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THERMOELASTIC OSCILLATIONS OF ANISOTROPIC BODIES

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Abstract The generalized radiation conditions at infinity of Sommerfeld–Kupradze type are established in the theory of thermoelasticity of anisotropic bodies. Applying the potential method and the theory of pseudodifferential equations on manifolds the uniqueness and existence theorems of solutions to the basic three–dimensional exterior boundary value problems are proved and representation formulas of solutions by potential type integrals are obtained.

INTRODUCTION

Boundary value problems (BVPs) of the theory of thermoelasticity have a long history. They encounter in many applications and mathematical models where the thermal stresses appear (for exhaustive historical and bibliographical material see [12], [18]).

Three–dimensional problems of statics, pseudo–oscillations, general dynamics and steady state oscillations of the thermoelasticity theory of isotropic elastic bodies are completely investigated by many authors (see, e.g., [7], [8], [9], [12], [18] and references therein). In particular, exterior steady state oscillation problems have been studied on the bases of Sommerfeld–Kupradze radiation conditions in the thermoelasticity and uniqueness theorems were proved with the help of a well–known Rellich's lemma, since components of a displacement vector and a temperature in the isotropic case can be represented as a sum of metaharmonic functions (for details see [12]).

Unfortunately the methods of investigation of thermoelastic steady oscillation problems developed for the isotropic case are not applicable in the case of general anisotropy. This is stipulated by a very complicated form of the corresponding characteristic equation which plays a significant role in the study of the far field behaviour of solutions to the oscillation equations (cf. [15], [19]).

We note that the basic and crack type BVPs for the pseudo–oscillation equations of the thermoelasticity theory (the anisotropic case) are considered in [3], [14].

To the best of the author's knowledge the problems of thermoelastic steady oscillations for anisotropic bodies have not been treated in the scientific literature.

In the present paper we will formulate thermoelastic radiation conditions for an anisotropic medium (the generalized Sommerfeld–Kupradze type radiation conditions) and prove the uniqueness theorems in the corresponding spaces. Afterwards the existence of solutions to the basic BVPs will be studied by the potential method and the theory of pseudodifferential equations (ΨDEs) on manifolds and representation formulas of the solutions by potential type integrals will be obtained.

1 CHARACTERISTIC EQUATION

1.1. The system of equations of linear thermoelastodynamics of homogeneous anisotropic elastic medium reads (see [18], Ch. V)

$$\begin{aligned} c_{kjpq}\, D_j D_q\, u_p(x,t) \,+\, X_k(x,t) \,=\, \varrho D_t^2 u_k(x,t) \,+\, \beta_{kj} D_j u_4(x,t), \\ \lambda_{pq} D_p D_q u_4(x,t) \,-\, c_0 D_t u_4(x,t) \,-\, T_0 \beta_{pq} D_t D_p u_q(x,t) \,=\, -Q(x,t), \end{aligned} \tag{1.1}$$

where $c_{kjpq} = c_{pqkj} = c_{jkpq}$ are elastic constants, $\lambda_{pq} = \lambda_{qp}$ are heat conductivity coefficients, c_0 is the thermal capacity, T_0 is the temperature of the medium in the natural state, $\beta_{pq} = \beta_{qp}$ are expressed in terms of the thermal and elastic constants, ϱ is the density of the medium; $u = (u_1, u_2, u_3)^\top$ is the displacement vector, u_4 is the temperature, $X = (X_1, X_2, X_3)^\top$ is the bulk force, Q is the heat source; $x = (x_1, x_2, x_3)$ denotes the spatial variable, while t is the time variable; $D_p = D_{x_p} = \partial/\partial x_p,\ \ D_t = \partial/\partial t$; here and in what follows the summation over repeated indices is meant from 1 to 3, unless otherwise stated; the superscript $\top$ denotes transposition.

In the sequel we consider the homogeneous version of equations (1.1), i.e., we assume $X = 0,\ Q = 0$. In addition, without any restriction of generality $\varrho = 1$ is assumed as well.

In (1.1) the term $-T_0 \beta_{pq} D_t D_p u_q(x,t)$ describes the coupling between the temperature and strain fields. It vanishes only for a stationary heat flow. In that case or if this term is neglected, we have the uncoupled thermoelasticity.

In the thermoelasticity theory the stress tensor $\{\sigma_{kj}\}$, the strain tensor $\{\varepsilon_{kj}\}$ and the temperature field u_4 are related by Duhamel–Neumann law

$$\sigma_{kj} = c_{kjpq}\varepsilon_{pq} - \beta_{kj} u_4, \quad \varepsilon_{kj} = 2^{-1}(D_k u_j + D_j u_k), \quad k, j = 1, 2, 3;$$

the $k-$th component of the vector of thermostresses, acting on a surface element with the unit normal vector $n = (n_1, n_2, n_3)$, is calculated by the formula

$$\sigma_{kj} n_j = c_{kjpq}\varepsilon_{pq} n_j - \beta_{kj} n_j u_4 = c_{kjpq} n_j\, D_q u_p - \beta_{kj} n_j u_4, \ \ k = 1, 2, 3. \tag{1.2}$$

The formal Laplace transform of equations (1.1) (with respect to t) leads to the so–called *pseudo-oscillation equations* of the thermoelasticity theory

$$\begin{aligned} c_{kjpq}\, D_j D_q u_p(x,\tau) = \tau^2 u_k(x,\tau) + \beta_{kj} D_j u_4(x,\tau), \\ \lambda_{pq} D_p D_q u_4(x,\tau) - c_0 \tau u_4(x,\tau) - \tau T_0 \beta_{pq} D_p u_q(x,\tau) = 0; \end{aligned} \tag{1.3}$$

here $\tau = \sigma - \mathrm{i}\omega$ is a complex parameter, $\omega \in I\!R$ and $\sigma \in I\!R \setminus \{0\}$.

If all data involved in (1.1) are harmonic time dependent, i.e.,

$$u_k(x,t) = \overset{1}{u}_k(x) \cos\omega t + \overset{2}{u}_k(x) \sin\omega t, \ \ k = 1, 2, 3, 4, \ \ \omega \in I\!R,$$

then we get the so–called *steady state oscillation equations* of the theory of thermoelasticity

$$\begin{aligned} &c_{kjpq} D_j D_q u_p(x) = -\omega^2 u_k(x) + \beta_{kj}\, D_j u_4(x), \\ &\lambda_{pq} D_p D_q u_4(x) + \mathrm{i}\omega c_0 u_4(x) + \mathrm{i}\omega T_0 \beta_{pq}\, D_p u_q(x) = 0, \end{aligned} \tag{1.4}$$

where the following notation $u_k(x) = \overset{1}{u}_k(x) + \mathrm{i}\, \overset{2}{u}_k(x)$, $k = 1, 2, 3, 4$, is employed.

It is evident that system (1.4) formally can be obtained from (1.3) provided $\sigma = 0$, but this similarity is a very formal one and it will become apparent later on.

