin{bmatrix} U_2^T & \\ & I_{n_2} \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} =: \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

analogously.

Step 3: Perform a simultaneous row and column compression of A_{43}:

$$\begin{bmatrix} I_{n_1} & & \\ & I_{\tau_1} & \\ & & U_3^T \end{bmatrix} \begin{bmatrix} A_{11} & \Sigma_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ \tilde{\Sigma}_{31} & 0 & A_{33} \\ 0 & 0 & A_{43} \end{bmatrix} \begin{bmatrix} I_{l_1} & \\ & V_3 \end{bmatrix}$$

$$=: \begin{array}{l} \\ l_1 - \tau_1 \\ n_1 + \tau_1 - l_1 \\ \tau_1 \\ \tau_2 \\ n_2 - \tau_1 - \tau_2 \end{array} \begin{array}{c} \begin{array}{cccc} \tau_1 & l_1 - \tau_1 & \tau_2 & l_2 - \tau_2 \end{array} \\ \begin{bmatrix} A_{11} & \Sigma_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ \tilde{\Sigma}_{31} & 0 & A_{33} & A_{34} \\ 0 & 0 & \tilde{\Sigma}_{43} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

with $\tilde{\Sigma}_{43}$ nonsingular. Partition

$$\begin{bmatrix} I_{n_1} & & \\ & I_{\tau_1} & \\ & & U_3^T \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} =: \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}$$

analogously.

Step 4: Perform a column compression of B_5 (note that by assumption B_5 has full row rank):

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} W_4 =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \\ \Sigma_{51} & 0 \end{bmatrix}.$$

with Σ_{51} nonsingular.

Step 5: Perform a row compression of $\begin{bmatrix} A_{21} & A_{23} \\ \tilde{\Sigma}_{31} & A_{33} \\ 0 & \tilde{\Sigma}_{43} \end{bmatrix}$:

$$\begin{bmatrix} I_{l_1 - \tau_1} & & \\ & U_5^T & \\ & & I_{n_2 - \tau_1 - \tau_2} \end{bmatrix} \begin{bmatrix} A_{11} & \Sigma_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ \tilde{\Sigma}_{31} & 0 & A_{33} & A_{34} \\ 0 & 0 & \tilde{\Sigma}_{43} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
=: \begin{array}{l} \\ l_1-\tau_1 \\ n_1+\tau_1-l_1 \\ \tau_1 \\ \tau_2 \\ n_2-\tau_1-\tau_2 \end{array}
\begin{array}{c} \begin{array}{cccc} \tau_1 & l_1-\tau_1 & \tau_2 & l_2-\tau_2 \end{array} \\
\left[\begin{array}{cccc} A_{11} & \Sigma_{12} & A_{13} & A_{14} \\ 0 & 0 & 0 & A_{24} \\ \Sigma_{31} & 0 & A_{33} & A_{34} \\ 0 & 0 & \Sigma_{43} & A_{44} \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}
$$

with Σ_{31} and Σ_{43} nonsingular. Partition

$$
\begin{bmatrix} I_{l_1-\tau_1} & & \\ & U_5^T & \\ & & I_{n_2-\tau_1-\tau_2} \end{bmatrix}
\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \\ \Sigma_{51} & 0 \end{bmatrix}
=: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \\ \Sigma_{51} & 0 \end{bmatrix}
$$

analogously.

Step 6: Perform a simultaneous row and column compression of A_{24}:

$$
\begin{bmatrix} I_{l_1-\tau_1} & & \\ & U_6^T & \\ & & I_{n_2} \end{bmatrix}
\begin{bmatrix} A_{11} & \Sigma_{12} & A_{13} & A_{14} \\ 0 & 0 & 0 & A_{24} \\ \Sigma_{31} & 0 & A_{33} & A_{34} \\ 0 & 0 & \Sigma_{43} & A_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} I_{l_1} & & \\ & I_{\tau_2} & \\ & & V_6 \end{bmatrix}
$$

$$
=: \begin{array}{l} \\ l_1-\tau_1 \\ \tau_3 \\ n_1+\tau_1-l_1-\tau_3 \\ \tau_1 \\ \tau_2 \\ n_2-\tau_1-\tau_2 \end{array}
\begin{array}{c} \begin{array}{cccccc} \tau_1 & l_1-\tau_1 & \tau_2 & n_2-\tau_1-\tau_2 & \tau_3 & l_2+\tau_1-n_2-\tau_3 \end{array} \\
\left[\begin{array}{cccccc} A_{11} & \Sigma_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ 0 & 0 & 0 & 0 & \Sigma_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \Sigma_{31} & 0 & A_{43} & A_{44} & A_{45} & A_{46} \\ 0 & 0 & \Sigma_{43} & A_{54} & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}
$$

with $\Sigma_{25} \in R^{\tau_3 \times \tau_3}$ nonsingular. Partition

$$
\begin{bmatrix} I_{l_1-\tau_1} & & \\ & U_6^T & \\ & & I_{n_2} \end{bmatrix}
\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \\ \Sigma_{51} & 0 \end{bmatrix}
=: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \\ B_{51} & B_{52} \\ \Sigma_{51} & 0 \end{bmatrix}
$$

analogously.

