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INTERACTION BETWEEN THERMOELASTIC AND SCALAR OSCILLATION FIELDS

(General Anisotropic Case)

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Abstract Three-dimensional mathematical problems of the interaction between thermoelastic and scalar oscillation fields are considered in a general anisotropic case. An elastic structure is assumed to be a bounded homogeneous anisortopic body occupying domain $\Omega^+ \subset I\!R^3$, where the thermoelastic field is defined, while in the physically anisotropic unbounded exterior domain $\Omega^- = I\!R^3 \setminus \overline{\Omega^+}$ there is defined the scalar field. These two fields satisfy the differential equations of steady state oscillations in the corresponding domains along with the transmission conditions of special type on the interface $\partial\Omega^\pm$. Uniqueness and existence theorems, for the non-resonance case, are proved by the reduction of the original interface problems to equivalent systems of boundary pseudodifferential equations (ΨDEs). The invertibility of the corresponding matrix pseudodifferential operators (ΨDO) in appropriate functional spaces is shown on the basis of generalized Sommerfeld-Kupradze type thermoradiation conditions for anisotropic bodies. In the resonance case, the co-kernels of the ΨDOs are analysed and the efficient conditions of solvability of the transmission problems are established.

Introduction

Problems connected with the interaction between vector fields of different dimension have received much attention in the mathematical and engineering scientific literature and have been intensively investigated for the past years. They arise in many physical and mechanical models describing the interaction of two different media where the whole process is characterized by a vector-function of dimension k in one medium and by a vector-function of dimension n in another one (for example, fluid-structure interaction when a streamlined body is an elastic obstacle, scattering of acoustic and electromagnetic waves by an elastic obstacle, interaction between an elastic body and seismic waves, etc.).

Quite many authors have considered and studied in detail the interaction between an elastic isotropic body, which occupies a bounded region Ω_1 and where a three-dimensional elastic vector field is to be defined, and some isotropic medium, say fluid, which occupies the unbounded exterior region, the complement of Ω_1 with respect to the whole space, where a scalar field is to be defined. The time-harmonic dependent unknown vector and scalar fields are coupled by some kinematic and dynamic conditions on the boundary $\partial\Omega_1$, which lead to various type of non-classical interface problems of steady state oscillations for a piecewise homogeneous isotropic medium. An exhaustive information in this direction concerning theoretical and numerical results can be found in [1–5], [7–10], [12], [15–16], [21–22], [24], [32–37]. The case, where the elastic body under consideration is anisotropic, has been treated in [29].

In the present paper we generalize the results of the above cited works into two directions: first, the both interacting media are assumed to be anisotropic and, second, in a solid body occupying the bounded region Ω_1 we consider the four-dimensional thermoelastic coupled field instead of the three-dimensional elastic one.

Moreover, we consider the most general interface conditions with arbitrary coefficients on the boundary $\partial\Omega_1$ which cover the natural coupling conditions of real physical fields in particular cases. We are mainly interested in the existence and uniqueness of solutions to the interface problems of steady state oscillations which involve a frequency parameter ω. When studying these problems there arise difficulties due to the above-mentioned anisotropy property of media in question. In this case for both fields we need non-trivial analogues of the classical Sommerfeld principle and the well-known Rellich's lemma [37].

We apply the potential method developed recently in [18] for the thermoelasticity theory of anisotropic bodies. In the non-resonance case, i.e. when there are no Jones modes (see subsection 1.5), we reduce the original interface problems to equivalent systems of boundary pseudodifferential equations (ΨDE) on $\partial\Omega_1$. The ellipticity, index problem and invertibility of the matrix boundary pseudodifferential operators (ΨDE) are studied on the basis of the generalized Sommerfeld-Kupradze type thermoradiation conditions for anisotropic bodies [18]. From these results follow the existence and uniqueness theorems for the original interface problems in question. The solutions are represented in the special form of potential type integrals.

Moreover, in the resonance case, i.e. when there exist Jones modes, we analyse the cokernels of the corresponding ΨDOs and establish efficient conditions of solvability for the non-homogeneous transmission problems. In particular, we have shown that the so-called direct scattering problems are solvable for arbitrary values of the frequency parameter ω.

1 Formulation of Problems and Uniqueness Results

1.1. Thermoelastic field. Let $\Omega_1 = \Omega^+ \subset I\!\!R^3$ be a bounded region $(\operatorname{diam}\Omega_1 < +\infty)$ with smooth, connected, non-self-intersecting boundary $S = \partial\Omega_1$; $\Omega_2 = \Omega^- = I\!\!R^3 \backslash \overline{\Omega}_1$, $\overline{\Omega}_1 = \Omega_1 \cup S$. Later we shall state the smoothness conditions required for S more precisely. The region $\overline{\Omega}_1$ is supposed to be filled up by a homogeneous anisotropic medium with the elastic coefficients $c_{kjpq} = c_{pqkj} = c_{jkpq}$, $k, j, p, q = 1, 2, 3$, and the density $\rho_1 = \text{const} > 0$.

The system of equations of linear thermoelastodynamics reads ([30], ch.V)

$$A(D, D_t)U(x,t) = X(x,t), \tag{1.1}$$

where $U = (u^\top, u_4)^\top$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, u_4 is the temperature, $X = (-F^\top, -Q)^\top$, $F = (F_1, F_2, F_3)^\top$ is the bulk force, Q is the heat source, $x = (x_1, x_2, x_3)$ denotes the spatial variable, while t is the time variable; $D = \nabla = (D_1, D_2, D_3)$, $D_p = D_{x_p} = \partial/\partial x_p$, $D_t = \partial/\partial t$;

$$A(D, D_t) := \begin{bmatrix} [C(D) - \rho_1 I_3 D_t^2]_{3\times 3} & [-\beta_{kj} D_j]_{3\times 1} \\ [-T_0 \beta_{kj} D_j D_t]_{1\times 3} & \Lambda(D) - c_0 D_t \end{bmatrix}_{4\times 4}, \tag{1.2}$$

$$C(D) := [c_{kjpq} D_j D_q]_{3\times 3}, \quad \Lambda(D) := \lambda_{pq} D_p D_p, \tag{1.3}$$

$\lambda_{pq} = \lambda_{qp}$ are heat conductivity coefficients, c_0 is the thermal capacity, T_0 is the temperature of the medium in the natural state, $\beta_{pq} = \beta_{qp}$ are expressed in terms of the thermal and elastic constants; here and in what follows the summation over repeated indices is meant from 1 to 3, unless otherwise stated; the superscript $\top$ denotes transposition; for the sake of simplicity the notation $[N]_{m\times n}$ is also used for the $m \times n$ matrix $[N_{kp}]_{m\times n}$; $I_m = [\delta_{kj}]_{m\times m}$ stands for the unit $m \times m$ matrix $(m \geq 1)$.

In the sequel without restriction of generality we consider the homogeneous version of equation (1.1) (i.e., $X = 0$).

In the thermoelasticity theory the stress tensor $\{\sigma_{kj}\}$, the strain tensor $\{\varepsilon_{kj}\}$ and the temperature field are related by Duhamel-Neumann law

$$\sigma_{kj} = c_{kjpq}\varepsilon_{pq} - \beta_{kj} u_4, \quad \varepsilon_{kj} = 2^{-1}(D_k u_j + D_j u_k), \quad k, j = 1, 2, 3.$$

The k-th component of the vector of thermostresses acting on a surface element with the unit normal vector $n = (n_1, n_2, n_3)$ is calculated by the formula

$$\sigma_{kj} n_j = c_{kjpq}\varepsilon_{pq} n_j - \beta_{kj} n_j u_4 = c_{kjpq} n_j D_q u_p - \beta_{kj} n_j u_4, \quad k = 1, 2, 3. \tag{1.4}$$

Further we introduce the classical stress operatopr

$$T(D, n) = [T_{kp}(D, n)]_{3\times 3} = [c_{kjpq} n_j D_q]_{3\times 3} \tag{1.5}$$

and the thermoelastic stress operator

$$P(D, n) = [[T(D, n)]_{3\times 3}, [-\beta_{kj} n_j]_{3\times 1}]_{3\times 4}\,. \tag{1.6}$$

Clearly, due to (1.4) we have

$$\sigma_{kj} n_j = [T(D,n)u]_k - \beta_{kj} n_j u_4 = [P(D,n)U]_k, \quad k = 1,2,3.$$

If all functions involved in (1.1) are time-harmonic dependent, i.e., $U(x,t) = U(x)\exp\{-\mathrm{i}\omega t\}$, then we get the so-called steady state oscillation equations of the theory of thermoelasticity

$$A(D, -\mathrm{i}\omega)U(x) = 0, \tag{1.7}$$

where $\omega > 0$ is the oscillation (frequency) parameter and $U(x)$ is a complex-valued vector function of the real variable x. The operator $A(D, -\mathrm{i}\omega)$ is defined by (1.2), where D_t is to be changed formally by $-\mathrm{i}\omega$. From the physical considerations it follows that (see [11], [30]):

a) the matrix $\tilde{\lambda} := [\lambda_{pq}]_{3\times 3}$ is positive definite, i.e.,

$$\tilde{\lambda}\zeta \cdot \zeta = \lambda_{pq}\zeta_q\overline{\zeta}_p \geq \delta_0|\zeta|^2, \quad \delta_0 = \mathrm{const} > 0; \tag{1.8}$$

b) $c_{kjpq}e_{kj}e_{pq}$ is a positive definite quadratic form in the real symmetric variables $e_{kj} = e_{jk}$, which implies the positive definiteness of the matrix $C(\xi)$ for $\xi \in I\!R^3\backslash\{0\}$ (see (1.3)), i.e.,

$$C(\xi)\zeta \cdot \zeta = C_{kp}(\xi)\zeta_p\overline{\zeta}_k \geq \delta_1|\xi|^2|\zeta|^2, \quad \delta_1 = \mathrm{const} > 0,$$

where ζ is an arbitrary three-dimensional complex vector $\zeta \in \mathbb{C}^3$ and the upper bar denotes complex conjugate.

Here and throughout the paper $a \cdot b = \sum_{k=1}^m a_k\overline{b}_k$ denotes the scalar product of two vectors in $\mathbb{C}^m$.

We will also use the following notations (when this causes no confusion):

a) if all elements of a vector $v = (v_1, \cdots, v_m)$ (matrix $N = [N_{kj}]_{m\times n}$) belong to one and the same space X, we will write $v \in X$ $(N \in X)$ instead of $v \in [X]^m$ $(N \in [X]_{m\times n})$;

b) if $K : X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_n$ and $X_1 = \cdots = X_m$, $Y_1 = \cdots = Y_n$, we will write $K : X \to Y$ instead of $K : [X]^m \to [Y]^n$.

1.2. Scalar field. We assume that the exterior domain Ω_2 is filled up with a homogeneous anisotropic medium with density ρ_2. Further, let some physical process in Ω_2 be described by a scalar function (scalar field) $w(x,t)$ being a solution of the homogeneous "wave equation"

$$\mathcal{A}(D, D_t)w(x,t) := a_{pq}D_pD_qw(x,t) - \rho_2D_t^2w(x,t) = 0,$$

where $a_{pq} = a_{qp}$ are real constants defining a positive definite matrix, i.e.,

$$\tilde{a} := [a_{pq}]_{3\times 3} \quad \text{and} \quad \tilde{a}\zeta \cdot \zeta = a_{pq}\zeta_q\overline{\zeta}_p \geq \delta_2|\zeta|^2, \quad \delta_2 = \mathrm{const} > 0, \tag{1.9}$$

for arbitrary $\zeta \in \mathbb{C}^3$.

As in the previous subsection, if $w(x,t) = w(x)\exp\{-\mathrm{i}\omega t\}$ is time-harmonic dependent, we get the steady oscillation equation for the complex-valued function $w(x)$:

$$\mathcal{A}(D, -\mathrm{i}\omega)w(x) := a(D)w(x) + \rho_2\omega^2w(x) = 0 \quad \text{with} \quad a(D) := a_{pq}D_pD_q, \tag{1.10}$$

where $\omega > 0$ is again a frequency parameter.

1.3. Radiation condition for the scalar field. In this subsection we shall apply results of [37] to derive Sommerfeld type radiation condition for the scalar field defined by equation (1.10). To this end let us introduce the characteristic function corresponding to the operator $\mathcal{A}(D, -\mathrm{i}\omega)$

$$\Phi_{\mathcal{A}}(\xi, -\mathrm{i}\omega) := -\mathcal{A}(-\mathrm{i}\xi, -\mathrm{i}\omega) = a_{pq}\xi_p\xi_q - \rho_2\omega^2, \quad \xi \in I\!R^3.$$

The characteristic surface (ellipsoid) in $I\!R^3$ defined by the equation

$$\Phi_{\mathcal{A}}(\xi, -\mathrm{i}\omega) = \tilde{a}\xi \cdot \xi - \rho_2\omega^2 = 0, \quad \xi \in I\!R^3, \tag{1.11}$$

we denote by $S_{\mathcal{A}}$.

It is evident that for an arbitrary vector $\eta \in I\!R^3$ with $|\eta| = 1$ there exists only one point $\xi(\eta) \in S_{\mathcal{A}}$ where the outward unit normal vector $n(\xi(\eta))$ has the same direction as η, i.e.,

$$n(\xi(\eta)) = \eta. \tag{1.12}$$

Note that the point $-\xi(\eta)$ belongs to $S_{\mathcal{A}}$ as well and $n(-\xi(\eta)) = -\eta$.