1.2. In order to rewrite the above equations in the matrix form, let us set

$$U = (u_1, u_2, u_3, u_4)^\top = (u, u_4)^\top, \quad u = (u_1, u_2, u_3)^\top,$$

$$C(D) = \|C_{kp}(D)\|_{3\times3}, \quad C_{kp}(D) = c_{kjpq}\, D_j D_q, \tag{1.5}$$

$$\Lambda(D) = \lambda_{pq}\, D_p D_q, \quad D = \nabla = (D_1, D_2, D_3). \tag{1.6}$$

For the sake of simplicity sometimes we will use also the notation either $[A]_{m\times n}$ or $[A_{kp}]_{m\times n}$ for the $m\times n$ matrix $A = \|A_{kp}\|_{m\times n}$.

Now we can represent equations (1.3) and (1.4) in the following form, respectively:

$$A(D, \tau)U(x, \tau) = 0, \tag{1.7}$$

$$A(D, -\mathrm{i}\omega)U(x) = 0, \tag{1.8}$$

where

$$A(D, \mu) = \left\| \begin{matrix} [C(D) - \mu^2 I_3]_{3\times3} & [-\beta_{kj} D_j]_{3\times1} \\ [-\mu T_0 \beta_{kj} D_j]_{1\times3} & \Lambda(D) - \mu c_0 \end{matrix} \right\|_{4\times4}, \tag{1.9}$$

$I_m = \|\delta_{kj}\|_{m\times m}$ stands for the unit $m\times m$ matrix, δ_{kj} is Kronecker's symbol.

Clearly, $\mu = \tau = \sigma - \mathrm{i}\,\omega$ corresponds to the pseudo–oscillations, while $\mu = -\mathrm{i}\omega$ corresponds to the steady oscillations.

Further we introduce the classical stress operator

$$T(D, n) = \|T_{kp}(D, n)\|_{3\times3} = \|c_{kjpq}\, n_j\, D_q\|_{3\times3} \tag{1.10}$$

and the thermoelastic stress operator

$$P(D, n) = \|[T(D, n)]_{3\times3},\ [-\beta_{kj} n_j]_{3\times1}\|_{3\times4}.$$

From (1.2) it follows that

$$[P(D, n)U]_k = \sigma_{kj} n_j = [T(D, n)u]_k - \beta_{kj} n_j u_4, \quad k = 1, 2, 3.$$

1.3. Let $\Omega^+ \subset I\!R^3$ be a bounded domain with a C^2–smooth connected boundary $S = \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$ and $\Omega^- = I\!R^3 \setminus \overline{\Omega^+}$. We assume that $\overline{\Omega^+}\,(\overline{\Omega^-})$ is occupied by a homogeneous anisotropic medium with the elastic and thermal characteristics described above.

From the physical considerations it follows that (see [6], [18]):

a) the matrix $||\lambda_{pq}||_{3\times 3}$ is positive definite, i.e.,

$$\Lambda(\xi) = \lambda_{pq}\xi_p\xi_q \geq \delta_0\,|\xi|^2, \;\; \xi \in I\!R^3, \;\; \delta_0 = \text{const} > 0; \tag{1.11}$$

b) $c_{kjpq}e_{kj}e_{pq}$ is a positive definite quadratic form in the real symmetric variables $e_{kj} = e_{jk}$, which implies positive definiteness of the matrix $C(\xi)$, $\xi \in I\!R^3 \setminus \{0\}$, defined by (1.5), i.e.,

$$C_{kj}(\xi)\eta_j\eta_k \geq \delta_1|\xi|^2|\eta|^2, \;\; \xi, \eta \in I\!R^3, \;\; \delta_1 = \text{const} > 0. \tag{1.12}$$

Inequalities (1.11) and (1.12) together with the symmetry properties of the matrices $||\lambda_{pq}||$ and $C(\xi)$ yield:

$$C(\xi)\eta \cdot \eta = C_{kj}(\xi)\eta_j\overline{\eta_k} \geq \delta_1|\xi|^2\,|\eta|^2, \;\; \xi \in I\!R^3, \tag{1.13}$$

$$\lambda_{pq}\eta_p\overline{\eta_q} \geq \delta_0|\eta|^2, \tag{1.14}$$

for an arbitrary complex vector $\eta \in \mathbb{C}^3$; $a \cdot b = \sum_{k=1}^m a_k\overline{b_k}$ denotes the scalar product of two vectors in $\mathbb{C}^m$ and upper bar denotes complex conjugate.

Let us note here that throughout of this paper we will use the following notations (when no confusion can be caused by this):

a) if all elements of a vector $v = (v_1, ..., v_m)$ (matrix $a = ||a_{kj}||_{m\times n}$) belong to one and the same space X, we will write $v \in X$ ($a \in X$) instead of $v \in [X]^m$ ($a \in [X]_{m\times n}$);

b) if $K : X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_n$ and $X_1 = \cdots = X_m$, $Y_1 = \cdots = Y_n$, we will write $K : X \to Y$ instead of $K : [X]^m \to [Y]^n$.

1.4. Our main goal is to investigate the basic BVPs for the equation (1.8). We will consider the following boundary conditions on S:

Problem $(\overset{\omega}{P}_1)^{\pm}$:

$$[u]^{\pm} = f = (f_1, f_2, f_3)^{\top}, \tag{1.15}$$

$$[u_4]^{\pm} = f_4, \tag{1.16}$$

i.e., the dicplacement vector and the temperature are prescribed on S;

Problem $(\overset{\omega}{P}_2)^{\pm}$:

$$[u]^{\pm} = f = (f_1, f_2, f_3)^{\top}, \tag{1.17}$$

$$[\partial_n u_4]^{\pm} = f_4, \;\;\; \partial_n = \lambda_{pq}n_p\,D_q = \lambda(D, n), \tag{1.18}$$

i.e., the dicplacement vector and the heat flux through the surface S are given on S (the case $[\partial_n u_4]^{\pm} = 0$ describes a thermal insulation over the surface bounding the body);

Problem $(\overset{\omega}{P}_3)^{\pm}$:

$$[P(D, n)U]^{\pm} = f, \tag{1.19}$$

$$[u_4]^{\pm} = f_4, \tag{1.20}$$

i.e., the vector of thermal stresses and the temperature are prescribed on S;

Problem $(\overset{\omega}{P}_4)^{\pm}$:

$$[P(D,n)U]^{\pm} = f, \tag{1.21}$$

$$[\partial_n u_4]^{\pm} = f_4, \tag{1.22}$$

i.e., the vector of thermal stresses and the heat flux are prescribed on S; here and throughout of this paper $n(x)$ denotes the exterior unit normal vector of S at the point $x \in S$; the symbols $[\cdot]^{\pm}$ denote limits on S from $\Omega^{\pm}$.