Step 7: Perform a column compression of B_{32}:

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \\ B_{51} & B_{52} \\ \Sigma_{51} & 0 \end{bmatrix} \begin{bmatrix} I_{n_2-\tau_1-\tau_2} & \\ & W_7 \end{bmatrix}$$

$$=: \begin{array}{c} \begin{array}{ccc} n_2-\tau_1-\tau_2 & \mu_b & m_1+\tau_1+\tau_2-n_2-\mu_b \end{array} \\ \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & 0 \\ B_{41} & B_{42} & B_{43} \\ B_{51} & B_{52} & B_{53} \\ \Sigma_{51} & 0 & 0 \end{bmatrix} \end{array}$$

with B_{32} of full column rank.
Step 8: Set

$$K := W_4 \begin{bmatrix} I_{n_2-\tau_1-\tau_2} & \\ & W_7 \end{bmatrix} \tilde{K} \begin{bmatrix} I_{\tau_2} & \\ & V_6^T \end{bmatrix} V_3^T$$

with

$$\tilde{K} = \begin{array}{c} \\ n_2-\tau_1-\tau_2 \\ \mu_b \\ m_1+\tau_1+\tau_2-n_2-\mu_b \end{array} \begin{array}{c} \begin{array}{cccc} \tau_2 & n_2-\tau_1-\tau_2 & \tau_3 & l_2+\tau_1-\tau_3-n_2 \end{array} \\ \begin{bmatrix} 0 & I_{n_2-\tau_1-\tau_2} & 0 & 0 \\ 0 & 0 & 0 & \tilde{K}_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

where

$$\tilde{K}_{24} = \begin{cases} \begin{bmatrix} I_{l_2+\tau_1-\tau_3-n_2} \\ 0 \end{bmatrix} & \text{if } l_2+\tau_1-\tau_3-n_2 \le \mu_b, \\ \begin{bmatrix} I_{\mu_b} & 0 \end{bmatrix} & \text{if } l_2+\tau_1-\tau_3-n_2 > \mu_b. \end{cases}$$

Set furthermore $Q := \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ with

$$Q_1 := V_3 \begin{bmatrix} I_{\tau_2} & \\ & V_6 \end{bmatrix} \begin{bmatrix} 0 \\ I_{l_2-t} \end{bmatrix}, \quad Q_2 := V_3 \begin{bmatrix} I_{\tau_2} & \\ & V_6 \end{bmatrix} \begin{bmatrix} I_t \\ 0 \end{bmatrix}.$$

Algorithm 2
Input: Matrices $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $\Gamma \in R^{q\times n}$ and an integer r, satisfying conditions (19)–(21) and (18).
Output: Feedback gains $F \in R^{m\times p}, G \in R^{m\times q}$ such that $(E + BG\Gamma, A + BFC)$ is regular, has index at most one, and $\operatorname{rank}(E + BG\Gamma) = r$.
Step 1: Compute orthogonal matrices U, V, P, W such that $(UEV, UBP, W\Gamma V)$ is in the condensed form (4) and partition UAV, CV accordingly as in (13). Set $\hat{s}_4 := r - t_1 - t_2 - t_3$.
Step 2: Use Algorithm 1 to compute a matrix K and an orthogonal matrix $\left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right] \in R^{\tilde{s}_4\times\tilde{s}_4}$ with $Q_2 \in R^{\tilde{s}_4\times(r_{e\gamma}-r)}$ such that the transformed submatrices

$$
\left[\begin{array}{c} E_{34} - E_{33}K \\ E_{44} \end{array}\right]\left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right] =: \begin{array}{cc} & \begin{array}{cc} \hat{s}_4 & r_{e\gamma}-r \end{array} \\ \begin{array}{c} t_3 \\ t_4 \end{array} & \left[\begin{array}{cc} \Theta_{34} & \Theta_{35} \\ \Theta_{44} & \Theta_{45} \end{array}\right] \end{array},
$$