It can be easily checked that the solution of the system of equations (1.11) and (1.12) is given by the formula

$$\xi(\eta) = \omega\sqrt{\rho_2}\,(\tilde{a}^{-1}\eta \cdot \eta)^{-1/2}\ \tilde{a}^{-1}\eta, \tag{1.13}$$

since the exterior unit normal of $S_{\mathcal{A}}$ at the point $\xi \in S_{\mathcal{A}}$ is defined by the equation

$$n(\xi) = \frac{\nabla_\xi \Phi_{\mathcal{A}}(\xi, -i\omega)}{|\nabla_\xi \Phi_{\mathcal{A}}(\xi, -i\omega)|} = \frac{\tilde{a}\xi}{|\tilde{a}\xi|}.$$

Here $\tilde{a}^{-1}$ is the matrix inverse to $\tilde{a}$.

Now we can define the class of functions $S_r(\Omega^-)$ satisfying the generalized Sommerfeld type radiation conditions (see [37]): a function w belongs to $S_r(\Omega^-)$ where either $r = 1$ or $r = 2$, if $w \in C^1(\Omega^-)$ and for sufficiently large $|x|$

$$w(x) = O(|x|^{-1}), \quad \frac{\partial w(x)}{\partial x_k} + (-1)^r \mathrm{i}\xi_k(\eta)w(x) = O(|x|^{-2}), \quad k = 1, 2, 3, \tag{1.14}$$

where $\eta = x/|x|$ and where $\xi(\eta) \in S_{\mathcal{A}}$ corresponds to η (i.e., $\xi(\eta)$ is defined by (1.13)).

Obviously, the conditions (1.14) are equivalent to the classical Sommerfeld radiation conditions for the Helmholtz equation if $a(D)$ is the Laplace operator (see, for example, [6], [38]). In the sequel functions satisfying the conditions (1.14) will be referred to as radiating functions.

Denote by $\gamma_{\mathcal{A}}(x, \omega, r)$ and $\gamma_a(x)$ the following fundamental solutions of the operators $\mathcal{A}(D, -\mathrm{i}\omega)$ and $a(D)$, respectively:

$$\gamma_{\mathcal{A}}(x, \omega, r) = -\frac{\exp\{(-1)^{r+1}\mathrm{i}\omega\sqrt{\rho_2}(\tilde{a}^{-1}x \cdot x)^{1/2}\}}{4\pi|\tilde{a}|^{1/2}(\tilde{a}^{-1}x \cdot x)^{1/2}}, \quad r = 1, 2, \tag{1.15}$$

$$\gamma_a(x) = -[4\pi|\tilde{a}|^{1/2}(\tilde{a}^{-1}x \cdot x)^{1/2}]^{-1}, \quad x \in I\!R^3 \backslash \{0\}, \tag{1.16}$$

where $|\tilde{a}| = \det \tilde{a}$.

LEMMA 1.1 *The fundamental solution* $\gamma_{\mathcal{A}}$ *defined by* (1.15) *satisfies the conditions:*

i) $\gamma_{\mathcal{A}}(x,\omega,r)\in C^{\infty}(\mathbb{R}^3\backslash\{0\})\cap S_r(\mathbb{R}^3\backslash\{0\})$;

ii) *in a neighbourhood of the origin*

$$|D_x^{\beta}\gamma_{\mathcal{A}}(x,\omega,r)-D_x^{\beta}\gamma_a(x)|\le c|x|^{1-|\beta|},\quad c=\text{const}>0,$$

where $\beta=(\beta_1,\beta_2,\beta_3)$ *is an arbitrary multi-index and* $|\beta|=\beta_1+\beta_2+\beta_3$;

iii) *for sufficiently large* $|x|$

$$\gamma_{\mathcal{A}}(x-y,\omega,r)=-\frac{|\tilde a\xi|}{4\pi\omega(\rho_2|\tilde a|)^{1/2}}\frac{\exp\{(-1)^{r+1}\mathrm{i}(\xi\cdot(x-y))\}}{|x|}+O(|x|^{-2}),\tag{1.17}$$

where y *varies in a bounded subset of* $\mathbb{R}^3$ *and* $\xi=\xi(\eta)\in S_{\mathcal{A}}$ *corresponds to the direction* $\eta=x/|x|$; *the asymptotic formula* (1.17) *can be differentiated any times with respect to* x *and* y.

Proof. The conditions i) and ii) follow directly from (1.15) and (1.16), while the item iii) is a consequence of equations (1.13), (1.15) and formulae

$$(\tilde a^{-1}(x-y)\cdot(x-y))^{1/2}=(\tilde a^{-1}x\cdot x)^{1/2}-(\tilde a^{-1}x\cdot x)^{-1/2}\,(\tilde a^{-1}x\cdot y)+O(|x|^{-1}),$$
$$\xi(\eta)=\omega(\rho_2)^{1/2}(\tilde a^{-1}x\cdot x)^{-1/2}\,\tilde a^{-1}x,\quad \eta=x/|x|,$$
$$\omega(\rho_2)^{1/2}(\tilde a^{-1}(x-y)\cdot(x-y))^{1/2}=(\xi\cdot x)-(\xi\cdot y)+O(|x|^{-1}).$$

∎

Now we formulate the analogue of Rellich's lemma.

LEMMA 1.2 *Let* $w\in S_r(\Omega^-)$ *be a solution of* (1.10) *in* Ω^- *and let*

$$\lim_{R\to\infty}\mathrm{Im}\int_{\Sigma_R}\nu(D,n)w(x)\overline{w(x)}d\Sigma_R=0,\tag{1.18}$$

where Σ_R *is the sphere centered at the origin and of radius* R,

$$\nu(D,n)=a_{pq}n_pD_q,\tag{1.19}$$

$n(x)=x/|x|=\eta$ *is the exterior unit normal vector to* Σ_R *at the point* x.

Then $w(x)=0$ *in* Ω^-.

Proof. The radiation conditions (1.14) along with (1.18) imply

$$\lim_{R\to\infty}\int_{\Sigma_R}|w(x)|^2(\tilde a^{-1}\eta\cdot\eta)^{-1/2}d\Sigma_R=0.\tag{1.20}$$

Note that w is a solution of the elliptic equation with constant coefficients and, therefore, it is an analytic function of the real variable x in the domain Ω^-. Moreover, since $w\in S_r(\Omega^-)$, we can write the following integral representation formula (cf. [37])

$$w(x)=\int_S\gamma_{\mathcal{A}}(x-y,\omega,r)[\nu(D_y,n(y))w(y)]^-dS_y$$
$$-\int_S[\nu(D_y,n(y))\gamma_{\mathcal{A}}(x-y,\omega,r)][w(y)]^-dS_y,\quad x\in\Omega^-,\tag{1.21}$$

where the operator $\nu(D,n)$ is defined by (1.19) and n is again the outward unit normal vector to S.

Due to Lemma 1.1, from (1.21) we then conclude

$$w(x) = |x|^{-1}b(\xi)\exp\{(-1)^{1+r}\mathrm{i}\xi x\} + O(|x|^{-2}), \tag{1.22}$$

for sufficiently large $|x|$, where $\xi = \xi(\eta) \in S_{\mathcal{A}}$ corresponds to $\eta = x/|x|$ and

$$b(\xi) = -|\tilde{a}\xi|\,[4\pi\omega(\rho_2|\tilde{a}|)^{1/2}]^{-1}\left\{\int\limits_S [\nu(D_y,n(y))w(y)]^- \exp\{(-1)^r\mathrm{i}\xi y\}dS_y \right.$$
$$\left. -(-1)^r\mathrm{i}\int\limits_S \nu(\xi,n(y))[w(y)]^-\exp\{(-1)^r\mathrm{i}\xi y\}dS_y\right\}.$$

Further, (1.20) and (1.22) yield

$$\int\limits_{\Sigma_1} |b(\xi(\eta))|^2(\tilde{a}^{-1}\eta\cdot\eta)^{-1/2}d\Sigma_1 = 0,$$

whence $b(\xi)=0$. Now by (1.22) we get $w(x) = O(|x|^{-2})$. This in turn implies that $w(x)$ has a compact support (see [37]) which, together with the above-mentioned analyticity of w in Ω^-, completes the proof. ∎

REMARK 1.3 *It is obvious that for the functions of the class $S_r(\Omega^-)$ the conditions* (1.18) *and* (1.20) *are equivalent.*

1.4. Formulation of interface problems. We are now in the position to formulate the basic mathematical transmission problems corresponding to the interaction of the fields introduced above.

Problem $(P_l^{(\omega,r)})$, $l=1,2,3$. Find a regular vector $U \in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})$ and a regular radiating function $w \in C^2(\Omega^-)\cap C^1(\overline{\Omega^-})\cap S_r(\Omega^-)$ which are solutions of equations (1.7) and (1.10), respectively, and satisty the following coupling conditions on $S = \partial\Omega^\pm$:

$$[P(D,n)U(x)]^+ = d_1[w(x)]^-n(x) + f(x), \quad f = (f_1,f_2,f_3)^\top, \tag{1.23}$$
$$[\lambda^{(l)}(D,n)u_4(x)]^+ = f_4(x), \tag{1.24}$$
$$[u(x)\cdot n(x)]^+ = d_2[\nu(D,n)w(x)]^- + f_5(x); \tag{1.25}$$

here and throughout this paper $n(x)$ denotes the exterior unit normal vector to S at the point $x \in S$, the symbols $[\cdot]^\pm$ denote limits on S from $\Omega^\pm$, $f_j\,(j=1,...,5)$ are the given functions on S;

$$\lambda^{(1)} := I_1, \quad \lambda^{(2)}(D,n) = \lambda(D,n) := \lambda_{pq}n_pD_q, \quad \lambda^{(3)}(D,n) := d_3'\lambda(D,n) + d_3''I_1, \tag{1.26}$$

where $d_3'\cdot d_3'' \neq 0$, and I_1 stands for the identical scalar operator, while d_1, d_2, d_3' and d_3'' are complex constants.

Note that $\lambda^{(1)}$ corresponds to the prescribed temperature on S, $\lambda^{(2)}$ – to the prescribed heat flow through the surface S, and $\lambda^{(3)}$ – to a free heat exchange over the surface S.

Conditions (1.23) and (1.25) in the real physical problems describe a dynamic and a kinematic coupling of the fields in question on the interface S (cf. [21], [23], [15]).

A pair (U, w) constructed by the regular vector U and the radiating regular scalar function w, occurred in the above formulation, in the sequel will be referred to as a solution of Problem $(P_l^{(\omega,r)})$.

1.5. Uniqueness results. We denote by J the set of values of the frequency parameter $\omega > 0$ for which the following boundary value problem

$$\begin{aligned} &C(D)u(x) + \rho_1\omega^2 u(x) = 0, \quad \beta_{kj}D_k u_j(x) = 0, \quad x \in \Omega^+, \\ &\qquad [T(D,n)u(x)]^+ = 0, \quad [(u(x)\cdot n(x))]^+ = 0, \quad x \in S, \end{aligned} \tag{1.27}$$

admits a nontrivial solution. Such solutions (vectors) are called *Jones modes.*

Clearly, J is at most countable [26]. In general $J \neq \emptyset$ ([20], [22]), however, as it will be shown latter on in Appendix, there exist domains for which $J = \emptyset$ (see also [13], [22]).

THEOREM 1.4 *Let S be C^2-smooth, $\omega \notin J$, and let*

$$(-1)^r d_1\overline{d}_2 > 0, \quad \mathrm{Re}\{\overline{d_3'}d_3''\} \geq 0. \tag{1.28}$$

Then the homogeneous Problems $(P_l^{(\omega,r)})$ $(l = 1,2,3)$ $(f_j = 0, j = 1,...5)$ have only the trivial solution.

Proof. Let some pair (U, w) be a solution to the homogeneous Problem $(P_l^{(\omega,r)})$.

We set $B_R = \{x \in I\!R^3 : |x| < R\}$ and $\Sigma_R = \partial B_R$, where R is supposed to be a sufficiently large positive number such that $\overline{\Omega^+} \subset B_R$. Further, let $\Omega_R^- = \Omega^- \cap B_R$. Clearly, $\partial\Omega_R^- = S \cup \Sigma_R$.

Since w and the components of the vector U are regular functions in the bounded domains Ω_R^- and Ω^+, respectively, we can write the following Green's formulae [18]:

$$\begin{aligned} &\int\limits_{\Omega_R^-} \{\overline{\mathcal{A}(D,-\mathrm{i}\omega)w(x)}\, w(x) - \overline{w(x)}\,\mathcal{A}(D,-\mathrm{i}\omega)w(x)\}\, dx \\ &\qquad = \int\limits_{\Sigma_R} \{\overline{\nu(D,n)w(x)}\, w(x) - \overline{w(x)}\,\nu(D,n)w(x)\}\, d\Sigma_R \\ &- \int\limits_S \{[\overline{\nu(D,n)w(x)}]^-[w(x)]^- - [\overline{w(x)}]^-[\nu(D,n)w(x)]^-\}dS, \end{aligned}$$

$$\begin{aligned} &\int\limits_{\Omega^+} \{[A(D,-\mathrm{i}\omega)U(x)]_k\overline{u_k(x)} - \frac{\mathrm{i}}{\omega T_0}[\overline{A(D,-\mathrm{i}\omega)U(x)}]_4[u_4(x)]\}dx \\ &= -\int\limits_{\Omega^+} \Big\{c_{kjpq}D_p u_q(x)\overline{D_k u_j(x)} - \omega^2|u(x)|^2 - \frac{\mathrm{i}}{\omega T_0}\lambda_{kj}D_k u_4(x)\overline{D_j u_4(x)} \\ &\qquad + \frac{c_0}{T_0}|u_4(x)|^2\Big\}\, dx + \int\limits_S \Big\{[P(D,n)U(x)]_k^+[\overline{u_k(x)}]^+ \\ &\qquad - \frac{\mathrm{i}}{\omega T_0}[u_4(x)]^+[\overline{\lambda(D,n)u_4(x)}]^+\Big\}\, dS, \end{aligned} \tag{1.29}$$

where $P(D,n)$, $\nu(D,n)$ and $\lambda(D,n)$ are boundary differential operators given by (1.6), (1.19) and (1.26), respectively.