Let us introduce matrix boundary operators corresponding to the above stated boundary conditions:

$$B^{(1)}(D,n) = I_4 = ||\delta_{kj}||_{4\times 4}, \quad B^{(2)}(D,n) = \left\| \begin{matrix} I_3 & [0]_{3\times 1} \\ [0]_{1\times 3} & \partial_n \end{matrix} \right\|_{4\times 4},$$

$$B^{(3)}(D,n) = \left\| \begin{matrix} [T(D,n)]_{3\times 3} & [-\beta_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & 1 \end{matrix} \right\|_{4\times 4},$$

$$B^{(4)}(D,n) = B(D,n) = \left\| \begin{matrix} [T(D,n)]_{3\times 3} & [-\beta_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & \partial_n \end{matrix} \right\|_{4\times 4}. \tag{1.23}$$

Clearly, the boundary condition

$$[B^{(k)}(D,n)U]^{\pm} = (f_1, f_2, f_3, f_4)^{\top}$$

corresponds to Problem $(\overset{\omega}{P}_k)^{\pm}$. The similar problems for equation (1.7) will be denoted by $(\overset{\tau}{P}_k)^{\pm}$.

It is well-known that in the case of unbounded domain (of type Ω^-) the following conditions at infinity (as $|x| \to +\infty$)

$$u_k(x) = \begin{cases} o(1) & \text{for} \quad \mu = 0, \\ O(|x|^N) & \text{for} \quad \mathrm{Re}\mu = \sigma > 0, \quad k = 1,2,3,4, \end{cases} \tag{1.24}$$

with an arbitrary fixed positive N, are sufficient for the uniqueness of solutions to the BVPs. In fact it can be proved that, if u is a solution of the corresponding homogeneous equation, then condition (1.24) implies

$$D^{\beta} u_k(x) = \begin{cases} O(|x|^{-1-|\beta|}) & \text{for} \quad \mu = 0, \\ O(|x|^{-\nu}) & \text{for} \quad \mathrm{Re}\mu = \sigma > 0, \quad k = 1,2,3,4, \end{cases} \tag{1.25}$$

where ν is an arbitrary positive number, $\beta = (\beta_1, \beta_2, \beta_3)$ is an arbitrary multi-index and $|\beta| = \beta_1 + \beta_2 + \beta_3$ (see [1], [11], [15]).

Concerning the case of steady oscillations (i.e., $\mu=-\mathrm{i}\omega$) to select the classes of uniqueness one needs special conditions at infinity which are essentialy connected with the characteristic surfaces of the operator $A(D,-\mathrm{i}\omega)$. Properties of these surfaces are analysed in subsections 1.6–1.8 below.

1.5 We note that the operator $A(D,\mu)$ defined by (1.9) is not formally self-adjoint. Denote by $A^*(D,\mu)$ the formally adjoint operator to $A(D,\mu)$

$$A^*(D,\mu)=\overline{A^\top(-D,\mu)}=A^\top(-D,\overline{\mu})=$$
$$=\left\|\begin{matrix}[C(D)-\overline{\mu}^2 I_3]_{3\times3} & [\overline{\mu}T_0\beta_{kj}D_j]_{3\times1}\\ [\beta_{kj}D_j]_{1\times3} & \Lambda(D)-\overline{\mu}c_0\end{matrix}\right\|_{4\times4}.$$

Let $U=(u_1,u_2,u_3,u_4)^\top$, $V=(v_1,v_2,v_3,v_4)^\top\in C^2(\Omega^+)\cap C^1(\overline{\Omega^+})$ (i.e., U and V are regular vectors in Ω^+) and $A(D,\mu)U$, $A^*(D,\mu)V\in \mathrm{L}_1(\Omega^+)$. Then the following equations hold for arbitrary $\mu\in\mathbb{C}$ (cf. [3], [15]):

$$\int\limits_{\Omega^+}A(D,\mu)U\cdot V\,dx=\int\limits_S[B(D,n)U]^+\cdot[V]^+\,dS-\int\limits_{\Omega^+}E(U,V)\,dx,$$

$$\int\limits_{\Omega^+}\{A(D,\mu)U\cdot V-U\cdot A^*(D,\mu)V\}\,dx=\int\limits_S\{[B(D,n)U]^+\cdot[V]^+-$$
$$-[U]^+\cdot[\overline{Q(D,n,\mu)}V]^+\}\,dS, \tag{1.26}$$

$$\int\limits_{\Omega^+}\{[A(D,\mu)U]_k\,\overline{u}_k+\frac{1}{\overline{\mu}T_0}[\overline{A(D,\mu)U}]_4\,u_4\}\,dx=$$
$$=-\int\limits_{\Omega^+}\{c_{kjpq}\,D_pu_q\,D_k\overline{u}_j+\mu^2|u|^2+\frac{1}{\overline{\mu}T_0}\lambda_{kj}\,D_ku_4\,D_j\overline{u}_4+\frac{c_0}{T_0}|u_4|^2\}\,dx+$$
$$+\int\limits_S\{[B(D,n)U]_k^+[\overline{u}_k]^++\frac{1}{\overline{\mu}T_0}[u_4]^+[\partial_n\overline{u}_4]^+\}\,dS, \tag{1.27}$$

where $B(D,n)$ is defined by (1.23), while

$$Q(D,n,\mu)=\left\|\begin{matrix}[T(D,n)]_{3\times3} & [\mu T_0\beta_{kj}n_j]_{3\times1}\\ [0]_{1\times3} & \partial_n\end{matrix}\right\|_{4\times4}, \tag{1.28}$$

$$E(U,V)=c_{kjpq}\,D_pu_q\,D_k\overline{v}_j+\mu^2u_k\overline{v}_k-\beta_{kj}u_4\,D_p\overline{v}_k+\lambda_{pq}\,D_qu_4\,D_p\overline{v}_4+$$
$$+c_0\mu u_4\overline{v}_4+\mu T_0\overline{v}_4\beta_{pq}\,D_pu_q.$$

The similar formulas hold valid also for the domain Ω^- when $\mu=0$ or $\mathrm{Re}\mu>0$ and the components of U and V satisfy conditions (1.25) (the superscript "+" must be changed by superscript "–" and the sign "+" in front of the surface integrals must be changed by the sign "–"). The case $\mu=-\mathrm{i}\omega$ will be considered later on.