$$
\left[\begin{array}{c} A_{14} - A_{13}K \\ A_{24} - A_{23}K \\ A_{44} - A_{43}K \\ A_{54} - A_{53}K \end{array}\right]\left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right] =: \begin{array}{cc} & \begin{array}{cc} \hat{s}_4 & r_{e\gamma}-r \end{array} \\ \begin{array}{c} t_1 \\ t_2 \\ t_4 \\ t_5 \end{array} & \left[\begin{array}{cc} \Phi_{14} & \Phi_{15} \\ \Phi_{24} & \Phi_{25} \\ \Phi_{44} & \Phi_{45} \\ \Phi_{54} & \Phi_{55} \end{array}\right] \end{array},
$$

$$
\Gamma_{14}\left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right] =: \begin{array}{cc} & \begin{array}{cc} \hat{s}_4 & r_{e\gamma}-r \end{array} \\ q - t_1 & \left[\begin{array}{cc} \Pi_{14} & \Pi_{15} \end{array}\right] \end{array},
$$

$$
(C_4 - C_3K)\left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right] =: \begin{array}{cc} & \begin{array}{cc} \hat{s}_4 & r_{e\gamma}-r \end{array} \\ p & \left[\begin{array}{cc} \Psi_4 & \Psi_5 \end{array}\right] \end{array}
$$

satisfy

$$
\operatorname{rank}\left[\begin{array}{cc} \Phi_{55} & A_{55} \end{array}\right] = n - r_{eb}
$$

and

$$
\operatorname{rank}\left[\begin{array}{cc|cc} 0 & 0 & \Phi_{15} & A_{15} \\ E_{22} & 0 & \Phi_{25} & A_{25} \\ E_{42} & E_{44} & \Phi_{45} & A_{45} \\ 0 & 0 & \Phi_{55} & A_{55} \\ \hline \Gamma_{12} & \Gamma_{14} & 0 & 0 \\ \hline 0 & 0 & \Psi_5 & C_5 \end{array}\right] = t_2 + \tilde{s}_4 + (n - r).
$$

Remark 2 If we set

$$Y := \begin{bmatrix} I_{t_1} & 0 & 0 & 0 & 0 \\ 0 & I_{t_2} & 0 & 0 & 0 \\ 0 & 0 & I_{t_3} & 0 & 0 \\ -\Gamma_{14}^{-1}\Gamma_{11} & -\Gamma_{14}^{-1}\Gamma_{12} & 0 & I_{\tilde{s}_4} & 0 \\ 0 & 0 & 0 & 0 & I_{\tilde{s}_5} \end{bmatrix} \begin{bmatrix} I_{t_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{t_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{t_3} & KQ_1 & KQ_2 & 0 \\ 0 & 0 & 0 & Q_1 & Q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\tilde{s}_5} \end{bmatrix}$$

and if for given $F \in R^{m\times p}$, $G \in R^{m\times q}$ we set

$$\begin{array}{c} t_3 \\ t_4 \end{array} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} := \begin{bmatrix} B_{31} & B_{32} \\ 0 & B_{42} \end{bmatrix} F$$

$$\begin{array}{cc} & \begin{array}{ccc} t_1 & \hat{s}_4 & r_{e\gamma}-r \end{array} \\ \begin{array}{c} t_3 \\ t_4 \end{array} & \begin{bmatrix} G_{31} & G_{34} & G_{35} \\ G_{41} & G_{44} & G_{45} \end{bmatrix} \end{array} := \begin{bmatrix} B_{31} & B_{32} \\ 0 & B_{42} \end{bmatrix} G \begin{bmatrix} 0 & \Pi_{14} & \Pi_{15} \\ \Gamma_{21} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Theta_{34} & \Theta_{35} \\ 0 & \Theta_{44} & \Theta_{45} \end{bmatrix},$$

then we have that

$$U(E+BG\Gamma)Y = \begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 & 0 \\ E_{31}+G_{31}-E_{34}\Gamma_{14}^{-1}\Gamma_{11} & E_{32}-E_{34}\Gamma_{14}^{-1}\Gamma_{12} & E_{33} & G_{34} & G_{35} & 0 \\ E_{41}+G_{41}-E_{44}\Gamma_{14}^{-1}\Gamma_{11} & E_{42}-E_{44}\Gamma_{14}^{-1}\Gamma_{12} & 0 & G_{44} & G_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U(A+BFC)Y = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & A_{13} & \Phi_{14} & \Phi_{15} & A_{15} \\ \hat{A}_{21} & \hat{A}_{22} & A_{23} & \Phi_{24} & \Phi_{25} & A_{25} \\ \hat{A}_{31} & \hat{A}_{32} & A_{33} & \Phi_{34}+F_1\Psi_4 & \Phi_{34}+F_1\Psi_5 & A_{35}+F_1C_5 \\ \hat{A}_{41} & \hat{A}_{42} & A_{43} & \Phi_{44}+F_2\Psi_4 & \Phi_{45}+F_2\Psi_5 & A_{45}+F_2C_5 \\ \hat{A}_{51} & \hat{A}_{52} & A_{53} & \Phi_{54} & \Phi_{55} & A_{55} \end{bmatrix}$$