Taking into account the conditions of the homogeneous Problem $(P_l^{(\omega,r)})$ and separating imaginary parts in (1.29), we get

$$\operatorname{Im}\int_{\Sigma_R}\overline{\nu(D,n)w(x)}\,w(x)d\Sigma_R=\operatorname{Im}\int_S[\overline{\nu(D,n)w(x)}]^-[w(x)]^-dS,$$

$$\frac{1}{\omega T_0}\int_{\Omega^+}\lambda_{kj}D_k u_4(x)\,\overline{D_j u_4(x)}dx+\frac{\delta_{3l}(\operatorname{Re}d_3)}{\omega T_0}\int_S|[u_4(x)]^+|^2dS$$
$$+\operatorname{Im}\left\{d_1\overline{d}_2\int_S[\overline{\nu(D,n)w(x)}]^-[w(x)]^-dS\right\}=0,$$

where δ_{kl} is Kronecker's symbol and $d_3=d_3''/d_3'$.

These equations, due to (1.28), imply

$$\frac{1}{\omega T_0}\int_{\Omega^+}\lambda_{kj}D_k u_4(x)\overline{D_j u_4(x)}dx+\frac{\delta_{3l}(\operatorname{Re}d_3)}{\omega T_0}\int_S|[u_4(x)]^+|^2dS$$
$$+d_1\overline{d}_2\operatorname{Im}\int_{\Sigma_R}\overline{\nu(D,\eta)w(x)}\,w(x)\,d\Sigma_R=0,\quad \eta=x/|x|. \tag{1.30}$$

For sufficiently large R and $x\in\Sigma_R$, by (1.14) we have

$$\overline{D_k w(x)}=(-1)^r\mathrm{i}\xi_k(\eta)\overline{w(x)}+O(R^{-2}),$$

where as above $\xi(\eta)\in S_{\mathcal{A}}$ corresponds to $\eta=x/|x|$ (see (1.13)). Therefore

$$\overline{\nu(D,\eta)w(x)}=\overline{a_{pq}\eta_p D_q w(x)}=\mathrm{i}(-1)^r(\tilde{a}\eta\cdot\xi(\eta))\,\overline{w(x)}+O(R^{-2})$$
$$=\mathrm{i}(-1)^r\omega(\rho_2)^{1/2}\,(\tilde{a}^{-1}\eta\cdot\eta)^{-1/2}\,\overline{w(x)}+O(R^{-2}).$$

Consequently, taking in (1.30) the limit as $R\to+\infty$, we obtain

$$\frac{1}{\omega T_0}\int_{\Omega^+}\lambda_{kj}D_k u_4(x)\overline{D_j u_4(x)}dx+\frac{\delta_{3l}(\operatorname{Re}d_3)}{\omega T_0}\int_S|[u_4(x)]^+|^2dS$$
$$+(-1)^r d_1\overline{d}_2\omega(\rho_2)^{1/2}\lim_{R\to+\infty}\int_{\Sigma_R}|w(x)|^2(\tilde{a}^{-1}\eta\cdot\eta)^{-1/2}d\Sigma_R=0, \tag{1.31}$$

where in the left-hand side we have only non-negative terms, due to the assumptions (1.28) of the theorem and condition (1.8). As a result, we see that each summand in (1.31) is zero.

Remark 1.3, Lemma 1.2 and the inequality (1.8) then yield

$$w(x)=0\ \text{ in }\ \Omega^-,\quad u_4(x)=\tilde{c}=\text{const}\ \text{ in }\ \Omega^+. \tag{1.32}$$

Further, we note that $\det A(D,-\mathrm{i}\omega)u_4(x)=0$ in Ω^+ which readily follows from (1.7). This equation for $u_4(x)=\tilde{c}=\text{const}$ implies $\det A(D,-\mathrm{i}\omega)\tilde{c}=(\omega\varrho_1)^3(\mathrm{i}\omega c_0)\tilde{c}=0$ and, consequently, we get $\tilde{c}=0$. Thus, $u_4(x)=0$ for $x\in\Omega^+$.

Due to (1.32), the homogeneous Problem $(P_l^{(\omega,r)})$ will be converted then into the problem (1.27) for the vector u, which completes the proof. ∎

REMARK 1.5 *From the above uniqueness results it follows that the homogeneous Problems* $(P_l^{(\omega,r)})$ *admit only nontrivial solutions* $(u, u_4; w)$ *where the exterior scalar field* w *and the interior temperature field* u_4 *vanish in* Ω^- *and* Ω^+*, respectively, while the nontrivial displacement vector* u *solves the boundary value problem* (1.27)*, i.e.,* u *is a Jones mode.*

In the next section we collect some auxiliary material which will be background to derive and analyse boundary integral (pseudodifferential) equations corresponding to the interface problems formulated above.

2 Thermoradiation conditions. Potential type operators

2.1. Radiation conditions in anisotropic thermoelasticity. When studying the solvability questions of the above interface problems by the BIE methods there arise, as an intermediate step, boundary value problems for the thermoelastic field in the exterior domain Ω^-. These problems for anisotropic elastic bodies were investigated quite recently in [18]. Mainly, we need the uniqueness results which are closely related to the radiation conditions at infinity (the so-called generalized Sommerfeld-Kupradze type radiation conditions [22]). To formulate these conditions let us introduce the characteristic polynomial of the operator $A(D, -\mathrm{i}w)$

$$M(\xi, -\mathrm{i}\omega) := \det A(-\mathrm{i}\xi, -\mathrm{i}\omega),\ \xi \in I\!R^3,\ \omega > 0.$$

The corresponding characteristic surface is defined by the equation

$$M(\xi, -\mathrm{i}\omega) = \Lambda(\xi)\Phi(\xi, \omega) - \mathrm{i}\omega c_0 \tilde{\Phi}(\xi, \omega) = 0,\ \xi \in I\!R^3, \tag{2.1}$$

where

$$A(-\mathrm{i}\xi, -\mathrm{i}\omega) = \begin{bmatrix} [\omega^2\rho_1 I_3 - C(\xi)]_{3\times 3} & [\mathrm{i}\beta_{kj}\xi_j]_{3\times 1} \\ [\omega T_0 \beta_{kj}\xi_j]_{1\times 3} & -\Lambda(\xi) + \mathrm{i}\omega C_0 \end{bmatrix}_{4\times 4},$$

$$\Phi(\xi, \omega) := \det[C(\xi) - \omega^2 \rho_1 I_3], \quad \widetilde{\Phi}(\xi, \omega) := \det[\widetilde{C}(\xi) - \omega^2\rho_1 I_3],$$
$$C(\xi) = [c_{kjpq}\xi_j\xi_q]_{3\times 3}, \quad \Lambda(\xi) = \lambda_{pq}\xi_p\xi_q, \quad \widetilde{C}(\xi) = C(\xi) + [c_0^{-1} T_0 \beta_{kj}\beta_{pq}\xi_j\xi_q]_{3\times 3}.$$

Clearly, (2.1) is equivalent to the system of equations

$$\Phi(\xi, \omega) = 0 \quad \text{and} \quad \widetilde{\Phi}(\xi, \omega) = 0 \quad \text{for} \quad \xi \in I\!R^3. \tag{2.2}$$

In what follows we consider the so-called regular case (cf.[37], [26], [28]):

I^0. $\nabla\Phi(\xi, \omega) \neq 0$ at real zeros of the polynomial $\Phi(\xi, \omega)$;

II^0. The Gaussian curvature of the surface, defined by the real zeros of the polynomial $\Phi(\xi, \omega)$, does not vanish anywhere.

Moreover, we assume the system (2.2) (i.e. the characteristic equation (2.1)) either to be inconsistent in $I\!R^3$ or to define a two-dimensional manifold. From the above requirements it follows that in the case under consideration the real zeros of the system (2.2) form non-self-intersecting, closed, convex two-dimensional surfaces $S_1, ..., S_m$, $1 \leq m \leq 3$, enveloping the origin. Therefore, for an arbitrary unit vector $\eta = x/|x|$ with $x \in I\!R^3\backslash\{0\}$, there exists exactly one point on each S_j (say $\xi^j = \xi^j(\eta) \in S_j$) at which the exterior unit normal $n(\xi^j)$ to S_j has the same direction as η. The point $\xi^j = \xi^j(\eta) \in S_j$ will be referred to as the point which corresponds to η. Note that if ξ^j corresponds to η, then $-\xi^j \in S_j$ and $n(-\xi^j) = -\eta$. If the system (2.2) is inconsistent in $I\!R^3$, then we have no real characteristic manifold and in this case we put $m = 0$.

Now we can formulate the thermoradiation conditions (for details see [18]).

A function v, which may be vector- or matrix-valued, belongs to the class $SK_r^m(\Omega^-)$, $r = 1, 2$, if it is C^1-smooth in Ω^- and for sufficiently large $|x|$ the following relations hold (no summation over j in the last equation):

$$
\begin{aligned}
v(x) = \textstyle\sum_{j=1}^m \overset{(j)}{v}(x), \quad \overset{(j)}{u}(x) = O(|x|^{-1}),\\
D_p \overset{(j)}{v}(x) + \mathrm{i}(-1)^r \xi_p^j \overset{(j)}{v}(x) = O(|x|^{-2}), \quad p = 1,2,3, \quad j = 1, ..., m,
\end{aligned}
\tag{2.3}
$$

where $\xi^j \in S_j$ corresponds to the vector $\eta = x/|x|$.

Clearly, this definition is essentially related to the operator $A(D, -\mathrm{i}\omega)$ and to its characteristic equation (2.1). It can be shown that in the isotropic case equations (2.3) are equivalent to the well-known radiation conditions in the thermoelastic oscillation theory of isotropic bodies (see [22]). Therefore (2.3) are called *the generalized Sommerfeld-Kupradze type radiation conditions in anisotropic thermoelasticity* [18]. A four-dimensional vector $U = (u_1, ..., u_4)^\top$ is said to be an (m, r)-*thermo-radiating vector* if it satisfies (2.3).

THEOREM 2.1 [*18*] *Let* $(-1)^{r+1}\omega > 0$ *and let* U *be a regular* (m, r)-*thermo-radiating solution of equation* (1.7) *in* Ω^- *satisfying the following boundary conditions on* $S = \partial\Omega^-$:

a) $[u(x)]^- = 0$ *or* $[P(D, n)U(x)]^- = 0,$

and

b) $[u_4(x)]^- = 0$ *or* $[\lambda(D, n)u_4(x)]^- = 0,$

where $P(D, n)$ *and* $\lambda(D, n)$ *are given by* (1.6) *and* (1.26), *respectively.*

Then $U(x) = 0$ *in* Ω^-.

2.2. Fundamental solutions. Let $\Gamma_C(x)$ be the homogeneous (of order -1) fundamental matrix of the operator $C(D)$ (see [26], [28]),

$$
\Gamma_C(x) = \mathcal{F}^{-1}_{\xi\to x}[C^{-1}(-\mathrm{i}\xi)] = -(8\pi^2|x|)^{-1}\int_0^{2\pi} C^{-1}(G\eta)d\varphi, \quad x \in I\!R^3\backslash\{0\},
$$

where $G = [g_{kj}]_{3\times3}$ is an orthogonal matrix with the property $G^\top x^\top = (0, 0, |x|)^\top$, while $\eta = (\cos\varphi, \sin\varphi, 0)^\top$; here $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the generalized Fourier transform and its inverse operators, respectively, which for smooth functions are defined as follows:

$$
\mathcal{F}_{x\to\xi}[f] = \int_{R^n} f(x)\exp\{\mathrm{i}x\xi\}dx, \quad \mathcal{F}^{-1}_{\xi\to x}[g] = (2\pi)^{-n}\int_{R^n} g(\xi)\exp\{-\mathrm{i}x\xi\}d\xi.
$$

By $\gamma_\Lambda(x)$ we denote the homogeneous fundamental solution of the operator $\Lambda(D)$ (cf. (1.16))

$$
\gamma_\Lambda(x) = -[4\pi|\tilde{\lambda}|^{1/2}(\tilde{\lambda}^{-1}x\cdot x)^{1/2}]^{-1}, \quad x \in I\!R^3\backslash\{0\},
$$

where $\tilde{\lambda} = [\lambda_{pq}]_{3\times 3}$ and $|\tilde{\lambda}| = \det \tilde{\lambda}$.

Applying the limiting absorption principle, in [18] there have been constructed the fundamental matrices $\Gamma_A(x, \omega, r) \in SK_r^m(I\!R^3 \backslash \{0\})$, $r = 1, 2$, for the operator $A(D, -\mathrm{i}\omega)$. They have the following properties (for details see [18], Section 2):

a) In a neighbourhood of the origin ($|x| < 1/2$)

$$|D_x^\beta \Gamma_{Akj}(x, \omega, r) - D_x^\beta \Gamma_{kj}(x)| \leq c\,\varphi_{|\beta|}^{(kj)}(x), \quad c = \text{const} > 0, \quad k, j = 1, ..., 4, \tag{2.4}$$

where

$$\Gamma(x) = [\Gamma_{kj}(x)]_{4\times 4} = \begin{bmatrix} [\Gamma_C(x)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \gamma_\Lambda(x) \end{bmatrix}_{4\times 4};$$

here

$$\varphi_0^{(kj)}(x) = 1, \quad \varphi_1^{(kj)}(x) = -\ln|x|, \quad \varphi_l^{(kj)}(x) = |x|^{1-l}, \quad l \geq 2,$$

for $1 \leq k, j \leq 3$, and $k = j = 4$, while

$$\varphi_0^{(k4)}(x) = \varphi_0^{(4k)}(x) = -\ln|x|, \quad \varphi_l^{(k4)}(x) = \varphi_l^{(4k)}(x) = |x|^{-l} \quad l \geq 1,$$

for $k = 1, 2, 3$; $\beta = (\beta_1, \beta_2, \beta_3)$ is an arbitrary multi-index.

b) For sufficiently large $|x|$ (and $1 \leq m \leq 3$)

$$\Gamma_A(x - y, \omega, r) = \sum_{j=1}^{m} C_r^{(j)} |x|^{-1} \exp\{(-1)^{r+1}\mathrm{i}(x - y)\xi^j\} + O(|x|^{-2}), \tag{2.5}$$

where y varies in a bounded subset of $I\!R^3$ and the asymptotic coefficients $C_r^{(j)} = [(C_r^{(j)})_{pq}]_{4\times 4}$ do not depend on $|x|$ (they depend on the thermoelastic constants, the frequency parameter ω and the direction $\eta = x/|x|$); the point $\xi^j \in S_j$ corresponds to the direction $\eta = x/|x|$; the equation (2.5) can be differentiated any times with respect to x and y.