1.6. Let us introduce the characteristic polynomial of the operator $A(D, \mu)$

$$M(\xi, \mu) = \det A(-\mathrm{i}\xi, \mu). \tag{1.29}$$

Denote by $N(-\mathrm{i}\xi, \mu)$ the adjoint matrix to $A(-\mathrm{i}\xi, \mu)$, i.e.,

$$A(-\mathrm{i}\xi, \mu)N(-\mathrm{i}\xi, \mu) = N(-\mathrm{i}\xi, \mu)A(-\mathrm{i}\xi, \mu) = M(\xi, \mu)I_4. \tag{1.30}$$

Therefore

$$A^{-1}(-\mathrm{i}\xi, \mu) = [M(\xi, \mu)]^{-1}N(-\mathrm{i}\xi, \mu). \tag{1.31}$$

Equations (1.29), (1.9) and (1.5) yield

$$\begin{aligned}
M(\xi, \mu) = \det \left\| \begin{matrix} [-C(\xi) - \mu^2\, I_3]_{3\times 3} & [\mathrm{i}\beta_{kj}\xi_j]_{3\times 1} \\ [\mathrm{i}\mu T_0 \beta_{kj}\xi_j]_{1\times 3} & -\mu c_0 \end{matrix} \right\|_{4\times 4} &+ \\
+\det \left\| \begin{matrix} [-C(\xi) - \mu^2\, I_3]_{3\times 3} & [\mathrm{i}\beta_{kj}\xi_j]_{3\times 1} \\ [0]_{1\times 3} & -\Lambda(\xi) \end{matrix} \right\|_{4\times 4} &= \\
= \Lambda(\xi)\ \det[C(\xi) + \mu^2\, I_3] - \mu T_0\ \det \left\| \begin{matrix} [-C(\xi) - \mu^2\, I_3]_{3\times 3} & [\beta_{kj}\xi_j]_{3\times 1} \\ [\beta_{kj}\xi_j]_{1\times 3} & c_0 T_0^{-1} \end{matrix} \right\|_{4\times 4} &= \\
= \Lambda(\xi)\ \det[C(\xi) + \mu^2\, I_3] &- \\
-\mu T_0\ \det \left\| \begin{matrix} [-C(\xi) - \mu^2\, I_3]_{3\times 3} - [c_0^{-1} T_0 \beta_{kj}\xi_j \beta_{pq}\xi_q]_{3\times 3} & [\beta_{kj}\xi_j]_{3\times 1} \\ [0]_{1\times 3} & c_0 T_0^{-1} \end{matrix} \right\|_{4\times 4} &= \\
= \Lambda(\xi)\ \det[C(\xi) + \mu^2\, I_3] + \mu c_0\ \det[\tilde{C}(\xi) + \mu^2\, I_3]&,
\end{aligned} \tag{1.32}$$

where $C(\xi)$ and $\Lambda(\xi)$ are defined by (1.5) and (1.6), respectively, and

$$\tilde{C}(\xi) = ||\tilde{C}_{kp}(\xi)||_{3\times 3} = C(\xi) + ||c_0^{-1} T_0 \beta_{kj}\beta_{pq}\xi_j\xi_q||_{3\times 3}, \tag{1.33}$$

$$\tilde{C}_{kp}(\xi) = (c_{kjpq} + c_0^{-1} T_0 \beta_{kj}\beta_{pq})\xi_j\xi_q, \quad k, p = 1, 2, 3.$$

Now let

$$\Psi(\xi, \mu) = \det[C(\xi) + \mu^2\, I_3], \tag{1.34}$$

$$\tilde{\Psi}(\xi, \mu) = \det[\tilde{C}(\xi) + \mu^2\, I_3]. \tag{1.35}$$

The relations (1.33) and (1.13) imply that the matrix $\tilde{C}(\xi)$ for any $\xi \in \mathbb{R}^3 \setminus \{0\}$ is positive definite and we have

$$\tilde{C}(\xi)\eta \cdot \eta = C(\xi)\eta \cdot \eta + c_0^{-1} T_0 |\beta_{kj}\xi_j\eta_k|^2 \geq \delta_1 |\xi|^2 |\eta|^2 \tag{1.36}$$

with an arbitrary $\eta \in \mathbb{C}^3$ and the same δ_1 as in (1.13).

Thus we have

$$M(\xi,\mu)=\Lambda(\xi)\Psi(\xi,\mu)+\mu c_0\tilde{\Psi}(\xi,\mu). \tag{1.37}$$

It is evident that, if $|\mu|<\mu_0$ with some positive μ_0, then there exists a positive number ϱ_0 such that

$$|\Psi(\xi,\mu)|\geq 1,\;\; |\tilde{\Psi}(\xi,\mu)|\geq 1,\;\; |M(\xi,\mu)|\geq 1, \tag{1.38}$$

for $|\xi|\geq\varrho_0$; ϱ_0 depends on μ_0 and the thermoelastic constants.

LEMMA 1.1 *Let* $\tau=\sigma-\mathrm{i}\omega$, $\mathrm{Re}\tau=\sigma>0$ *and* $\xi\in I\!R^3$.
Then $M(\xi,\tau)\neq 0$ *for any* $\omega\in I\!R$.

Proof. Let us suppose that the assertion of the lemma is false, i.e., $M(\xi,\tau)=0$. Then the homogeneous system of linear algebraic equations

$$A(-\mathrm{i}\xi,\tau)\;a=0 \tag{1.39}$$

has some non–trivial solution $a=(a_1,\cdots,a_4)^\top\in \mathbb{C}^4\setminus\{0\}$.

Multiplying the $k-$th equation of (1.39) by $\overline{a}_k$ and summing the first three equations we get

$$\begin{aligned}&-c_{kjpq}\xi_j\xi_q a_p\overline{a}_k-\tau^2\delta_{kp}a_p\overline{a}_k+\mathrm{i}\beta_{kj}\xi_j a_4\overline{a}_k=0,\\ &\;\;\mathrm{i}\tau T_0\beta_{kj}\xi_j a_k\overline{a}_4-\lambda_{pq}\xi_p\xi_q|a_4|^2-\tau c_0|a_4|^2=0.\end{aligned}$$

Deviding the latter equation by τT_0, taking the complex conjugate and adding to the first one, we arrive at

$$c_{kjpq}\xi_j\xi_q a_p\overline{a}_k+\tau^2 a_k\overline{a}_k+\tau[|\tau|^2T_0]^{-1}\lambda_{pq}\xi_p\xi_q|a_4|^2+c_0T_0^{-1}|a_4|^2=0.$$

Due to (1.13) we deduce by separating the real and imaginary parts

$$\begin{cases} C(\xi)\,\tilde{a}\cdot\tilde{a}+(\sigma^2-\omega^2)|\tilde{a}|^2+\sigma[|\tau|^2T_0]^{-1}\Lambda(\xi)|a_4|^2+c_0T_0^{-1}|a_4|^2=0,\\ \omega\{2\sigma|\tilde{a}|^2+[|\tau|^2T_0]^{-1}\Lambda(\xi)|a_4|^2\}=0,\end{cases}$$

with $\tilde{a}=(a_1,a_2,a_3)^\top$.