with

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \\ \hat{A}_{31} & \hat{A}_{32} \\ \hat{A}_{41} & \hat{A}_{42} \\ \hat{A}_{51} & \hat{A}_{52} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \\ A_{41} & A_{42} \\ A_{51} & A_{52} \end{bmatrix} - \begin{bmatrix} A_{14} \\ A_{24} \\ A_{34} \\ A_{44} \\ A_{54} \end{bmatrix} \Gamma_{14}^{-1} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \end{bmatrix}.$$

It is easy to see that if

$$G_{35} = 0, \quad G_{45} = 0$$

and if the rank of

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 & \Phi_{15} & A_{15} \\ E_{21} & E_{22} & 0 & 0 & \Phi_{25} & A_{25} \\ E_{31} + G_{31} - E_{34}\Gamma_{14}^{-1}\Gamma_{11} & E_{32} - E_{34}\Gamma_{14}^{-1}\Gamma_{12} & E_{33} & G_{34} & \Phi_{35} + F_1\Psi_5 & A_{35} + F_1C_5 \\ E_{41} + G_{41} - E_{44}\Gamma_{14}^{-1}\Gamma_{11} & E_{42} - E_{44}\Gamma_{14}^{-1}\Gamma_{12} & 0 & G_{44} & \Phi_{45} + F_2\Psi_5 & A_{45} + F_2C_5 \\ 0 & 0 & 0 & 0 & \Phi_{55} & A_{55} \end{bmatrix} \tag{39}$$

is n, then we have that $(E + BG\Gamma, A + BFC)$ is regular, of index at most one and $\mathrm{rank}(E + BG\Gamma) = r$. Note that (39) holds if and only if

$$\mathrm{rank} \begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 & \Phi_{15} & A_{15} \\ E_{21} & E_{22} & 0 & 0 & 0 & \Phi_{25} & A_{25} \\ E_{31} + G_{31} & E_{32} & E_{33} & E_{34} & G_{34} & \Phi_{35} + F_1\Psi_5 & A_{35} + F_1C_5 \\ E_{41} + G_{41} & E_{42} & 0 & E_{44} & G_{44} & \Phi_{45} + F_2\Psi_5 & A_{45} + F_2C_5 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{55} & A_{55} \\ \Gamma_{11} & \Gamma_{12} & 0 & \Gamma_{14} & 0 & 0 & 0 \end{bmatrix} = n + \tilde{s}_4$$

or equivalently that the rank of

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 & \Phi_{15} & A_{15} \\ E_{21} & E_{22} & 0 & 0 & \Phi_{25} & A_{25} \\ E_{41} + G_{41} & E_{42} & E_{44} & G_{44} & \Phi_{45} + F_2\Psi_5 & A_{45} + F_2C_5 \\ 0 & 0 & 0 & 0 & \Phi_{55} & A_{55} \\ \Gamma_{11} & \Gamma_{12} & \Gamma_{14} & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

$$= \begin{bmatrix} I_{t_1} & & & & & & & \\ & I_{t_2} & & & & & & \\ & & I_{t_4} & 0 & 0 & G_{41} & G_{44} & F_2 \\ & & & I_{t_5} & & & & \\ & & & & I_{q-t_1} & & & \end{bmatrix} \begin{bmatrix} E_{11} & 0 & 0 & 0 & \Phi_{15} & A_{15} \\ E_{21} & E_{22} & 0 & 0 & \Phi_{25} & A_{25} \\ E_{41} & E_{42} & E_{44} & 0 & \Phi_{45} & A_{45} \\ 0 & 0 & 0 & 0 & \Phi_{55} & A_{55} \\ \Gamma_{11} & \Gamma_{12} & \Gamma_{14} & 0 & 0 & 0 \\ I_{t_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Psi_5 & C_5 \end{bmatrix}$$

is $n + \tilde{s}_4 - t_3$. For the construction of F, G, it therefore suffices to study the submatrix product (40). The following construction is therefore only based on this submatrix.