Note that if $m = 0$, then the elements of the corresponding fundamental matrix

$$\Gamma_A(x, \omega) = \mathcal{F}^{-1}_{\xi \to x}[A^{-1}(-\mathrm{i}\xi, -\mathrm{i}\omega)]$$

of the operator $A(D, -\mathrm{i}\omega)$ together with all its derivatives decrease more rapidly than any negative power of $|x|$ as $|x| \to \infty$. Therefore, in this case any solution of the equation (1.7) in Ω^-, which increases at infinity no more than some polynomial, actually decrease more rapidly than any negative power of $|x|$ as $|x| \to \infty$. Note that $\Gamma_A(x, \omega)$ has again the matrix $\Gamma(x)$ as a dominant singular part in the vicinity of the origin ($|x| < 1/2$) (cf. (2.4)).

In what follows we will mainly consider the case $1 \leq m \leq 3$, although all the results obtained remain valid for $m = 0$ as well.

2.3. Potentials. In what follows $V_N(g)(x)$ and $W_N(g)(x)$ denote the single and double layer potentials, respectively, which are constructed by means of the fundamental solution (matrix) of the differential operator N. In particular,

$$V_{\mathcal{A}}(g)(x) = \int_S \gamma_{\mathcal{A}}(x - y, \omega, r) g(y) dS_y, \tag{2.6}$$

$$W_{\mathcal{A}}(g)(x) = \int_S \nu(D_y, n(y)) \gamma_{\mathcal{A}}(x - y, \omega, r) g(y) dS_y, \tag{2.7}$$

$$V_{\mathcal{A}}(h)(x) = \int_S \Gamma_{\mathcal{A}}(x-y,\omega,r)h(y)dS_y, \tag{2.8}$$

$$W_{\mathcal{A}}(h)(x) = \int_S [Q(D_y,n(y),-\mathrm{i}\omega)\Gamma_{\mathcal{A}}^\top(x-y,\omega,r)]^\top h(y)dS_y, \tag{2.9}$$

where g is a scalar function, while $h = (\tilde{h}^\top, h_4)^\top$ is a four-dimensional vector with $\tilde{h} = (h_1,h_2,h_3)^\top$,

$$Q(D,n,-\mathrm{i}\omega) = \begin{bmatrix} [T(D,n)]_{3\times 3} & [-\mathrm{i}\omega T_0\beta_{kj}n_j]_{3\times 1} \\ [0]_{1\times 3} & \lambda(D,n) \end{bmatrix}_{4\times 4},$$

$T(D,n)$, $\nu(D,n)$ and $\lambda(D,n)$ are given by (1.5), (1.19) and (1.26).

Lemma 1.1 and inequalities (2.4) imply that the dominant singular parts of the potential operators of oscillations (2.6) and (2.7) are given by the potential operators of statics $V_a(g)$ and $W_a(g)$, respectively, constructed by the fundamental solution $\gamma_a(x)$:

$$V_a(g)(x) = \int_S \gamma_a(x-y)g(y)dS_y, \quad W_a(g)(x) = \int_S \nu(D_y,n(y))\gamma_a(y-x)g(y)dS_y. \tag{2.10}$$

Analogously, the dominant singular parts of the potential operators of thermoelastic oscillations (2.8) and (2.9) are given by the corresponding static potentials $(V_C(\tilde{h})^\top, V_\Lambda(h_4))^\top$ and $(W_C(\tilde{h})^\top, W_\Lambda(h_4))^\top$, respectively, where

$$V_C(\tilde{h})(x) = \int_S \Gamma_C(x-y)\tilde{h}(y)dS_y, \quad V_\Lambda(h_4)(x) = \int_S \gamma_\Lambda(x-y)h_4(y)dS_y, \tag{2.11}$$

$$W_C(\tilde{h})(x) = \int_S [T(D_y,n(y))\Gamma_C(y-x)]^\top \tilde{h}(y)dS_y, \tag{2.12}$$

$$W_\Lambda(h_4)(x) = \int_S \lambda(D_y,n(y))\gamma_\Lambda(y-x)h_4(y)dS_y. \tag{2.13}$$

Due to Lemma 1.1 and the relations (2.4) it is obvious that the oscillation potentials (2.6)–(2.9) and their dominant static approximations (2.11)–(2.13) have the same smoothness properties and jump relations. To establish the asymptotic behaviour of the oscillation potentials at infinity we have to apply the asymptotic formulae (1.17) and (2.5).

LEMMA 2.2 [*18*] *Let $S \in C^{k+1+\alpha'}$ with integer $k \geq 0$ and $0 < \alpha < \alpha' < 1$, and let N be one of the differential operators $\mathcal{A}$, a, Λ, C, A. Then*

i) *the operators*

$$V_N : C^{k+\alpha}(S) \to C^{k+1+\alpha}(\overline{\Omega^\pm}), \quad W_N : C^{k+\alpha}(S) \to C^{k+\alpha}(\overline{\Omega^\pm}),$$

are bounded; moreover, $V_{\mathcal{A}}(g)$, $W_{\mathcal{A}}(g) \in S_r(\Omega^-)$ and $V_{\mathcal{A}}(h), W_{\mathcal{A}}(h) \in SK_r^m(\Omega^-)$;

ii) *for arbitrary* $z \in S$, $g \in C^{k+\alpha}(S)$ *and* $h = (\tilde{h}^\top, h_4)^\top \in C^{k+\alpha}(S)$ *the following jump relations hold on* S*:*

$$
\begin{aligned}
[V_{\mathcal{A}}(g)(z)]^\pm &= \int\limits_S \gamma_{\mathcal{A}}(z-y,\omega,r)g(y)dS_y =: \mathcal{H}_{\mathcal{A}}\, g(z), \quad k \ge 0, \\
[\nu(D_z,n(z))V_{\mathcal{A}}(g)(z)]^\pm &= \mp 2^{-1}g(z) + \int\limits_S \nu(D_z,n(z))\gamma_{\mathcal{A}}(z-y,\omega,r)g(y)\, dS_y \\
&=: [\mp 2^{-1}I_1 + \mathcal{K}^{(1)}_{\mathcal{A}}]g(z), \quad k \ge 0, \\
[W_{\mathcal{A}}(g)(z)]^\pm &= \pm 2^{-1}g(z) + \int\limits_S \nu(D_y,n(y))\gamma_{\mathcal{A}}(y-z,\omega,r)g(y)\, dS_y \\
&=: [\pm 2^{-1}I_1 + \mathcal{K}^{(2)}_{\mathcal{A}}]g(z), \quad k \ge 0, \qquad (2.14) \\
[\nu(D_z,n(z))W_{\mathcal{A}}(g)(z)]^\pm &=: \mathcal{L}_{\mathcal{A}}\, g(z), \quad k \ge 1, \\
[V_A(h)(z)]^\pm &= \int\limits_S \Gamma_A(x-y,\omega,r)h(y)dS_y =: \mathcal{H}_A\, h(z), \quad k \ge 0, \\
[B(D_z,n(z))V_A(h)(z)]^\pm &= \mp 2^{-1}h(z) + \int\limits_S B(D_z,n(z))\Gamma_A(z-y,\omega,r)h(y)dS_y \\
&=: [\mp 2^{-1}I_4 + \mathcal{K}^{(1)}_A]h(z), \quad k \ge 0, \\
[W_A(h)(z)]^\pm = \pm 2^{-1}h(z) &+ \int\limits_S [Q(D_y,n(y),-\mathrm{i}\omega)\Gamma^\top_A(z-y,\omega,r)]^\top h(y)\, dS_y \\
&=: [\pm 2^{-1}I_4 + \mathcal{K}^{(2)}_A]h(z), \quad k \ge 0, \\
[B(D_z,n(z))W_A(h)(z)]^\pm &=: \mathcal{L}_A\, h(z), \quad k \ge 1;
\end{aligned}
$$

here

$$
B(D,n) = \begin{bmatrix} [T(D,n)]_{3\times 3} & [\beta_{kj}n_j]_{3\times 1} \\ [0]_{1\times 3} & \lambda(D,n) \end{bmatrix}_{4\times 4}; \qquad (2.15)
$$

iii) *the operators*

$$
\begin{aligned}
\mathcal{H}_N &: C^{k+\alpha}(S) \to C^{k+1+\alpha}(S), \\
\mathcal{K}^{(1)}_N,\ \mathcal{K}^{(2)}_N &: C^{k+\alpha}(S) \to C^{k+\alpha}(S), \\
\mathcal{L}_N &: C^{k+1+\alpha}(S) \to C^{k+\alpha}(S),
\end{aligned}
$$

where N *is either* $\mathcal{A}$ *or* A*, are bounded.*

LEMMA 2.3 [*18*], [*28*], [*27*] *Let* k, α, α', g, $\tilde{h}$ *and* S *be the same as in Lemma 2.2. Then:*

i) *the dominant singular parts of the scalar operators of steady oscillations* $\mathcal{H}_{\mathcal{A}}$, $\mp 2^{-1}I_1 + \mathcal{K}^{(1)}_{\mathcal{A}}$, $\pm 2^{-1}I_1 + \mathcal{K}^{(2)}_{\mathcal{A}}$, $\mathcal{L}_{\mathcal{A}}$ *are the corresponding static operators defined by the equations, respectively:*

$$
\begin{aligned}
\mathcal{H}_a g(z) &:= [V_a(g)(z)]^\pm = \int\limits_S \gamma_a(x-y)g(y)dS_y, \qquad (2.16) \\
(\mp 2^{-1}I_1 + \mathcal{K}_a)g(z) &:= [\nu(D_z,n(z))V_a(g)(z)]^\pm
\end{aligned}
$$

$$
\begin{aligned}
&= \mp 2^{-1} g(z) + \int_S \nu(D_z, n(z)) \gamma_a(z-y) g(y) dS_y, &(2.17)\\
(\pm 2^{-1} I_1 + \mathcal{K}_a^*) g(z) &:= [W_a(g)(z)]^{\pm} = \pm 2^{-1} g(z) \\
&+ \int_S \nu(D_y, n(y)) \gamma_a(y-z) g(y) dS_y, &(2.18)\\
\mathcal{L}_a g(z) &:= [\nu(D_z, n(z)) W_a(g)(z)]^{\pm}; &(2.19)
\end{aligned}
$$

ii) *the dominant singular parts of the matrix operators of thermoelastic oscillations* $\mp 2^{-1} I_4 + \mathcal{K}_A^{(1)}$, $\mathcal{H}_A$, $\pm 2^{-1} I_4 + \mathcal{K}_A^{(2)}$, *and* $\mathcal{L}_A$ *are the following static operators, respectively*:

$$
\begin{bmatrix} [\mp 2^{-1} I_3 + \mathcal{K}_C]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mp 2^{-1} I_1 + \mathcal{K}_\Lambda \end{bmatrix}_{4\times 4}, \quad \begin{bmatrix} [\mathcal{H}_C]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{H}_\Lambda \end{bmatrix}_{4\times 4},
$$

$$
\begin{bmatrix} [\pm 2^{-1} I_3 + \mathcal{K}_C^*]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \pm 2^{-1} I_1 + \mathcal{K}_\Lambda^* \end{bmatrix}_{4\times 4}, \quad \begin{bmatrix} [\mathcal{L}_C]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_\Lambda \end{bmatrix}_{4\times 4},
$$

where

$$
\begin{aligned}
\mathcal{H}_C \tilde{h}(z) &:= [V_C(\tilde{h})(z)]^{\pm} = \int_S \Gamma_C(z-y) \tilde{h}(y) dS_y, \\
[\mp 2^{-1} I_3 + \mathcal{K}_C] \tilde{h}(z) &:= [T(D_z, n(z)) V_C(\tilde{h})(z)]^{\pm} \\
&= \mp 2^{-1} \tilde{h}(z) + \int_S T(D_z, n(z)) \Gamma_C(z-y) \tilde{h}(y) dS_y, \\
[\pm 2^{-1} I_3 + \mathcal{K}_C^*] \tilde{h}(z) &:= [W_C(\tilde{h})(z)]^{\pm} \\
&= \pm 2^{-1} \tilde{h}(z) + \int_S T(D_y, n(y)) \Gamma_C(y-z) \tilde{h}(y) dS_y, \\
\mathcal{L}_C \tilde{h}(z) &:= [T(D_z, n(z)) W_C(\tilde{h})(z)]^{\pm};
\end{aligned}
$$

the boundary operators $\mathcal{H}_\Lambda$, $\mp 2^{-1} I_1 + \mathcal{K}_\Lambda$, $\pm 2^{-1} I_1 + \mathcal{K}_\Lambda^*$ *and* $\mathcal{L}_\Lambda$ *are defined by* (2.16)–(2.19) *with* a *and* $\nu(D, n)$ *replaced by* Λ *and* $\lambda(D, n)$, *respectively*;

iii) $\mathcal{H}_N$, $(\mp 2^{-1} I_n + \mathcal{K}_N)$, $(\pm 2^{-1} I_n + \mathcal{K}_N^*)$ *and* $\mathcal{L}_N$, ($n = 1$ *for* $N = \mathcal{A}, a, \Lambda$, $n = 3$ *for* $N = C$ *and* $n = 4$ *for* $N = A$) *are elliptic pseudodifferential operators* (ΨDOs) *of order* $-1, 0, 0$ *and* 1, *respectively, with the index zero.*

Moreover, $\mathcal{K}_N$ *and* $\mathcal{K}_N^*$ *are formally mutually adjoint singular integral operators* (*SIO*), *while* $-\mathcal{H}_N$ *and* $\mathcal{L}_N$ *are formally self-adjoint non-negative operators with positive definite principal homogeneous symbols for* $N = C, a, \Lambda$;

iv) *the operators* $\mathcal{H}_N$, $2^{-1} I_n + \mathcal{K}_N$, $-2^{-1} I_n + \mathcal{K}_N + \mathcal{H}_N$, $\mathcal{L}_N + (2^{-1} I_n + \mathcal{K}_N)$ *and their adjoints have the trivial null-spaces for* $N = a$, Λ *and* $N = C$.