From this system and (1.11) it follows that $a_1=\cdots=a_4=0$, for any $\xi\in I\!R^3$, $\omega\in I\!R$ and $\sigma>0$. This contradiction completes the proof. ∎

1.7. Now we will analyse the characteristic polynomial $M(\xi,-\mathrm{i}\omega)$ of the operator $A(D,-\mathrm{i}\omega)$. It is evident that (see (1.34), (1.35), (1.37))

$$M(\xi,-\mathrm{i}\omega)=\Lambda(\xi)\Phi(\xi,\omega)-\mathrm{i}\omega c_0\tilde{\Phi}(\xi,\omega) \tag{1.40}$$

with

$$\Phi(\xi,\omega)=\det[C(\xi)-\omega^2\,I_3]=\Psi(\xi,-\mathrm{i}\omega), \tag{1.41}$$

$$\tilde{\Phi}(\xi,\omega)=\det[\tilde{C}(\xi)-\omega^2\,I_3]=\tilde{\Psi}(\xi,-\mathrm{i}\omega). \tag{1.42}$$

Characteristic surfaces of the operator $A(D, -\mathrm{i}\omega)$ are defined by the equation

$$M(\xi, -\mathrm{i}\omega) = 0, \ \ \xi \in I\!R^3, \tag{1.43}$$

which in turn, due to (1.40), is equivalent to the following sysytem

$$\begin{cases} \Phi(\xi, \omega) = 0, \\ \tilde{\Phi}(\xi, \omega) = 0, \ \ \xi \in I\!R^3. \end{cases} \tag{1.44}$$

Passing on the spherical co–ordinates

$$\xi_1 = \varrho \cos\varphi \sin\theta, \ \ \xi_2 = \varrho \sin\varphi \sin\theta, \ \ \xi_3 = \varrho \cos\theta,$$
$$0 \le \varrho < +\infty, \ \ 0 \le \varphi < 2\pi, \ \ 0 \le \theta \le \pi,$$

and, taking into account formulas (1.41), (1.42), (1.13) and (1.36), we conclude that each equation of the system (1.44) has three positive roots with respect to ϱ^2. These roots are proportional to ω^2 and polynomials $\Phi(\xi, \omega)$ and $\tilde{\Phi}(\xi, \omega)$ can be represented in the form:

$$\Phi(\xi, \omega) = \Phi(\eta, 0)\, [\varrho^2 - \omega^2 \varrho_1^2(\theta, \varphi)][\varrho^2 - \omega^2 \varrho_2^2(\theta, \varphi)][\varrho^2 - \omega^2 \varrho_3^2(\theta, \varphi)], \tag{1.45}$$

$$\tilde{\Phi}(\xi, \omega) = \tilde{\Phi}(\eta, 0)\, [\varrho^2 - \omega^2 \tilde{\varrho}_1^2(\theta, \varphi)][\varrho^2 - \omega^2 \tilde{\varrho}_2^2(\theta, \varphi)][\varrho^2 - \omega^2 \tilde{\varrho}_3^2(\theta, \varphi)],$$

where $\eta = \xi/\varrho$, $|\xi| = \varrho$, $\Phi(\eta, 0) = \det C(\eta) > 0$, $\tilde{\Phi}(\eta, 0) = \det \tilde{C}(\eta) > 0$; here $\{\varrho_k^2(\theta, \varphi)\}_{k=1}^3$ and $\{\tilde{\varrho}_k^2(\theta, \varphi)\}_{k=1}^3$ do not depend on ω and are solutions of the following equations (with respect to ϱ^2):

$$\Phi(\xi, 1) = \Phi(\eta, 0)\varrho^6 + \Phi^{(2)}(\eta)\varrho^4 + \Phi^{(1)}(\eta)\varrho^2 - 1 = 0, \tag{1.46}$$

$$\tilde{\Phi}(\xi, 1) = \tilde{\Phi}(\eta, 0)\varrho^6 + \tilde{\Phi}^{(2)}(\eta)\varrho^4 + \tilde{\Phi}^{(1)}(\eta)\varrho^2 - 1 = 0, \tag{1.47}$$

where $\Phi^{(j)}(\eta)$ and $\tilde{\Phi}^{(j)}(\eta)$ are even, homogeneous functions of order $2j$ in η (see (1.41), (1.42)).

We assume the following conditions to be fulfilled (cf. [15], [19]):

I^0. *$\nabla\Phi(\xi, \omega) \neq 0$ at real zeros of the polynomial $\Phi(\xi, \omega)$;*

II^0. *Full curvature of the surface, defined by the real zeros of the polynomial $\Phi(\xi, \omega)$, does not vanish anywhere.*

From the above conditions $I^0 - II^0$ it follows that the real zeros of the polynomial $\Phi(\xi, \omega)$ form non–self–intersecting, closed, convex two–dimensional surfaces S_j^0, $j = 1, 2, 3$, enveloping the origin of co–ordinates. For an arbitrary vector $x \in I\!R^3 \setminus \{0\}$ there exist exactly two points on each S_j^0, namely $\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j)$ and $\xi_*^j = -\xi^j$, at which the exterior unit normal is parallel to the vector x. We provide that at ξ^j the normal vector $n(\xi^j)$ and x have the same direction, while at ξ_*^j they are opposite directed. Note that, if $\xi^j \in S_j^0$ and $\xi^k \in S_k^0$ correspond to the same vector x, then (due to the convexity property of the above surfaces)

$$(\xi^j \cdot x) \neq (\xi^k \cdot x) \ \text{ for } \ k \neq j.$$

In the sequel the $\xi^j \in S_j^0$ will be referred to as the point which corresponds to the vector x (i.e., to the direction $x/|x|$).

Clearly,

$$\varrho = |\omega| \; \varrho_k(\theta,\varphi) > 0, \quad k = 1,2,3,$$

represent the equations of the surfaces S_k^0 in the spherical co–ordinates.

The set of points in $I\!R^3$ defined by the system of equations (1.44) may have a very complicated geometric form. Among these forms we single out and study the following *regular case*: The system (1.44) is either inconsistent in $I\!R^3$ (i.e., it defines the empty set) or it defines a two–dimensional manifold, i.e., equations (1.46) and (1.47) have m $(1 \leq m \leq 3)$ common roots and, if $1 \leq m < 3$, the remaining two groups of roots form disjoint sets for arbitrary values of θ and φ. We denote these common roots by $\nu_1(\theta,\varphi), \cdots, \nu_m(\theta,\varphi)$ $(1 \leq m \leq 3)$ and without loss of generality assume that

$$0 < \varrho_1(\theta,\varphi) < \varrho_2(\theta,\varphi) < \varrho_3(\theta,\varphi), \;\; 0 < \nu_1(\theta,\varphi) < \cdots < \nu_m(\theta,\varphi). \tag{1.48}$$

Thus in this case the characteristic equation (1.43) (i.e., the system (1.44)) defines analytic (characteristic) surfaces $S_1, \cdots, S_m$, whose equations in the spherical co–ordinates read

$$\varrho = |\omega| \; \nu_k(\theta,\varphi) > 0, \quad k = 1, \cdots, m.$$

The exterior BVPs corresponding to the case $m = 0$ turned out to be very similar to those of the pseudo–oscillation ones (see Remark 2.7) and therefore in what follows we will mainly consider the case $1 \leq m \leq 3$.