Step 3. Perform a column compression $\begin{bmatrix} \Phi_{55} & A_{55} \end{bmatrix} V_3 = \begin{bmatrix} 0 & \Sigma_4 \end{bmatrix}$ and set

$$
\begin{bmatrix}
E_{11} & 0 & 0 & 0 & \Phi_{15} & A_{15} \\
E_{21} & E_{22} & 0 & 0 & \Phi_{25} & A_{25} \\
E_{41} & E_{42} & E_{44} & 0 & \Phi_{45} & A_{45} \\
0 & 0 & 0 & 0 & \Phi_{55} & A_{55} \\
\Gamma_{11} & \Gamma_{12} & \Gamma_{14} & 0 & 0 & 0 \\
I_{t_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_5 & C_5
\end{bmatrix}
\begin{bmatrix}
I_{t_1} & & & & \\
& I_{t_2} & & & \\
& & I_{\tilde{s}_4} & & \\
& & & I_{\hat{s}_4} & \\
& & & & V_3
\end{bmatrix}
$$

$$
=: \begin{array}{c} \\ t_1 \\ t_2 \\ t_4 \\ t_5 \\ \tilde{s}_4 \\ t_1 \\ \hat{s}_4 \\ p \end{array}
\begin{array}{c}
\begin{array}{cccccc} t_1 & t_2 & \tilde{s}_4 & \hat{s}_4 & n-r-t_5 & t_5 \end{array} \\
\begin{bmatrix}
E_{11} & 0 & 0 & 0 & \Phi_{15} & \Phi_{16} \\
E_{21} & E_{22} & 0 & 0 & \Phi_{25} & \Phi_{26} \\
E_{41} & E_{42} & E_{44} & 0 & \Phi_{35} & \Phi_{36} \\
0 & 0 & 0 & 0 & 0 & \Sigma_4 \\
\Gamma_{11} & \Gamma_{12} & \Gamma_{14} & 0 & 0 & 0 \\
I_{t_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_5 & \Psi_6
\end{bmatrix}
\end{array}
$$

with Σ_4 nonsingular.

Step 4: Perform a column compression of $\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{14} \end{bmatrix}$:

$$
\begin{bmatrix}
E_{11} & 0 & 0 & 0 & \Phi_{15} & \Phi_{16} \\
E_{21} & E_{22} & 0 & 0 & \Phi_{25} & \Phi_{26} \\
E_{41} & E_{42} & E_{44} & 0 & \Phi_{35} & \Phi_{36} \\
0 & 0 & 0 & 0 & 0 & \Sigma_4 \\
\Gamma_{11} & \Gamma_{12} & \Gamma_{14} & 0 & 0 & 0 \\
I_{t_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_5 & \Psi_6
\end{bmatrix}
\begin{bmatrix}
V_4 & & & \\
& I_{\hat{s}_4} & & \\
& & I_{n-r-t_5} & \\
& & & I_{t_5}
\end{bmatrix}
$$

$$
=: \begin{array}{c@{\,}c}
 & \begin{array}{cccccc} t_1 & t_2 & \tilde{s}_4 & \hat{s}_4 & n-r-t_5 & t_5 \end{array} \\
\begin{array}{c} t_1 \\ t_2 \\ t_4 \\ t_5 \\ \tilde{s}_4 \\ t_1 \\ \hat{s}_4 \\ p \end{array} &
\left[\begin{array}{cccccc}
\Theta_{11} & \Theta_{12} & \Theta_{13} & 0 & \Phi_{15} & \Phi_{16} \\
\Theta_{21} & \Theta_{22} & \Theta_{23} & 0 & \Phi_{25} & \Phi_{26} \\
\Theta_{31} & \Theta_{32} & \Theta_{33} & 0 & \Phi_{35} & \Phi_{36} \\
0 & 0 & 0 & 0 & 0 & \Sigma_4 \\
0 & 0 & \Sigma_5 & 0 & 0 & 0 \\
\Lambda_1 & \Lambda_2 & \Lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_5 & \Psi_6
\end{array}\right]
\end{array}
$$

with Σ_5 nonsingular.

Remark 3 Since $\left[\begin{array}{ccc} E_{11} & 0 & 0 \\ E_{21} & E_{22} & 0 \\ \Gamma_{11} & \Gamma_{12} & \Gamma_{14} \end{array}\right]$ is nonsingular, it follows that also $\left[\begin{array}{cc} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{array}\right]$ is nonsingular.