The principal homogeneous symbols (symbol matrices) of the above operators can be written explicitly in terms of curve integrals (for details see [18], Section 5).

3 Existence results

3.1. Reduction to BIEs. We start with Problem $(P_3^{(\omega,r)})$. It is obvious that by virtue of (1.24) and (1.26), if we formally put $d_3' = 0$ in the formulation of Problem $(P_3^{(\omega,r)})$, we get then Problem $(P_1^{(\omega,r)})$, while $d_3'' = 0$ gives Problem $(P_2^{(\omega,r)})$. From now on, without loss of generality, we assume that $\omega > 0$ and we look for a solution (U, w) in the form of single and double layer potentials

$$U(x) = V_A(h)(x), \quad x \in \Omega^+ = \Omega_1, \tag{3.1}$$

$$w(x) = W_{\mathcal{A}}(g)(x) + p_0 V_{\mathcal{A}}(g)(x), \quad x \in \Omega^- = \Omega_2, \tag{3.2}$$

where $h = (\tilde{h}^\top, h_4)^\top$, $\tilde{h} = (h_1, h_2, h_3)$ and g are unknown densities, and $p_0 = p_1 + ip_2$ is a complex number with $p_2 \neq 0$.

We note that the single layer potential (3.1) is constructed by the fundamental matrix $\Gamma_A(x, \omega, 1)$ (since the case $\omega > 0$ is treated), while the potentials in the representation (3.2) are constructed by means of the fundamental solution $\gamma_{\mathcal{A}}(x, \omega, r)$.

Due to the interface conditions (1.23)–(1.25) and Lemma 2.2, we obtain the following system of ΨDEs for the unknown densities on S:

$$\begin{aligned}
\{[-2^{-1}I_4 + \mathcal{K}_A^{(1)}]h\}_k - d_1 n_k[-2^{-1}I_1 + \mathcal{K}_{\mathcal{A}}^{(2)} + p_0\mathcal{H}_{\mathcal{A}}]g &= f_k, \quad k = 1, 2, 3, \\
d_3'\{[-2^{-1}I_4 + \mathcal{K}_A^{(1)}]h\}_4 + d_3''\{\mathcal{H}_A h\}_4 &= f_4(z), \\
\sum_{j=1}^{3}\{\mathcal{H}_A h\}_j n_j - d_2\{\mathcal{L}_{\mathcal{A}} + p_0[2^{-1}I_1 + \mathcal{K}_{\mathcal{A}}^{(1)}]\}g &= f_5.
\end{aligned} \tag{3.3}$$

Put $\varphi = (\tilde{h}^\top, h_4, g)^\top$ and $\psi = (f_1, ..., f_5)^\top$, and rewrite (3.3) in the matrix form

$$\mathcal{P}_3\varphi = \psi$$

where

$$\mathcal{P}_3 = \begin{bmatrix} [(-2^{-1}I_4 + \mathcal{K}_A^{(1)})_{pq}]_{3\times4} & [-d_1 n_k\, \mathcal{S}_{\mathcal{A}}^{(0)}]_{3\times1} \\ [\{d_3'(-2^{-1}I_4 + \mathcal{K}_A^{(1)}) + d_3''\mathcal{H}_A\}_{4q}]_{1\times4} & 0 \\ [\sum_{j=1}^3 n_j(\mathcal{H}_A)_{jq}]_{1\times4} & -d_2\, \mathcal{S}_{\mathcal{A}}^{(1)} \end{bmatrix}_{5\times5} \tag{3.4}$$

with

$$\mathcal{S}_{\mathcal{A}}^{(0)} = -2^{-1}I_1 + \mathcal{K}_{\mathcal{A}}^{(2)} + p_0\mathcal{H}_{\mathcal{A}} \quad \text{and} \quad \mathcal{S}_{\mathcal{A}}^{(1)} = \mathcal{L}_{\mathcal{A}} + p_0(2^{-1}I_1 + \mathcal{K}_{\mathcal{A}}^{(1)}). \tag{3.5}$$

It is evident that because of Lemma 2.2, item iii) $\mathcal{P}_3$ possesses the mapping property

$$\mathcal{P}_3 : [C^{k+1+\alpha}(S)]^5 \longrightarrow [C^{k+1+\alpha}(S)]^4 \times [C^{k+\alpha}(S)] \tag{3.6}$$

provided $S \in C^{k+2+\alpha'}$with $0 < \alpha < \alpha' < 1$. Denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ the boundary ΨDOs generated by the representations (3.1)–(3.2) in the case of Problems $(P_1^{(\omega,r)})$ and $(P_2^{(\omega,r)})$. From the above arguments it follows that $\mathcal{P}_1$ and $\mathcal{P}_2$ are represented by (3.4) with $d_3' = 0$,

$d_3'' = 1$, and $d_3' = 1$, $d_3'' = 0$, respectively. Therefore $\mathcal{P}_2$ possesses the mapping property (3.6) as well, while we have for $\mathcal{P}_1$

$$\mathcal{P}_1 : [C^{k+1+\alpha}(S)]^5 \to [C^{k+1+\alpha}(S)]^3 \times [C^{k+2+\alpha}(S)] \times [C^{k+\alpha}(S)].$$

3.2. Two auxiliary lemmata. We present now two lemmata which will essentially be used in the study of properties of the boundary operators $\mathcal{P}_l$.

LEMMA 3.1 *Let $g \in C^{1+\alpha}(S)$ with $S \in C^{2+\alpha'}$, $0 < \alpha < \alpha' < 1$, and let w be given by formula (3.2) with p_0 as above. Moreover, let w vanish in Ω^-. Then $g = 0$ on S.*

Proof. Due to Lemma 2.2, item ii), we have

$$[w(z)]^+ - [w(z)]^- = g(z), \quad [\nu(D_z, n(z))w(z)]^+ - [\nu(D_z, n(z))w(z)]^- = -p_0 g(z), \tag{3.7}$$

whence, by assumption of the lemma, we derive

$$[\nu(D_z, n(z))w(z)]^+ + p_0[w(z)]^+ = 0, \quad z \in S. \tag{3.8}$$

Since w is a regular solution of equation (1.10) in Ω^+, by applying the divergence theorem for w and $\overline{w}$, we obtain

$$\int_{\Omega^+} [a_{pq} D_p w \overline{D_q w} - \rho_2 \omega^2 |w|^2] \, dx = \int_S [\nu(D, n)w]^+ [\overline{w}]^+ \, dS.$$

This equation together with the conditions (3.8) and (1.9) implies

$$\mathrm{Im} \left\{ p_0 \int_S |[w]^+|^2 dS \right\} = p_2 \int_S |[w]^+|^2 dS = 0.$$

Now the proof immediately follows from the condition $p_2 \neq 0$ and equation (3.7). ∎

REMARK 3.2 *In fact, we have proved that the homogeneous interior boundary value problem (1.10) and (3.8) with $p_0 = p_1 + \mathrm{i} p_2$, $p_2 \neq 0$, has only the trivial solution in the class of regular functions.*

The next lemma gives the explicit form for the equivalent pseudodifferential scalar lifting operators in the case of an arbitrary surface S.

LEMMA 3.3 [27], [17] *Let*

$$\mathcal{R}_{(-1)} g(z) = \frac{1}{2\pi} \int_S \frac{1}{|z-y|} g(y) dS_y, \tag{3.9}$$

$$\begin{aligned} \mathcal{R}_{(1)} g(z) &= \frac{1}{2\pi} \int_S \sum_{j=1}^{3} \left[\tau_j(D_z, n(z), S) \frac{1}{|z-y|} \right] [\tau_j(D_y, n(y), S) g(y)] dS_y \\ &+ \left\{ -g(z) + \frac{1}{2\pi} \int_S \frac{\partial}{\partial n(y)} \frac{1}{|z-y|} g(y) dS_y \right\}, \quad z \in S, \end{aligned} \tag{3.10}$$

where $\partial/\partial n$ denotes the usual directional normal derivative and $\tau_j(D_z, n(z), S)$, $j=1,2,3$, stand for the directional derivatives in the tangent plane to S at the point z :

$$\tau_1 = n_2 D_3 - n_3 D_2,\ \ \tau_2 = n_3 D_1 - n_1 D_3,\ \ \tau_3 = n_1 D_2 - n_2 D_1.$$

Then

i) *the operators $\mathcal{R}_{(m)} : C^{k+1+\alpha}(S) \to C^{k+1-m+\alpha}(S)$ with $S \in C^{k+2+\alpha'}$ and their formal adjoint ones are isomorphisms between $C^{k+1+\alpha}(S)$ and $C^{k+1-m+\alpha}(S)$;*

ii) *the principal homogeneous symbol of the operator $\mathcal{R}_{(m)}$ at any point $z \in S$ is $|\tilde{\xi}|^m$, where $\tilde{\xi} \in I\!R^2 \setminus \{0\}$; moreover, $\mathcal{R}_{(-1)}$ is formally self-adjoint;*

iii) *$\mathcal{R}_{(m)}$ is an equivalent lifting pseudodifferential operator of order m on S.*

3.3. Properties of operators $\mathcal{P}_l$. Here we will analyse the solvability of the boundary pseudodiferential equation

$$\mathcal{P}_l \varphi = \psi \tag{3.11}$$

which corresponds to Problem $(P_l^{(\omega,r)})$ due to the representations (3.1) and (3.2). Throughout the subsection we assume that the conditions

$$S \in C^{2+k+\alpha'}, \quad 0 < \alpha < \alpha' < 1, \quad k \geq 0, \quad \omega \notin J, \tag{3.12}$$

are fulfilled.

LEMMA 3.4 *The homogeneous version of equation* (3.11) ($\psi = 0$) *has only the trivial solution in the class $[C^\alpha(S)]^4 \times [C^{1+\alpha}(S)]$.*

Proof. Let $\varphi = (\tilde{h}^\top, h_4, g)^\top \in [C^\alpha(S)]^4 \times [C^{1+\alpha}(S)]$ be some solution of the homogeneous equation (3.11) ($\psi = 0$). Then the pair (U, w), constructed by the solution φ via the representations (3.1) and (3.2), is a regular solution to the homogeneous Problem $(P_l^{(\omega,r)})$. Therefore, by the uniqueness Theorem 1.4, we deduce $U(x) = V_A(h)(x) = 0$ in Ω^+ and $w(x) = w_{\mathcal{A}}(g)(x) + p_0 V_{\mathcal{A}}(g)(x) = 0$ in Ω^-. By Lemma 3.1 in view of the last equation we derive $g = 0$ on S.

Further, we have $U(x) = V_A(h)(x) \in C^{1+\alpha}(\overline{\Omega^-}) \cap SK_1^m(\Omega^-)$ and $[V_A(h)]^- = 0$ on S (due to Lemma 2.2, item ii). Theorem 2.1 then implies $V_A(h)(x) = 0$ in Ω^-, and, consequently, $h = 0$ on S. ∎

Now let us introduce the following diagonal matrix operators

$$Q_1 = \begin{bmatrix} [I_3]_{3\times3} & [0]_{3\times1} & [0]_{3\times1} \\ [0]_{1\times3} & \mathcal{R}_{(1)} & 0 \\ [0]_{1\times3} & 0 & \mathcal{R}_{(-1)} \end{bmatrix}_{5\times5}, \quad Q_2 = Q_3 = \begin{bmatrix} [I_4]_{4\times4} & [0]_{4\times1} \\ [0]_{1\times4} & \mathcal{R}_{(-1)} \end{bmatrix}_{5\times5}, \tag{3.13}$$

with $\mathcal{R}_{(m)}$ defined by (3.9) and (3.10).

From (3.4) and (3.13) we have:

$$Q_1\mathcal{P}_1 = \begin{bmatrix} [(-2^{-1}I_4+\mathcal{K}_A^{(1)})_{pq}]_{3\times 4} & [-d_1 n_k\,\mathcal{S}_{\mathcal{A}}^{(0)}]_{3\times 1} \\ [\mathcal{R}_{(1)}\mathcal{H}_{A4q}]_{1\times 4} & 0 \\ [\mathcal{R}_{(-1)}\sum_{j=1}^{3} n_j(\mathcal{H}_A)_{jq}]_{1\times 4} & -d_2\mathcal{R}_{(-1)}\,\mathcal{S}_{\mathcal{A}}^{(1)} \end{bmatrix}_{5\times 5}, \tag{3.14}$$

$$Q_l\mathcal{P}_l = \begin{bmatrix} [(-2^{-1}I_4+\mathcal{K}_A^{(1)})_{pq}]_{3\times 4} & [-d_1 n_k\mathcal{S}_{\mathcal{A}}^{(0)}]_{3\times 1} \\ [(d_3'(-2^{-1}I_4+\mathcal{K}_A^{(1)})+d_3''\mathcal{H}_A)_{4q}]_{1\times 4} & 0 \\ [\mathcal{R}_{(-1)}\sum_{j=1}^{3} n_j(\mathcal{H}_A)_{jq}]_{1\times 4} & -d_2\mathcal{R}_{(-1)}\mathcal{S}_{\mathcal{A}}^{(1)} \end{bmatrix}_{5\times 5}, \tag{3.15}$$

where $l=2,3$ ($d_3'=1$ and $d_3''=0$ for $l=2$); here $\mathcal{S}_{\mathcal{A}}^{(0)}$ and $\mathcal{S}_{\mathcal{A}}^{(1)}$ are defined by (3.5).