1.8. From the above arguments it follows that

$$\Psi(\xi,\mu) = \Phi(\eta,0)\,[\varrho^2 + \mu^2\varrho_1^2(\theta,\varphi)][\varrho^2 + \mu^2\varrho_2^2(\theta,\varphi)][\varrho^2 + \mu^2\varrho_3^2(\theta,\varphi)], \tag{1.49}$$

$$\tilde{\Psi}(\xi,\mu) = \tilde{\Phi}(\eta,0)\,[\varrho^2 + \mu^2\tilde{\varrho}_1^2(\theta,\varphi)][\varrho^2 + \mu^2\tilde{\varrho}_2^2(\theta,\varphi)][\varrho^2 + \mu^2\tilde{\varrho}_3^2(\theta,\varphi)], \tag{1.50}$$

for any $\xi \in I\!R^3$ and $\mu \in C\!\!\!\!I$.

Consequently, according to (1.37) we have

$$\begin{aligned} M(\xi,\mu) = \Phi(\eta,0)\,\Lambda(\xi)\,[\varrho^2 + \mu^2\varrho_1^2(\theta,\varphi)][\varrho^2 + \mu^2\varrho_2^2(\theta,\varphi)][\varrho^2 + \mu^2\varrho_3^2(\theta,\varphi)] + \\ +\mu c_0\,\tilde{\Phi}(\eta,0)\,[\varrho^2 + \mu^2\tilde{\varrho}_1^2(\theta,\varphi)][\varrho^2 + \mu^2\tilde{\varrho}_2^2(\theta,\varphi)][\varrho^2 + \mu^2\tilde{\varrho}_3^2(\theta,\varphi)] = \\ = \Phi_m(\varrho,\theta,\varphi;\mu)\,\Psi_m(\varrho,\theta,\varphi;\mu), \end{aligned} \tag{1.51}$$

where

$$\begin{aligned} \Phi_m(\varrho,\theta,\varphi;\mu) = \Phi_m(\xi,\mu) = \Phi_m(-\xi,\mu) = \Phi_m(\xi,-\mu) = \\ = (-1)^m\,[\varrho^2 + \mu^2\nu_1^2(\theta,\varphi)]\cdots[\varrho^2 + \mu^2\nu_m^2(\theta,\varphi)], \end{aligned} \tag{1.52}$$

$$\begin{aligned} \Psi_m(\varrho,\theta,\varphi;\mu) = \Psi_m(\xi,\mu) = \Psi_m(-\xi,\mu) = \\ = (-1)^m\,\{\Phi(\eta,0)\,\Lambda(\xi)\;[\varrho^2 + \mu^2\lambda_1^2(\theta,\varphi)]\cdots[\varrho^2 + \mu^2\lambda_{3-m}^2(\theta,\varphi)] + \\ +\mu c_0\,\tilde{\Phi}(\eta,0)\,[\varrho^2 + \mu^2\tilde{\lambda}_1^2(\theta,\varphi)]\cdots[\varrho^2 + \mu^2\tilde{\lambda}_{3-m}^2(\theta,\varphi)]; \end{aligned} \tag{1.53}$$

here $\lambda_j^2(\theta,\varphi)$ and $\tilde{\lambda}_j^2(\theta,\varphi)$ denote different (non–common) roots of the equations (1.46) and (1.47), respectively. Note that formulas (1.49), (1.50), (1.51), (1.52) and (1.53) are valid for arbitrary $\xi \in I\!R^3$ and $\mu \in C\!\!\!\!I$.

The multiplier $(-1)^m$ in (1.52) ensures the inequality

$$\Phi_m(0,-\mathrm{i}\omega)>0, \tag{1.54}$$

which will be employed later on.

REMARK 1.2 *Note that the polynomial $\Phi_m(\varrho,\theta,\varphi;-\mathrm{i}\omega)$ in ϱ vanishes on S_j, $j=1,\cdots,m$ (i.e., when $\varrho=|\omega|\nu_j(\theta,\varrho)$) while the other one $\Psi_m(\varrho,\theta,\varphi;-\mathrm{i}\omega)$ is different from zero for any real ϱ and ω. Therefore there exists a positive number ε_0 such that*

$$|\Psi_m(\varrho,\theta,\varphi;\mu)|>0$$

for $|\mathrm{Im}\varrho|\le\varepsilon_0$ and $|\mathrm{Re}\mu|\le\varepsilon_0$, where $\varrho=\varrho'+\mathrm{i}\varrho''$, $\mu=\sigma-\mathrm{i}\omega$ and $|\varrho|\le 2\varrho_0$ with arbitrary ω and ϱ_0 fixed.

Now from equations (1.51) and (1.52) it follows that, if $|\mathrm{Re}\mu|=|\sigma|<\varepsilon_0$ and $|\sigma\,\nu_j(\theta,\varphi)|<\varepsilon_0$, then the complex numbers $\pm(\omega+\mathrm{i}\sigma)\nu_j(\theta,\varphi)=\pm\,\mathrm{i}\mu\nu_j(\theta,\varphi)$, $j=1,\cdots,m$ are the only zeros of the polynomial (1.51) with respect to ϱ in the strip $|\mathrm{Im}\varrho|=|\varrho''|<\varepsilon_0$. As a consequence we have: $M(\xi,\mu)\neq 0$ for $\xi\in I\!R^3$ and $0<|\sigma|=|\mathrm{Re}\mu|<\varepsilon_0$.

2 FUNDAMENTAL MATRICES

In this section we will construct maximally decreasing fundamental matrices of the steady state oscillation operator by limiting absorption principle (cf. [15]).

Denote by $\Gamma(x,\tau)$ a fundamental matrix of the operator $A(D,\tau)$: $A(D,\tau)\Gamma(x,\tau)=I_4\,\delta(x)$, $\tau=\sigma-\mathrm{i}\omega$, $\sigma\neq 0$, where $\delta(x)$ is Dirac's distribution.

Let $0<|\mathrm{Re}\tau|=|\sigma|<\varepsilon_0$ with $\varepsilon_0>0$ from Remark 1.2. Then due to representation (1.51), Remark 1.2 and equation (1.31) we have

$$M(\xi,\tau)\neq 0,\ \ \xi\in I\!R^3, \qquad A^{-1}(-\mathrm{i}\xi,\tau)\in \mathrm{L}_2(I\!R^3),$$

and we can represent $\Gamma(x,\tau)$ by Fourier integral [16]

$$\Gamma(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-\mathrm{i}\xi,\tau)]. \tag{2.1}$$

By $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ we denote the generalized Fourier and inverse Fourier transforms which for summable functions are defined as follows

$$\mathcal{F}_{x\to\xi}[f]=\int\limits_{R^n} f(x)\,e^{\mathrm{i}x\xi}\,dx, \qquad \mathcal{F}^{-1}_{\xi\to x}[g]=(2\pi)^{-n}\int\limits_{R^n} g(\xi)\,e^{-\mathrm{i}x\xi}\,d\xi.$$

Let h be a cutoff function with properties

$$h(\xi)=h(-\xi),\ \ h\in \mathrm{C}^\infty(I\!R^3),\ \ h(\xi)=1 \quad \text{for} \quad |\xi|<\varrho_0,$$

$$h(\xi)=0 \quad \text{for} \quad |\xi|>2\varrho_0 \tag{2.2}$$

with ϱ_0 from (1.38).