Step 5: Perform an RQ decomposition (see [11]) of $\left[\begin{array}{ccc} \Theta_{11} & \Theta_{12} & \Phi_{15} \\ \Theta_{21} & \Theta_{22} & \Phi_{25} \end{array}\right]$:

$$
\left[\begin{array}{cccccc}
\Theta_{11} & \Theta_{12} & \Theta_{13} & 0 & \Phi_{15} & \Phi_{16} \\
\Theta_{21} & \Theta_{22} & \Theta_{23} & 0 & \Phi_{25} & \Phi_{26} \\
\Theta_{31} & \Theta_{32} & \Theta_{33} & 0 & \Phi_{35} & \Phi_{36} \\
0 & 0 & 0 & 0 & 0 & \Sigma_4 \\
0 & 0 & \Sigma_5 & 0 & 0 & 0 \\
\Lambda_1 & \Lambda_2 & \Lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & \Psi_5 & \Psi_6
\end{array}\right]
\left[\begin{array}{cccccc}
V_{11} & V_{12} & 0 & 0 & V_{15} & 0 \\
V_{21} & V_{22} & 0 & 0 & V_{25} & 0 \\
0 & 0 & I_{\tilde{s}_4} & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
V_{51} & V_{52} & 0 & 0 & V_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & I_{t_6}
\end{array}\right]
$$

$$
=: \begin{array}{c@{\,}c}
 & \begin{array}{cccccc} t_1 & t_2 & \tilde{s}_4 & \hat{s}_4 & n-r-t_5 & t_5 \end{array} \\
\begin{array}{c} t_1 \\ t_2 \\ t_4 \\ t_5 \\ \tilde{s}_4 \\ t_1 \\ \hat{s}_4 \\ p \end{array} &
\left[\begin{array}{cccccc}
\Sigma_1 & 0 & \Theta_{13} & 0 & 0 & \Phi_{16} \\
\Theta_{21} & \Sigma_2 & \Theta_{23} & 0 & 0 & \Phi_{26} \\
\Theta_{31} & \Theta_{32} & \Theta_{33} & 0 & \Phi_{35} & \Phi_{36} \\
0 & 0 & 0 & 0 & 0 & \Sigma_4 \\
0 & 0 & \Sigma_5 & 0 & 0 & 0 \\
\Lambda_1 & \Lambda_2 & \Lambda_3 & 0 & \Lambda_5 & 0 \\
0 & 0 & 0 & I_{\hat{s}_4} & 0 & 0 \\
\Psi_1 & \Psi_2 & \Psi_3 & 0 & \Psi_5 & \Psi_6
\end{array}\right]
\end{array}
$$

with Σ_1 and Σ_2 nonsingular.

Remark 4 It remains to determine G_{41}, G_{44} and F_2 such that the matrix

$$\begin{bmatrix} 0 & \Phi_{35} \end{bmatrix} + \begin{bmatrix} G_{41} & G_{44} & F_2 \end{bmatrix} \begin{bmatrix} 0 & \Lambda_5 \\ I_{\hat{s}_4} & 0 \\ 0 & \Psi_5 \end{bmatrix}$$

is nonsingular.

Step 6: Set

$$z_1 := \operatorname{rank}(\Phi_{35})$$

and perform a simultaneous row and column compression of Φ_{35}:

$$\begin{bmatrix} U_6 & & & \\ & I_{t_1} & & \\ & & I_{\hat{s}_4} & \\ & & & I_p \end{bmatrix} \begin{bmatrix} 0 & \Phi_{35} \\ 0 & \Lambda_5 \\ I_{\hat{s}_4} & 0 \\ 0 & \Psi_5 \end{bmatrix} \begin{bmatrix} I_{\hat{s}_4} & \\ & V_6 \end{bmatrix}$$

$$=: \begin{array}{cc} & \begin{array}{ccc} \hat{s}_4 & n-r-t_5-z_1 & z_1 \end{array} \\ \begin{array}{c} z_1 \\ t_4-z_1 \\ t_1 \\ \hat{s}_4 \\ p \end{array} & \begin{bmatrix} 0 & 0 & \Sigma_3 \\ 0 & 0 & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} \\ I_{\hat{s}_4} & 0 & 0 \\ 0 & \Psi_{52} & \Psi_{53} \end{bmatrix} \end{array}.$$

Step 7: Set

$$z_2 := \operatorname{rank}(\Lambda_{52}), \quad z_3 := n-r-t_5-z_1-z_2$$

and perform a simultaneous row and column compression of Λ_{52}:

$$\begin{bmatrix} I_{z_1} & & & & \\ & I_{t_4-z_1} & & & \\ & & U_7 & & \\ & & & I_{\hat{s}_4} & \\ & & & & I_p \end{bmatrix} \begin{bmatrix} 0 & 0 & \Sigma_3 \\ 0 & 0 & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} \\ I_{\hat{s}_4} & 0 & 0 \\ 0 & \Psi_{52} & \Psi_{53} \end{bmatrix} \begin{bmatrix} I_{\hat{s}_4} & & \\ & V_7 & \\ & & I_{z_1} \end{bmatrix}$$