LEMMA 3.5 *Operators $Q_l\mathcal{P}_l$ ($l=1,2,3$) are singular integral operators of normal type with the index equal to zero. Moreover,*

i) *equations $Q_l\mathcal{P}_l\varphi = Q_l\psi$ and $\mathcal{P}_l\varphi=\psi$ are equivalent under the following assumptions on ψ:*

$$\psi\in[C^{k+\alpha}(S)]^3\times[C^{k+1+\alpha}(S)]\times[C^{k+\alpha}(S)] \ \text{ for } \ l=1,$$
$$\psi\in[C^{k+\alpha}(S)]^5 \ \text{ for } \ l=2,3;$$

ii) *null-spaces of the operators $Q_l\mathcal{P}_l$ are trivial.*

Proof. The fact that $Q_l\mathcal{P}_l\,(l=1,2,3)$ are singular integral operators (i.e., ΨDOs of order 0) is a ready consequence of Lemma 3.3 and formulae (3.13)–(3.15).

For a scalar (matrix) pseudodifferential operator $\mathcal{N}$ on S we denote by $\boldsymbol{\sigma}(\mathcal{N})(z,\tilde{\xi})$ the corresponding principal homogeneous symbol (symbol matrix) with $z\in S$ and $\tilde{\xi}\in I\!R^2\setminus\{0\}$. Further, we show that $Q_l\mathcal{P}_l$ are elliptic ΨDOs, that is, their principal (homogeneous of order 0) symbol matrices are not singular.

Applying Lemmata 2.3 and 3.3, equations (3.14) and (3.15), and the theorem on a principal symbol of composition of ΨDOs [14], we arrive at the formulae

$$\boldsymbol{\sigma}(Q_1\mathcal{P}_1) = [\{\boldsymbol{\sigma}(Q_1\mathcal{P}_1)\}_{kj}]_{5\times 5}$$
$$= \begin{bmatrix} [\boldsymbol{\sigma}(-2^{-1}I_3+\mathcal{K}_C)]_{3\times 3} & [0]_{3\times 1} & [-d_1 n_k(z)\boldsymbol{\sigma}(-2^{-1}I_1+\mathcal{K}_a^*)]_{3\times 1} \\ [0]_{1\times 3} & |\tilde{\xi}|\boldsymbol{\sigma}(\mathcal{H}_\Lambda) & 0 \\ [0]_{1\times 3} & 0 & -d_2|\tilde{\xi}|^{-1}\boldsymbol{\sigma}(\mathcal{L}_a) \end{bmatrix},$$

$$\boldsymbol{\sigma}(Q_l\mathcal{P}_l) = [\{\boldsymbol{\sigma}(Q_l\mathcal{P}_l)\}_{kj}]_{5\times 5}$$

$$= \begin{bmatrix} [\boldsymbol{\sigma}(-2^{-1}I_3+\mathcal{K}_C)]_{3\times 3} & [0]_{3\times 1} & [-d_1 n_k(z)\boldsymbol{\sigma}(-2^{-1}I_1+\mathcal{K}_a^*)]_{3\times 1} \\ [0]_{1\times 3} & d_3'\boldsymbol{\sigma}(-2^{-1}I_1+\mathcal{K}_\Lambda) & 0 \\ [0]_{1\times 3} & 0 & -d_2|\tilde{\xi}|^{-1}\boldsymbol{\sigma}(\mathcal{L}_a) \end{bmatrix}$$

where $l=2,3$.

Due to the homogeneity of the above matrices it is sufficient to consider only the case $|\tilde{\xi}| = 1$. The latter equations yield

$$\begin{aligned}
\det \sigma(Q_1\mathcal{P}_1) &= -d_2\sigma(\mathcal{H}_\Lambda)\sigma(\mathcal{L}_a)\det\sigma(-2^{-1}I_3 + \mathcal{K}_C), && (3.16)\\
\det \sigma(Q_2\mathcal{P}_2) &= -d_2\,\sigma(-2^{-1}I_1 + \mathcal{K}_\Lambda)\,\sigma(\mathcal{L}_a)\det\sigma(-2^{-1}I_3 + \mathcal{K}_C), \\
\det \sigma(Q_3\mathcal{P}_3) &= -d_2\,d_3'\,\sigma(-2^{-1}I_1 + \mathcal{K}_\Lambda)\,\sigma(\mathcal{L}_a)\det\sigma(-2^{-1}I_3 + \mathcal{K}_C). && (3.17)
\end{aligned}$$

From Lemma 2.3, item iii) it follows that $\sigma(Q_l\mathcal{P}_l)$, $(l = 1, 2, 3)$ are non-singular matrices, since all the multipliers in the right-hand sides of the above equations do not vanish anywhere for $z \in S$ and $|\tilde{\xi}| = 1$. Thus, the operators $Q_l\mathcal{P}_l$ are the SIOs of normal type [22], [24].

To study the index problem, let us note that the dominant singular parts of the operators $Q_1\mathcal{P}_1$ and $\mathcal{N}_1$, where

$$\mathcal{N}_1 := \begin{bmatrix} [-2^{-1}I_3 + \mathcal{K}_C + \mathcal{H}_C]_{3\times3} & [0]_{3\times1} & [-d_1 n_k(z)(-2^{-1}I_1 + \mathcal{K}_a^*)]_{3\times1} \\ [0]_{1\times3} & \mathcal{R}_{(1)}\mathcal{H}_\Lambda & [0] \\ [0]_{1\times3} & 0 & -d_2\mathcal{R}_{(-1)}[\mathcal{L}_a + (2^{-1}I_1 + \mathcal{K}_a^*)] \end{bmatrix}_{5\times5},$$

are the same, i.e., $\sigma(\mathcal{N}_1) = \sigma(Q_1\mathcal{P}_1)$. Moreover, they have the same index since the index is invariant to a compact perturbation (see [22], [24]).

Now we will prove that the homogeneous equation

$$\mathcal{N}_1\mu = 0 \quad \text{with} \quad \mu = (\mu_1, ..., \mu_5)^\top, \tag{3.18}$$

and its adjoint one

$$\mathcal{N}_1^*\mu^* = 0 \quad \text{with} \quad \mu^* = (\mu_1^*, ..., \mu_5^*)^\top, \tag{3.19}$$

have only the trivial solution.

Applying the embedding theorems (see, e.g. [22], Ch.IV) we conclude that the solutions μ and μ^* of (3.18) and (3.19), respectively, belong to the space $C^{1+\alpha}(S)$. Further, the equation (3.18) is equivalent to the following system:

$$\begin{gathered}
[-2^{-1}I_3 + \mathcal{K}_C + \mathcal{H}_C]\tilde{\mu} - d_1 n(z)(-2^{-1}I_1 + \mathcal{K}_a^*)\mu_5 = 0,\\
\mathcal{H}_\Lambda\mu_4 = 0, \quad [\mathcal{L}_a + (2^{-1}I_1 + \mathcal{K}_a^*)]\mu_5 = 0,
\end{gathered}$$

where $\tilde{\mu} = (\mu_1, \mu_2, \mu_3)^\top$, since $\mathcal{R}_{(1)}$ and $\mathcal{R}_{(-1)}$ have the trivial null-spaces (see Lemma 3.3, item i). From Lemma 2.3, item iv) it then follows that $\mu_5 = 0$, $\mu_4 = 0$ and $\tilde{\mu} = 0$, i.e., equation (3.22) possesses only the trivial solution.

It is also obvious that (3.19) is equivalent to the system of equations

$$\begin{gathered}
(-2^{-1}I_3 + \mathcal{K}_C^* + \mathcal{H}_C)\tilde{\mu}^* = 0, \quad \mathcal{H}_\Lambda\mathcal{R}_{(1)}^*\tilde{\mu}_4^* = 0,\\
-d_1(-2^{-1}I_1 + \mathcal{K}_a)(\tilde{\mu}^* \cdot n) - d_2[\mathcal{L}_a + (2^{-1}I_1 + \mathcal{K}_a)]\mathcal{R}_{(-1)}\tilde{\mu}_5^* = 0,
\end{gathered}$$

where $\tilde{\mu}^* = (\tilde{\mu}_1^*, \tilde{\mu}_2^*, \tilde{\mu}_3^*)^\top$.

Now Lemmata 2.3, item iv) and 3.3 imply that $\tilde{\mu}^* = 0$. Thus, equation (3.19) has only the trivial solution as well. Therefore, $\mathrm{Ind}\,\mathcal{N}_1 = \mathrm{Ind}(Q_1\mathcal{R}_1) = 0$. The equalities $\mathrm{Ind}(Q_l\mathcal{R}_l) = 0$ $(l = 2, 3)$ can be shown by quite the same arguments.

To finish the prove we note that the operators

$$Q_1 : [C^{k+\alpha}(S)]^3 \times [C^{k+1+\alpha}(S)] \times [C^{k+\alpha}(S)] \to [C^{k+\alpha}(S)]^4 \times [C^{k+1+\alpha}(S)],$$

$$Q_2,\ Q_3 : [C^{k+\alpha}(S)]^5 \to [C^{k+\alpha}(S)]^4 \times [C^{k+1+\alpha}(S)],$$

are isomorphisms due to Lemma 3.3. item i). Moreover, $Q_l\psi \in [C^{k+\alpha}(S)]^4 \times [C^{k+1+\alpha}(S)]$ implies that $\varphi \in [C^{k+\alpha}(S)]^4 \times [C^{k+1+\alpha}(S)]$. Now, the assertions i) and ii) of the lemma follow from Lemma 3.4. The proof is completed. ∎

COROLLARY 3.6 *Let the conditions* (3.12) *be fulfilled and let* ψ *be as in Lemma* 3.5. *Then the* ΨDE (3.11)*, which is equivalent to the singular integral equation* $Q_l\mathcal{P}_l\varphi = Q_l\psi$*, is uniquely solvable in the class* $[C^{k+\alpha}(S)]^4 \times [C^{k+1+\alpha}(S)]$.

Proof. It easily follows from Lemma 3.5 and Fredholm theorems for SIEs of normal type on closed manifolds [22], [24]. ∎

3.4. Existence theorems. We now can formulate the main results of this paper concerning the existence and regularity of solutions to the original interface problems.

THEOREM 3.7 *Let the condition* (3.12) *be fulfilled and let*

$$f_1,\ f_2,\ f_3,\ f_5 \in C^{k+\alpha}(S), \quad f_4 \in C^{k+1+\alpha}(S).$$

Then Problem $(P_1^{(\omega,r)})$ *has a unique regular solution* (U, w) *representable in the form of surface potentials* (3.1) *and* (3.2)*, where the densities* h *and* g *are defined by the uniquely solvable* ΨDE (3.3) *with* $d_3' = 0$ *and* $d_3'' = 1$. *Moreover,* $U \in C^{k+1+\alpha}(\overline{\Omega^+})$ *and* $w \in C^{k+1+\alpha}(\overline{\Omega^-}) \cap S_r(\Omega^-)$.

THEOREM 3.8 *Let the conditions* (3.12) *be fulfilled and* $f_j \in C^{k+\alpha}(S)$, $j = \overline{1,5}$. *Then Problem* $(P_l^{(\omega,r)})$ $(l = 2, 3)$ *has a unique regular solution* (U, w) *representable in the form of surface potentials* (3.1) *and* (3.2)*, where the densities* h *and* g *are defined by the uniquely solvable* ΨDE (3.3) (*with* $d_3' = 1$ *and* $d_3'' = 0$ *for* $l = 2$). *Moreover,* $U \in C^{k+1+\alpha}(\overline{\Omega^+})$ *and* $w \in C^{k+1+\alpha}(\overline{\Omega^-}) \cap S_r(\Omega^-)$.

Proof of these theorems readily follows from the results of the previous Subsection 3.3.

3.5. An alternative approach. If we look for a solution (U, w) to Problem $(P_l^{(\omega,r)})$ in the form of potentials

$$\begin{aligned} U(x) &= W_A(h)(x), \quad x \in \Omega^+ = \Omega_1, && (3.20)\\ w(x) &= W_{\mathcal{A}}(g)(x) + p_0 V_{\mathcal{A}}(g)(x), \quad x \in \Omega^- = \Omega_2, && (3.21) \end{aligned}$$

we arrive then at the following system of ΨDEs on S for the unknown densities $h = (h_1, ..., h_4)^\top$ and g:

$$\begin{aligned} [\mathcal{L}_A h]_k - d_1 n_k[-2^{-1}I_1 + \mathcal{K}_{\mathcal{A}}^{(2)} + p_0\mathcal{H}_{\mathcal{A}}]g &= \psi_k, \quad k = 1, 2, 3,\\ d_3'[\mathcal{L}_A h]_4 + d_3''\{[2^{-1}I_4 + \mathcal{K}_A^{(2)}]h\}_4 &= \psi_4,\\ \sum_{j=1}^{3} n_j\{[2^{-1}I_4 + \mathcal{K}_A^{(2)}]h\}_j - d_2\{\mathcal{L}_{\mathcal{A}} + p_0(2^{-1}I_1 + \mathcal{K}_{\mathcal{A}}^{(1)})\}g &= \psi_5; \qquad (3.22) \end{aligned}$$

here $d_3' = 0$ and $d_3'' = 1$ for $l = 1$, $d_3' = 1$ and $d_3'' = 0$ for $l = 2$, and $d_3' \, d_3'' \neq 0$ for $l = 3$.