Now we decompose (2.1) into two parts

$$\Gamma(x,\tau)=\Gamma^{(1)}(x,\tau)+\Gamma^{(2)}(x,\tau),$$

where

$$\Gamma^{(1)}(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}[(1-h(\xi))A^{-1}(-\mathrm{i}\xi,\tau)], \tag{2.3}$$

$$\Gamma^{(2)}(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}[h(\xi)A^{-1}(-\mathrm{i}\xi,\tau)]=(2\pi)^{-3}\int\limits_{|\xi|<2\varrho_0} h(\xi)A^{-1}(-\mathrm{i}\xi,\tau)e^{-\mathrm{i}x\xi}\,d\xi. \tag{2.4}$$

The main result of this section will follow from two lemmas which we now present.

Let $\Gamma^{(0)}(x)$ be the homogeneous (of order -1) fundamental matrix of the operator $C(D)$ (see [14], [15])

$$\Gamma^{(0)}(x)=\mathcal{F}^{-1}_{\xi\to x}[C^{-1}(-\mathrm{i}\xi)]=(-8\pi^2|x|)^{-1}\int\limits_0^{2\pi} C^{-1}(a\eta)\,d\varphi,$$

$x\in I\!\!R^3\setminus\{0\}$, $a=||a_{kj}||_{3\times3}$ is an orthogonal matrix with property $a^\top x^\top=(0,0,|x|)^\top$, $\eta=(\cos\varphi,\sin\varphi,0)^\top$, and $\gamma^{(0)}(x)$ be the homogeneous (of order -1) fundamental function of the operator $\Lambda(D)$ (see [13])

$$\gamma^{(0)}(x)=\mathcal{F}^{-1}_{\xi\to x}[\Lambda^{-1}(-\mathrm{i}\xi)]=-[4\pi\,|L|^{1/2}(L^{-1}x\cdot x)^{1/2}]^{-1},$$

$$L=||\lambda_{pq}||_{3\times3},\quad |L|=\det L.$$

LEMMA 2.1 *The entries of the matrix $\Gamma^{(1)}(x,\tau)$ belong to $\mathrm{C}^\infty(I\!\!R^3\setminus\{0\})$ and for an arbitrary $\sigma\in[-\varepsilon_0,\varepsilon_0]$ together with all derivatives decrease more rapidly than any negative power of $|x|$ as $|x|\to+\infty$.*

The limit

$$\lim_{\sigma\to0}D_x^\beta\Gamma^{(1)}(x,\sigma-\mathrm{i}\omega)=D_x^\beta\Gamma^{(1)}(x,-\mathrm{i}\omega)$$

exists uniformly for $|x|>\delta$ with an arbitrary δ and in the neigbourhood of the origin ($|x|<1/2$) the following inequalities

$$|D_x^\beta\Gamma^{(1)}_{kj}(x,\sigma-\mathrm{i}\omega)-D_x^\beta\Gamma^{(1)}_{kj}(x,-\mathrm{i}\omega)|\le|\sigma|\,c\,\varphi^{(kj)}_{|\beta|}(x),$$

$$|D_x^\beta\Gamma^{(1)}_{kj}(x,\tau)-D_x^\beta\Gamma_{kj}(x)|\le c\,\varphi^{(kj)}_{|\beta|}(x),$$

hold, where $c=const>0$ does not depend on σ,

$$\Gamma(x)=\left\|\begin{matrix}[\Gamma^{(0)}(x)]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & \gamma^{(0)}(x)\end{matrix}\right\|_{4\times4},$$

$$\varphi^{(kj)}_0(x)=1,\ \ \varphi^{(kj)}_1(x)=-\ln|x|,\ \ \varphi^{(kj)}_l(x)=|x|^{1-l},\ \ l\ge2,$$

for $1\le k,j\le3$ and $k=j=4$;

$$\varphi^{(k4)}_0(x)=\varphi^{(4k)}_0(x)=-\ln|x|,\ \ \varphi^{(k4)}_m(x)=\varphi^{(4k)}_m(x)=|x|^{-m},\ \ m\ge1,$$

for $k=1,2,3$; β is an arbitrary multi-index.

Proof. It is quite similar to the proof of Lemma 3.1 of [15]. ∎

Now we will analyse properties of the matrix $\Gamma^{(2)}(x,\tau)$.

Going to the spherical co–ordinates in the integral (2.4) we get

$$\Gamma^{(2)}(x,\tau) = (2\pi)^{-3} \int\limits_{\Sigma_1} d\Sigma_1 \left\{ \int\limits_0^{\varrho_0} + \int\limits_{\varrho_0}^{2\varrho_0} \right\} h(\xi) A^{-1}(-\mathrm{i}\xi,\tau) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho, \tag{2.5}$$

where Σ_1 is the unit sphere in $I\!R^3$ centered at the origin.

Taking into account Remark 1.2 and the analyticity of the integrand with respect to ϱ in the interval $(0,\varrho_0)$ and introducing the complex $\varrho = \varrho' + \mathrm{i}\varrho''$ plane we can rewrite (2.5) by Cauchy theorem as follows

$$\begin{aligned}\Gamma^{(2)}(x,\tau) = (2\pi)^{-3} \int\limits_{\Sigma_1} d\Sigma_1 \Bigg\{ & \int\limits_{l_0^{\pm}} A^{-1}(-\mathrm{i}\xi,\tau) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho + \\ & + \int\limits_{\varrho_0}^{2\varrho_0} h(\xi) A^{-1}(-\mathrm{i}\xi,\tau) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho \Bigg\},\end{aligned} \tag{2.6}$$

where $l_0^{\pm} = [0, |\omega|\nu_1 - \delta] \cup l_{1,\delta}^{\pm} \cup [|\omega|\nu_1 + \delta, |\omega|\nu_2 - \delta] \cup l_{2,\delta}^{\pm} \cup \cdots \cup l_{m,\delta}^{\pm} \cup [|\omega|\nu_m + \delta, \varrho_0]$, $\delta > 0$ is a sufficiently small number, $l_{j,\delta}^{+}\,[l_{j,\delta}^{-}]$ is a semicircle in the upper [lower] half-plane centered at $|\omega|\nu_j$ and radius δ oriented clockwise [counter–clockwise]; in (2.6) the contour $l_0^{+}\,[l_0^{-}]$ corresponds to the case $\sigma\omega < 0$ $[\sigma\omega > 0]$.