$$=: \begin{array}{cc} & \begin{array}{cccc} \hat{s}_4 & z_3 & z_2 & z_1 \end{array} \\ \begin{array}{c} z_1 \\ t_4-z_1 \\ z_2 \\ t_1-z_2 \\ \hat{s}_4 \\ p \end{array} & \begin{bmatrix} 0 & 0 & 0 & \Sigma_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_7 & \Lambda_{54} \\ 0 & 0 & 0 & \Lambda_{64} \\ I_{\hat{s}_4} & 0 & 0 & 0 \\ 0 & \Psi_{52} & \Psi_{53} & \Psi_{54} \end{bmatrix} \end{array}$$

with Σ_7 nonsingular and Ψ_{52} of full column rank.
Step 8: Perform a row compression of Ψ_{52}:

$$\begin{bmatrix} I_{z_1} & & & & & \\ & I_{t_4-z_1} & & & & \\ & & I_{z_2} & & & \\ & & & I_{t_1-z_2} & & \\ & & & & I_{\hat{s}_4} & \\ & & & & & U_8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \Sigma_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_7 & \Lambda_{54} \\ 0 & 0 & 0 & \Lambda_{64} \\ I_{\hat{s}_4} & 0 & 0 & 0 \\ 0 & \Psi_{52} & \Psi_{53} & \Psi_{54} \end{bmatrix}$$

$$=: \begin{array}{c} \\ z_1 \\ t_4-z_1 \\ z_2 \\ t_1-z_2 \\ \hat{s}_4 \\ z_3 \\ p-z_3 \end{array} \begin{array}{c} \begin{array}{cccc} \hat{s}_4 & z_3 & z_2 & z_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \Sigma_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_7 & \Lambda_{54} \\ 0 & 0 & 0 & \Lambda_{64} \\ I_{\hat{s}_4} & 0 & 0 & 0 \\ 0 & \Sigma_8 & \Psi_{53} & \Psi_{54} \\ 0 & 0 & \Psi_{63} & \Psi_{64} \end{bmatrix} \end{array}$$

with Σ_8 nonsingular.
Step 9: Since $\hat{s}_4 + z_1 + z_2 + z_3 = t_4$, choose

$$Z := \begin{array}{c} \\ z_1 \\ t_4-z_1 \end{array} \begin{array}{c} \begin{array}{ccccc} z_2 & t_1-z_2 & \hat{s}_4 & z_3 & p-z_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ Z_{21} & 0 & Z_{23} & Z_{24} & 0 \end{bmatrix} \end{array}$$

with

$$\begin{bmatrix} Z_{21} & Z_{23} & Z_{24} \end{bmatrix} = I_{t_4-z_1}.$$

Remark 5 With Z chosen as in Step 9, we have that the matrix

$$\begin{bmatrix} 0 & 0 & 0 & \Sigma_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} + Z \begin{bmatrix} 0 & 0 & \Sigma_7 & \Lambda_{54} \\ 0 & 0 & 0 & \Lambda_{64} \\ I_{\hat{s}_4} & 0 & 0 & 0 \\ 0 & \Sigma_8 & \Psi_{53} & \Psi_{54} \\ 0 & 0 & \Psi_{63} & \Psi_{64} \end{bmatrix}$$

is nonsingular.

Step 10: Perform the following partitioning:

$$U_6^T Z \begin{bmatrix} I_{z_2} & & & \\ & I_{t_1-z_2} & & \\ & & I_{\hat{s}_4} & \\ & & & U_8 \end{bmatrix} \begin{bmatrix} U_7 & & \\ & I_{\hat{s}_4} & \\ & & I_p \end{bmatrix} =: \begin{array}{c} \begin{array}{ccc} t_1 & \hat{s}_4 & p \end{array} \\ \begin{bmatrix} G_{41} & G_{44} & F_2 \end{bmatrix} \end{array}$$

and determine F, G by solving the linear systems

$$\begin{bmatrix} B_{31} & B_{32} \\ 0 & B_{42} \end{bmatrix} F = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

and

$$\begin{bmatrix} B_{31} & B_{32} \\ 0 & B_{42} \end{bmatrix} G \begin{bmatrix} 0 & \Pi_{14} & \Pi_{15} \\ \Gamma_{21} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ G_{41} & G_{44} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \Theta_{34} & \Theta_{35} \\ 0 & \Theta_{44} & \Theta_{45} \end{bmatrix}.$$