Denote the matrix operator generated by the left-hand side of (3.22) by

$$\widetilde{\mathcal{P}}_l = \begin{bmatrix} [(\mathcal{L}_A)_{pq}]_{3\times 4} & [-d_1 n_k \, \mathcal{S}_{\mathcal{A}}^{(0)}]_{3\times 1} \\ [\{d_3' \mathcal{L}_A + d_3''(2^{-1} I_4 + \mathcal{K}_A^{(2)})\}_{4q}]_{1\times 4} & 0 \\ [\sum_{j=1}^{3} n_j (2^{-1} I_4 + \mathcal{K}_A^{(2)})_{jq}]_{1\times 4} & -d_2 \mathcal{S}_{\mathcal{A}}^{(1)} \end{bmatrix}_{5\times 5}, \tag{3.23}$$

where $\mathcal{S}_{\mathcal{A}}^{(0)}$ and $\mathcal{S}_{\mathcal{A}}^{(1)}$ are defined by (3.5).

Applying the arguments, quite similar to above, we can prove that the operator $\widetilde{\mathcal{P}}_l$ is an elliptic (in the sense of Douglis-Nirenberg) ΨDO with the index equal to zero; moreover, if the conditions (3.12) are fulfilled, then the operators

$$\begin{aligned} \widetilde{\mathcal{P}}_1 \;&:\; [C^{k+1+\alpha}(S)]^5 \to [C^{k+\alpha}(S)]^3 \times [C^{k+1+\alpha}(S)] \times [C^{k+\alpha}(S)], \\ \widetilde{\mathcal{P}}_l \;&:\; [C^{k+1+\alpha}(S)]^5 \to [C^{k+\alpha}(S)]^5 \quad \text{for} \quad l = 2, 3, \end{aligned}$$

are isomorphisms. Therefore, one can easily formulate the analogues of Theorems 7 and 8 (with only very slight modifications related to the representation of a solution by formulae (3.20) and (3.21)).

REMARK 3.9 *Note that the so-called direct boundary integral equation approach leads to the system of boundary ΨDEs of type considered above but with the extended spectral set J_1 with respect to ω: $J_1 = J \cup J_0$, where J is the same as above, while J_0 is the set of eigenvalues of the interior Dirichlet problem for equation* (1.10) *in Ω^+.*

3.6. Discussion on excluded eigenfrequences. Let us examine what happens when $\omega \in J$. It is evident that the homogeneous Problem $(P_l^{(\omega,r)})$ in this case has a nontrivial solution (U, w) with $U = (u, 0)^\top$ in Ω^+ and $w = 0$ in Ω^-, where u solves the homogeneous boundary value problem (1.27) (see Theorem 1.4 and Remark 1.5). The boundary conditions of the problem (1.27) together with the equation $u_4 = 0$ in Ω^+ imply that $[B(D,n)U]^+ = 0$ on S (see (2.15)). Therefore from the general integral representation of a solution of the equation (1.7) (see [18], Section 3)

$$U(x) = W_A([U]^+)(x) - V_A([BU]^+)(x), \quad x \in \Omega^+,$$

it follows that the four-dimensional vector $U = (u, 0)^\top$, where u is an arbitrary solution of the problem (1.27), is representable as a double layer potential $U(x) = W_A(h)(x)$ with the density $h = (\tilde{h}^\top, 0)$, where $\tilde{h} = (u)^+$ and $(\tilde{h} \cdot n) = 0$.

It can also be proved that if $\omega \in J$ and $(h, g)^\top$ is an arbitrary solution of the homogeneous version of system (3.22), then $g = 0$, $h_4 = 0$ and $\tilde{h} = (h_1, h_2, h_3)^\top$ possesses the same properties as above. Note that if $(h, g)^\top$ is an arbitrary solution of the homogeneous system (3.3), we can conclude only that $g = 0$. Therefore the alternative approach described in the previous subsection seems to be more natural for spectral frequences.

Thus, if $\omega \in J$ and the non-homogeneous system (3.3) [(3.22)] is solvable (that is, ψ is orthogonal to the null-space of the corresponding adjoint operator), then the density g of the exterior field w is defined uniquely, while the density h of the interior thermoelastic field U is determined within a summand defining a nontrivial solution of the problem (1.27).

In the wave scattering and fluid-solid interaction problems the vector-function f in the condition (1.23) has the form $f_0 n$, where f_0 is a scalar function and n is the unit normal vector of S at the point $x \in S$ (see [6], [23]).

In what follows we will show that for the vector-function $\psi = (f_0 n, f_4, f_5)^\top$, where f_0 and n are as above, and f_4 and f_5 are arbitrary functions, the interface Problems $(P_l^{(\omega,r)})$ are solvable for all $\omega \in J$ by the both approaches described above.

To this end we have to prove that the vector-function ψ is orthogonal to the null-spaces of the corresponding adjoint operators.

Denote by $\mathcal{P}_l^*$ and $\widetilde{\mathcal{P}}_l^*$ the operators adjoint to $\mathcal{P}_l$ and $\widetilde{\mathcal{P}}_l$, respectively (see (3.4), (3.23)):

$$\begin{gathered}
\mathcal{P}_l^* = [(\mathcal{P}_l^*)_{pq}]_{5\times 5},\\
(\mathcal{P}_l^*)_{pq} = -2^{-1} I_4 + \mathcal{K}_{A^*}^{(2)})_{pq}, \ \ p = \overline{1,4},\ q = \overline{1,3},\\
(\mathcal{P}_l^*)_{p4} = \overline{d_3'}(-2^{-1} I_4 + \mathcal{K}_{A^*}^{(2)})_{p4} + \overline{d_3''}(\mathcal{H}_{A^*})_{p4}, \ \ p = \overline{1,4},\\
(\mathcal{P}_l^*)_{p5} = \textstyle\sum_{j=1}^{3} (\mathcal{H}_{A^*})_{pj} n_j, \ \ p = \overline{1,4},\\
(\mathcal{P}_l^*)_{5q} = -\overline{d}_1(-2^{-1} I_1 + \mathcal{K}_{\mathcal{A}^*}^{(1)} + \overline{p}_0 \mathcal{H}_{\mathcal{A}^*}) n_q, \ \ q = \overline{1,3},\\
(\mathcal{P}_l^*)_{54} = 0, \quad (\mathcal{P}_l)_{55} = -\overline{d}_2 [\mathcal{L}_{\mathcal{A}^*} + \overline{p}_0 (2^{-1} I_1 + \mathcal{K}_{\mathcal{A}^*}^{(2)})];
\end{gathered} \tag{3.24}$$

$$\begin{gathered}
\widetilde{\mathcal{P}}_l^* = [(\widetilde{\mathcal{P}}_l^*)_{pq}]_{5\times 5},\\
(\widetilde{\mathcal{P}}_l^*)_{pq} = (\mathcal{L}_{A^*})_{pq}, \ \ p = \overline{1,4}, \ \ q = \overline{1,3},\\
(\widetilde{\mathcal{P}}_l^*)_{p4} = \overline{d_3'}(\mathcal{L}_{A^*})_{p4} + \overline{d_3''}(2^{-1} I_4 + \mathcal{K}_{A^*}^{(1)})_{p4}, \ \ p = \overline{1,4},\\
(\widetilde{\mathcal{P}}_l^*)_{p5} = \textstyle\sum_{j=1}^{3} (2^{-1} I_4 + \mathcal{K}_{A^*}^{(1)})_{pj} n_j, \ \ p = \overline{1,4},\\
(\widetilde{\mathcal{P}}_l^*)_{5q} = -\overline{d}_1(-2^{-1} I_1 + \mathcal{K}_{\mathcal{A}^*}^{(1)} + \overline{p}_0 \mathcal{H}_{\mathcal{A}^*}) n_q, \ \ q = \overline{1,3},\\
(\widetilde{\mathcal{P}}_l^*)_{54} = 0, \quad (\widetilde{\mathcal{P}}_l^*)_{55} = -\overline{d}_2 [\mathcal{L}_{\mathcal{A}^*} + \overline{p}_0 (2^{-1} I_1 + \mathcal{K}_{\mathcal{A}^*}^{(2)})];
\end{gathered}$$

here the following notations are used:

$$\begin{gathered}
(\pm 2^{-1} I_4 + \mathcal{K}_{A^*}^{(2)}) h(z) := [W_{A^*}(h)(z)]^{\pm}\\
= \pm 2^{-1} h(z) + \int\limits_S \{B(D_y, n(y)) \Gamma_{A^*}^\top (z-y,\omega,r)\}^\top h(y)\, dS_y,\\
\mathcal{H}_{A^*} h(z) := [V_{A^*}(h)(z)]^{\pm} = \int\limits_S \Gamma_{A^*}(z-y,\omega,r) h(y)\, dS_y,\\
(\mp 2^{-1} I_4 + \mathcal{K}_{A^*}^{(1)}) h(z) := [Q(D_z, n(z), \mathrm{i}\omega) V_{A^*}(h)(z)]^{\pm}\\
= \mp 2^{-1} h(z) + \int\limits_S \{Q(D_z, n(z), \mathrm{i}\omega) \Gamma_{A^*}(z-y,\omega,r)\} h(y)\, dS_y,\\
\mathcal{L}_{A^*} h(z) := \lim_{\Omega^{\pm} \ni x \to z \in S} Q(D_x, n(z), \mathrm{i}\omega) W_{A^*}(h)(x),
\end{gathered}$$

with

$$W_{A^*}(h)(x) = \int\limits_S \{B(D_y, n(y)) \Gamma_{A^*}^\top (x-y,\omega,r)\}^\top h(y)\, dS_y, \tag{3.25}$$

$$V_{A^*}(h)(x)=\int\limits_S \Gamma_{A^*}(x-y,\omega,r)h(y)dS_y, \tag{3.26}$$

where $\Gamma_{A^*}(x-y,\omega,r)=\overline{\Gamma_A^\top(y-x,\omega,r)}$ is a fundamental matrix of the adjoint operator $A^*(D,-\mathrm{i}\omega)=\overline{A^\top(-D,-\mathrm{i}\omega)}$ and $h=(\tilde h^\top,h_4)^\top$ with $\tilde h=(h_1,h_2,h_3)^\top$. Since the scalar operator $\mathcal{A}(D,-\mathrm{i}\omega)$ is formally self-adjoint (see (1.10)), we have

$$\mathcal{H}_{\mathcal{A}^*}=\overline{\mathcal{H}}_{\mathcal{A}},\quad \mathcal{K}^{(1)}_{\mathcal{A}^*}=\overline{\mathcal{K}^{(1)}_{\mathcal{A}}},\quad \mathcal{K}^{(2)}_{\mathcal{A}^*}=\overline{\mathcal{K}^{(2)}_{\mathcal{A}}},\quad \mathcal{L}_{\mathcal{A}^*}=\overline{\mathcal{L}}_{\mathcal{A}},$$

and they are generated by the potentials

$$W_{\mathcal{A}^*}(g)(x)=\overline{W}_{\mathcal{A}}(g)(x)=\int\limits_S \nu(D_y,n(y))\overline{\gamma_{\mathcal{A}}(x-y,\omega,r)}\,g(y)dS_y,$$

$$V_{\mathcal{A}^*}(g)(x)=\overline{V}_{\mathcal{A}}(g)(x)=\int\limits_S \overline{\gamma_{\mathcal{A}}(x-y,\omega,r)}\,g(y)dS_y.$$

Note that

$$\mathcal{H}_{A^*}=(\mathcal{H}_A)^*,\quad \mathcal{K}^{(1)}_{A^*}=(\mathcal{K}^{(2)}_A)^*,\quad \mathcal{K}^{(2)}_{A^*}=(\mathcal{K}^{(1)}_A)^*,\quad \mathcal{L}_{A^*}=(\mathcal{L}_A)^*,$$

where $(\mathcal{M})^*$ denotes the operator formally adjoint to $\mathcal{M}$.