Algorithm 3
Input: Matrix

$$\Phi = \begin{array}{c} \\ \tau_1 \\ \tau_2 \end{array} \begin{array}{c} \begin{array}{cc} \mu_1 & \mu_2 \end{array} \\ \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Psi_1 & \Psi_2 \end{bmatrix} \end{array}$$

satisfying

$$\operatorname{rank} \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix} = \tau_1.$$

Output: Orthogonal matrices U_1, U_2, V_1, V_2 such that Φ is transformed to the form (37).
Step 1: Perform a row compression of Φ_2:

$$\begin{bmatrix} U_1 & \\ & I_{\tau_2} \end{bmatrix} \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Psi_1 & \Psi_2 \end{bmatrix} =: \begin{array}{c} k_1 \\ \tau_1 - k_1 \\ \tau_2 \end{array} \begin{array}{c} \begin{array}{cc} \mu_1 & \mu_2 \end{array} \\ \begin{bmatrix} \Phi_{11} & 0 \\ \Phi_{21} & \Phi_{22} \\ \Psi_1 & \Psi_2 \end{bmatrix} \end{array}$$

with Φ_{22} of full row rank.
Step 2: Perform a row compression of $\begin{bmatrix} \Phi_{22} \\ \Psi_2 \end{bmatrix}$:

$$\begin{bmatrix} I_{k_1} & \\ & U_2 \end{bmatrix} \begin{bmatrix} \Phi_{11} & 0 \\ \Phi_{21} & \Phi_{22} \\ \Psi_1 & \Psi_2 \end{bmatrix} =: \begin{array}{c} k_1 \\ k_2 \\ \tau_1 + \tau_2 - k_1 - k_2 \end{array} \begin{array}{c} \begin{array}{cc} \mu_1 & \mu_2 \end{array} \\ \begin{bmatrix} \Phi_{11} & 0 \\ \Psi_{21} & \Sigma_2 \\ \Psi_{31} & 0 \end{bmatrix} \end{array}$$

with Σ_2 full row rank.
Step 3: Perform a column compression of Φ_{11}:

$$\begin{bmatrix} \Phi_{11} & 0 \\ \Psi_{21} & \Sigma_2 \\ \Psi_{31} & 0 \end{bmatrix} \begin{bmatrix} V_1 & \\ & I_{\mu_2} \end{bmatrix} =: \begin{array}{c} k_1 \\ k_2 \\ \tau_1 + \tau_2 - k_1 - k_2 \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & \mu_1 - k_1 & \mu_2 \end{array} \\ \begin{bmatrix} \Sigma_1 & 0 & 0 \\ \Psi_{21} & \Psi_{22} & \Sigma_2 \\ \Psi_{31} & \Psi_{32} & 0 \end{bmatrix} \end{array}$$

with Σ_1 nonsingular.
Step 4: Perform a column compression of Ψ_{32}:

$$\begin{bmatrix} \Sigma_1 & 0 & 0 \\ \Psi_{21} & \Psi_{22} & \Sigma_2 \\ \Psi_{31} & \Psi_{32} & 0 \end{bmatrix} \begin{bmatrix} I_{k_1} & & \\ & V_2 & \\ & & I_{\mu_2} \end{bmatrix}$$

$$=: \begin{array}{l} \\ k_1 \\ k_2 \\ \tau_1+\tau_2-k_1-k_2 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_3 & \mu_1-k_1-k_3 & \mu_2 \end{array} \\ \begin{bmatrix} \Sigma_1 & 0 & 0 & 0 \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Sigma_2 \\ \Psi_{31} & \Sigma_3 & 0 & 0 \end{bmatrix} \end{array}$$

with Σ_3 of full column rank.

Appendix B–A list of subspaces.

Assume that matrices (E, A, B, C, Γ) are in the forms (7) and (13). Then

$$S_{e\gamma} = \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ \tilde{s}_4 \\ \tilde{s}_5 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_{\tilde{s}_5} \end{bmatrix},$$

$$T_{eb} = \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_{t_5} \end{bmatrix},$$

$$\tilde{S}_{eb} = \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ \tilde{s}_4 \\ \tilde{s}_5 \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_{t_3} & 0 & 0 \\ 0 & I_{\tilde{s}_4} & 0 \\ 0 & 0 & I_{\tilde{s}_5} \end{bmatrix},$$

$$\tilde{T}_{e\gamma} = \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{bmatrix} I_{t_1} & 0 & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{t_4} & 0 \\ 0 & 0 & I_{t_5} \end{bmatrix},$$

$$\tilde{T}^0_{e\gamma} = \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ p \end{array} \begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{t_4} & 0 & 0 \\ 0 & 0 & I_{t_5} & 0 \\ 0 & 0 & 0 & I_p \end{bmatrix}.$$