LEMMA 3.10 *Let $\varphi^*=(h^{*\top},g^*)^\top$, where $h^*=(\tilde h^{*\top},h_4^*)^\top$ and $\tilde h^*=(h_1^*,h_2^*,h_3^*)^\top$, be a solution to the equation either $\mathcal{P}_l^*\varphi^*=0$ or $\widetilde{\mathcal{P}}_l^*\varphi^*=0$. Then*

$$\tilde h^*\cdot n=0,\quad h_4^*=0,\quad g^*=0, \tag{3.27}$$

provided the conditions (1.28) *are fulfilled.*

Proof. Let $\varphi^*=(h^{*\top},g^*)^\top$ be an arbitrary solution to the equation $\mathcal{P}_l^*\varphi^*=0$. Equation (3.24) then implies

$$\mathcal{P}_l^*\varphi^*=\begin{bmatrix} \{W_{A^*}(\tilde h^*,\overline{d_3'}h_4^*)+V_{A^*}(ng^*,\overline{d_3''}h_4^*)\}^- \\ \{[\nu(D,n)+\overline{p}_0][\overline{V}_{\mathcal{A}}(-\overline{d}_1\tilde h^*\cdot n)+\overline{W}_{\mathcal{A}}(-\overline{d}_2g^*)]\}^+ \end{bmatrix}_{5\times 1}=0. \tag{3.28}$$

In turn, (3.28) yields that the vector-function

$$U^*(x)=(u^{*\top},u_4^*)^\top=W_{A^*}(\tilde h^*,\overline{d_3'}h_4^*)+V_{A^*}(ng^*,\overline{d_3''}h_4^*) \tag{3.29}$$

and the scalar function

$$w^*(x)=\overline{V}_{\mathcal{A}}(-\overline{d}_1\tilde h^*\cdot n)+\overline{W}_{\mathcal{A}}(-\overline{d}_2g^*) \tag{3.30}$$

solve the following BVPs:

$$\begin{gathered} A^*(D,-\mathrm{i}\omega)U^*(x)=0\quad\text{in}\quad\Omega^-,\quad [U^*(z)]^-=0\quad\text{on}\quad S; \\ \mathcal{A}^*(D,-\mathrm{i}\omega)w^*(x)=0\quad\text{in}\quad\Omega^+,\quad \{[\nu(D_z,n(z))+\overline{p}_0]w^*(z)\}^+=0\quad\text{on}\quad S. \end{gathered} \tag{3.31}$$

It can be easily checked that if U^* is a regular solution of BVP (3.31), then the vector $U = (\overline{u^{*\top}}, \mathrm{i}\omega T_0 \overline{u_4^*})$ is a regular (m,r)-thermo-radiating solution to the equation (1.7) in Ω^- and $[U]^- = 0$ on S. Therefore $U(x) = 0$ in Ω^- due to Theorem 2.1. Consequently, $U^*(x) = 0$ in Ω^-. We can also show that $w^*(x) = 0$ in Ω^+ (see the proof of Lemma 3.1).

Further, the properties of the potentials involved in (3.29) and (3.30), together with the results just obtained, lead to the equations

$$
\begin{gathered}
[U^*]^+ = (\tilde{h}^{*\top}, \overline{d_3'} h_4^*)^\top, \quad [Q(D,n,\mathrm{i}\omega)U^*]^+ = -(ng^*, \overline{d_3''} h_4^*)^\top, \\
[w^*]^- = \overline{d}_2 g^*, \quad [\nu(D,n) w^*]^- = -\overline{d}_1 (\tilde{h}^* \cdot n).
\end{gathered} \tag{3.32}
$$

Hence we get the following transmission problem for the vector U^* (defined by (3.29) in Ω^+) and the scalar function w^* (defined by (3.30) in Ω^-):

$$
\begin{gathered}
A^*(D, -\mathrm{i}\omega) U^*(x) = 0 \quad \text{in} \quad \Omega^+, \\
\mathcal{A}^*(D, -\mathrm{i}\omega) w^*(x) = 0 \quad \text{in} \quad \Omega^-, \\
\left.
\begin{aligned}
&(u^* \cdot n)^+ = -(\overline{d}_1)^{-1} [\nu(D,n) w^*]^- \\
&[(T(D,n)u^*)_k + \mathrm{i}\omega T_0 \beta_{kj} n_j u_4^*]^+ = -(\overline{d}_2)^{-1} n_k [w^*]^-, \ k = 1,2,3, \\
&[\overline{d_3'} \lambda(D,n) u_4^* + \overline{d_3''} u_4^*]^+ = 0
\end{aligned}
\right\} \text{ on } S.
\end{gathered} \tag{3.33}
$$

If we introduce the vector $U = (\overline{u^{*\top}}, \mathrm{i}\omega T_0 \overline{u^*}_4)$ and the function $w = \overline{w^*}$, we can reduce the transmission problem (3.33) to the homogeneous Problem $(P_l^{(\omega,r)})$ with constants $d_1' = -d_2^{-1}$ and $d_2' = -d_1^{-1}$ instead of d_1 and d_2 (see (1.23)–(1.25)). Since the constants d_1' and d_2' satisfy the first inequality in (1.28), from (3.33) it follows that $w^*(x) = 0$ in Ω^-, $u_4^*(x) = 0$ in Ω^+ and u^* solves the problem (1.27) (see Remark 1.5). Therefore equations (3.32) imply (3.27).

The proof for the equation $\widetilde{\mathcal{P}}_l^* \varphi^* = 0$ is verbatim. ■

REMARK 3.11 *In fact, we have proved that the only nontrivial part $\tilde{h}^*$ of the nontrivial solution φ^* to the homogeneous equation* (3.28) *coincides with the trace on the boundary S of some Jones mode u^*. The inverse assertion is aslo valid.*

From the above results it follows that the vector-function $\psi = (f_0 n, f_4, f_5)^\top$ with arbitrarily given f_0, f_4 and f_5, is orthogonal (even pointwise) to the null-spaces of the operators $\mathcal{P}_l^*$ and $\widetilde{\mathcal{P}}_l^*$. Therefore the non-homogeneous interface Problem $(P_l^{(\omega,r)})$ with the above boundary data vector-function ψ is always solvable (but not uniquely for $\omega \in J$).

REMARK 3.12 *In the present paper we have confined ourselves to the classical formulation of the transmission problems and obtained the uniqueness and existence results in Hölder spaces. Applying the same analysis as in [17] and [19], we can study the above problems in the Sobolev W_p^1, Bessel-potential H_p^s and Besov $B_{p,q}^s$ spaces by the above approaches and obtain the corresponding uniqueness and existence results for weak solutions.*

APPENDIX: Domains Free of Jones Modes

Here we describe a class of domains with a special geometry for which there exist no Jones modes.

Let Ω^+ be a bounded, simply connected Lipschitz domain with a piecewise C^2-smooth boundary $S = \partial\Omega^+$. We assume that Ω^+ is filled up by an isotropic elastic material with the Lamé constants λ and μ. In this case $\beta_{kj} = (2\mu + 3\lambda)\alpha\delta_{kj}$, where α is a positive constant. Moreover, we have (cf. [22], [30]):

$$C(D)u = \mu\Delta u + (\lambda + \mu)\,\mathrm{grad\,div}\,u, \quad \beta_{kj}D_k u_j = (2\mu + 3\lambda)\alpha\,\mathrm{div}\,u,$$
$$T(D,n)u = 2\mu\partial_n u + \lambda\,\mathrm{div}\,u + \mu[n \times \mathrm{curl}\,u],$$

where Δ is the Laplacian, ∂_n stands for the usual normal derivative and $\cdot \times \cdot$ denotes the vector product.

Therefore the conditions of problem (1.28) for the isotropic case can be written as

$$(\Delta + k_1^2)u(x) = 0 \quad \text{in} \quad \Omega^+, \tag{A.1}$$
$$\mathrm{div}\,u(x) = 0 \quad \text{in} \quad \Omega^+, \tag{A.2}$$
$$\{2\partial_n u(x) + [n(x) \times \mathrm{curl}\,u(x)]\}^+ = 0 \quad \text{on} \quad S, \tag{A.3}$$
$$\{u(x)\cdot n(x)\}^+ = 0 \quad \text{on} \quad S, \tag{A.4}$$

where $k_1^2 = \varrho_1\omega^2\mu^{-1}$.

In what follows we assume that the displacement vector $u = (u_1, u_2, u_3)^\top$ belongs to the Sobolev space $W_2^1(\Omega^+)$. Therefore, due to the regularity theorems of weak solutions (see, e.g. [11]) we have

$$u \in W_2^1(\Omega^+) \cap C^\infty(\Omega^+) \cap C^2(\Omega^+ \cup S^0),$$

where S^0 is the union of C^2-smooth open submanifolds of S. We note that if some open submanifold $S^* \subset S$ is C^∞-smooth then $u \in C^\infty(\Omega^+ \cup S^*)$.

Further, we recall that for the components u_j $(j = 1,2,3)$ of the displacement vector u, which are solutions of the Helmholtz equation (A.1) in the domain Ω^+, there holds the following integral representation formula (see, e.g. [22], [6]):

$$\int\limits_S \Big\{\partial_{n(y)}\gamma(x-y,\omega)\,[u_j(y)]^+ - \gamma(x-y,\omega)\,[\partial_n u_j(y)]^+\Big\}\, dS_y$$
$$= \begin{cases} u_j(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-, \end{cases} \tag{A.5}$$

where $\gamma(x,\omega) = -(4\pi|x|)^{-1}\exp\{\mathrm{i}k|x|\}$ is a fundamental solution of the Helmholtz equation.

THEOREM A.1 *Let the boundary surface $S = \partial\Omega^+$ contain two plane, open submanifolds S_1 and S_2 with non-parallel normal vectors.*

Then the homogeneous problem $(A.1) - (A.4)$ has only the trivial solution.

Proof. Denote by $a^{(1)} = (a_{11}, a_{12}, a_{13})$ and $a^{(2)} = (a_{21}, a_{22}, a_{23})$ the outward unit normal vectors of S_1 and S_2, respectively. Clearly, a_{kj} $(k = 1,2,\ j = 1,2,3)$ are real constants and due to the assumptions of the theorem

$$a := a^{(1)} \times a^{(2)} = (a_1, a_2, a_3) \neq 0. \tag{A.6}$$

Further, we introduce the metaharmonic functions in Ω^+, related to the displacement vector u as follows:

$$v_p(x) = u(x) \cdot a^{(p)} = a_{p1} u_1(x) + a_{p2} u_2(x) + a_{p3} u_3(x), \quad p = 1, 2. \tag{A.7}$$

Obviously,

$$v_p \in W_2^1(\Omega^+) \cap C^\infty(\Omega^+ \cup S_1 \cup S_2) \cap C^2(\Omega^+ \cup S^0),$$

where S^0 is the same as above. It is also evident that the representation formula (A.5) remains valid for v_p $(p = 1, 2)$.

Let us show that v_p $(p = 1, 2)$ vanish in Ω^+. To this end, for definiteness, we put $p = 1$ and rewrite the boundary conditions (A.3) as

$$\{2\partial_n u + [n \times \operatorname{curl} u]\}_q^+ = \left\{ \partial_n u_q + \sum_{j=1}^{3} n_j \, D_q u_j \right\}^+ = 0 \quad \text{on} \quad S, \tag{A.8}$$

where $q = 1, 2, 3$ and $D_q = \partial / \partial x_q$. These equations imply

$$\left\{ \partial_n u_q + D_q \left(\sum_{j=1}^{3} a_{1j} \, u_j \right) \right\}^+ = 0 \quad \text{on} \quad S_1, \quad q = 1, 2, 3,$$

since $n_j = a_{1j}$ on S_1.

By multiplying the q-th equation by $n_q = a_{1q}$ and summing , we get

$$\left\{ \partial_n \left(\sum_{q=1}^{3} a_{1q} \, u_q \right) + \partial_n \left(\sum_{j=1}^{3} a_{1j} \, u_j \right) \right\}^+ = 0 \quad \text{on} \quad S_1,$$

i.e.,

$$\{\partial_n v_1(x)\}^+ = 0 \quad \text{on} \quad S_1.$$

Moreover, from (A.4) we have

$$\{v_1(x)\}^+ = 0 \quad \text{on} \quad S_1.$$

Therefore, due to the integral representation formula (A.5), we arrive at the equation for v_1

$$\int\limits_{S \setminus S_1} \left\{ \partial_{n(y)} \gamma(x - y, \omega) \, [v_1(y)]^+ - \gamma(x - y, \omega) \, [\partial_n v_1(y)]^+ \right\} dS_y$$

$$= \begin{cases} v_1(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-. \end{cases} \tag{A.9}$$

Taking into account that the left-hand side integral in (A.9) defines an analytic function of the real variable x in the connected domain $I\!R^3 \setminus \overline{S_1}$ which vanishes in $\Omega^- \subset I\!R^3 \setminus \overline{S_1}$, we conclude that this integral is zero in the whole domain of analyticity $I\!R^3 \setminus \overline{S_1}$. Hence $v_1(x) = 0$ in Ω^+. By the same arguments we can show that $v_2(x) = 0$ in Ω^+.

Now from (A.7) we see that the displacement vector $u(x)$ for arbitrary $x \in \Omega^+$ is parallel to the constant vector a given by (A.6). Therefore there exists a scalar function $v(x)$ such that

$$u(x) = a\, v(x), \quad \text{i.e.,} \quad u_j(x) = a_j\, v(x) \quad \text{for} \quad x \in \Omega^+, \quad j = 1,2,3. \tag{A.10}$$

It is easy to check that the function v satisfies the conditions (cf. (A.1), (A.3), (A.4), (A.8)):

$$\begin{aligned} (\Delta + k_1^2)v(x) &= 0 \quad \text{in} \quad \Omega^+, \\ (a \cdot n(x))\{v(x)\}^+ &= 0 \quad \text{on} \quad S, \\ \left\{ a_q \partial_n v(x) + \sum_{j=1}^{3} n_j a_j\, D_q v(x) \right\}^+ &= 0 \quad \text{on} \quad S, \quad q = 1,2,3. \end{aligned}$$

The last two equations yield

$$b(x)\{v(x)\}^+ = 0 \quad \text{and} \quad b(x)\{\partial_n v(x)\}^+ = 0 \quad \text{on} \quad S \tag{A.11}$$

where

$$b(x) = a \cdot n(x) = a_1 n_1(x) + a_2 n_2(x) + a_3 n_3(x), \quad x \in S, \tag{A.12}$$

with $n(x)$ being again the outward unit normal to S and a given by (A.6). From (A.12) it follows that there exists an open C^2-smooth submanifold $S^* \subset S$ such that

$$b(x) \neq 0 \quad \text{for} \quad x \in S^*,$$

since the vector a is constant and S is a piecewise C^2-smooth closed surface. Therefore (A.11) implies

$$\{v(x)\}^+ = 0 \quad \text{and} \quad \{\partial_n v(x)\}^+ = 0 \quad \text{on} \quad S^*,$$

whence, by the same arguments as above, we deduce that $v(x) = 0$ in Ω^+. Finally, the equations (A.10) show that the displacement vector $u(x)$ vanishes in Ω^+. ∎

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