Now passing to the limit in (2.6) as $\sigma \to 0\pm$ we get

$$\begin{aligned}\sigma\omega > 0: \quad \lim_{\sigma\to 0} \Gamma^{(2)}(x,\sigma - \mathrm{i}\omega) = (2\pi)^{-3} \int\limits_{\Sigma_1} d\Sigma_1 \Bigg\{ & \int\limits_{l_0^{-}} A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho + \\ & + \int\limits_{\varrho_0}^{2\varrho_0} h(\xi) A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho \Bigg\} \equiv \Gamma_{+}^{(2)}(x,-\mathrm{i}\omega),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\sigma\omega < 0: \quad \lim_{\sigma\to 0} \Gamma^{(2)}(x,\sigma - \mathrm{i}\omega) = (2\pi)^{-3} \int\limits_{\Sigma_1} d\Sigma_1 \Bigg\{ & \int\limits_{l_0^{+}} A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho + \\ & + \int\limits_{\varrho_0}^{2\varrho_0} h(\xi) A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega) e^{-\mathrm{i}x\xi} \varrho^2 \, d\varrho \Bigg\} \equiv \Gamma_{-}^{(2)}(x,-\mathrm{i}\omega),\end{aligned} \tag{2.8}$$

These limits exist uniformly for $|x| < R_0$ with an arbitrary R_0.

Such type of integrals have been studied in [15]. Applying the arguments quite similar to that of [15] we arrive to the formulas

$$\Gamma_{\pm}^{(2)}(x,-\mathrm{i}\omega) = (2\pi)^{-3} \Bigg[\lim_{\delta\to 0} \int\limits_{|\Phi_m|>\delta} h(\xi) A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega) e^{-\mathrm{i}x\xi} \, d\xi \pm$$

$$\pm\mathrm{i}\pi \sum_{j=1}^{m} \int\limits_{\Sigma_1} \left\{ \frac{N(-\mathrm{i}\xi, -\mathrm{i}\omega)e^{-\mathrm{i}x\xi}\varrho^2}{[\partial/\partial\varrho\, \Phi_m(\varrho,\theta,\varphi;-\mathrm{i}\omega)]\Psi_m(\varrho,\theta,\varphi;-\mathrm{i}\omega)} \right\}_{\varrho=|\omega|\nu_j} d\Sigma_1 \Bigg], \tag{2.9}$$

where Φ_m and Ψ_m are defined by (1.52) and (1.53), respectively.

We need to go over to the integrals over S_j in the last summand of (2.9). To this end let us note that the exterior unit normal of S_j is defined by equation

$$n(\xi) = (-1)^j \frac{\nabla_\xi \Phi_m(\xi,-\mathrm{i}\omega)}{|\nabla_\xi \Phi_m(\xi,-\mathrm{i}\omega)|}, \ \xi \in S_j, \ j = 1, ..., m,$$

since due to (1.52), (1.48) and(1.54)

$$(-1)^j [\partial/\partial\varrho \Phi_m(\xi,-\mathrm{i}\omega)]_{\varrho=|\omega|\nu_j} > 0, \ \ j = 1, ..., m. \tag{2.10}$$

Further

$$d\Sigma_1 = \left[\frac{\xi/|\xi| \cdot n(\xi)}{\varrho^2}\right]_{\varrho=|\omega|\nu_j} dS_j = (-1)^j \left[\frac{\partial/\partial\varrho\Phi_m(\xi,-\mathrm{i}\omega)}{\varrho^2|\nabla\Phi_m(\xi,-\mathrm{i}\omega)|}\right]_{\varrho=|\omega|\nu_j} dS_j.$$

Therefore (2.9) implies

$$\Gamma^{(2)}_{\pm}(x,-\mathrm{i}\omega) = (2\pi)^{-3} \Bigg[\mathrm{V.P.} \int\limits_{\boldsymbol{R}^3} h(\xi) A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega) e^{-\mathrm{i}x\xi}\, d\xi \, \pm$$

$$\pm\mathrm{i}\pi \sum_{j=1}^{m} (-1)^j \int\limits_{S_j} \frac{N(-\mathrm{i}\xi,-\mathrm{i}\omega)e^{-\mathrm{i}x\xi}}{|\nabla\Phi_m(\xi,-\mathrm{i}\omega)|\Psi_m(\xi,-\mathrm{i}\omega)}\, dS_j \Bigg], \tag{2.11}$$

where

$$\mathrm{V.P.} \int\limits_{\boldsymbol{R}^3} h(\xi)A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)e^{-\mathrm{i}x\xi}\, d\xi = \lim_{\delta\to 0} \int\limits_{|\Phi_m(\xi,-\mathrm{i}\omega)|>\delta} h(\xi)A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)e^{-\mathrm{i}x\xi}\, d\xi.$$

Existence and asymptotic behaviour of similar integrals are investigated in [5], [20]. Namely, in [20] there are analysed the following functions ($n-$dimensional version of the case in question)

$$I_j(x) = \int\limits_{S_j} \frac{f(\xi)e^{\mathrm{i}x\xi}}{|\nabla\Phi_m(\xi)|}\, dS_j, \ \ j = 1, ...m, \tag{2.12}$$

$$J(x) = \mathrm{V.P.} \int\limits_{\boldsymbol{R}^n} \frac{f(\xi)e^{\mathrm{i}x\xi}}{\Phi_m(\xi)}\, d\xi, \ \ n \geq 2, \tag{2.13}$$

where

i) $\mathrm{diam}(\mathrm{supp}\, f) < \infty$; $f, \Phi_m \in \mathrm{C}^\infty(I\!R^n)$,

ii) the equation $\Phi_m(\xi) = 0$, $\xi \in I\!R^n$, defines $(n-1)-$dimensional closed non–self–intersecting surfaces S_j, $j = 1, ..., m$, with a full curvature different from zero everywhere; $\nabla\Phi_m(\xi) \neq 0$ for $\xi \in S_j$;

iii) for an arbitrary unit vector η the system

$$\begin{cases} \Phi_m(\xi) = 0, \\ \nabla\Phi_m(\xi)|\nabla\Phi_m(\xi)|^{-1} = \pm\eta, \end{cases} \tag{2.14}$$

has only a finite number of solutions with respect to ξ.

Clearly, in the case under consideration the above conditions for the functions occured in (2.11) are fulfilled due to (2.2) and $I^0 - II^0$. Moreover,

$$\Phi_m(\xi, -\mathrm{i}\omega) = \Phi_m(\xi, \mathrm{i}\omega) = \Phi_m(-\xi, \mathrm{i}\omega)$$

and the corresponding system of type (2.14) defines $2m$ points $\pm\xi^j \in S_j$ $j = 1, ..., m$ (the so–called stationary points); we emphasize also that the unit exterior normal vector $n(\xi^j)$ has the same direction as η, while $n(-\xi^j)$ is opposite directed.

We assume the function $\Phi_m(\xi)$ from (2.12), (2.13) to possess the analogous symmetry property with respect to ξ.

Now let $|x|$ be sufficiently large, $\eta = x/|x|$ and let $\pm\xi^j \in S_j$, $j = 1, ..., m$, be the stationary points corresponding to η: $n(\xi^j) = \eta$, $n(-\xi^j) = -n(\xi^j) = -\eta$. Then according to the results [5], [20] we have the following asymptotic formulas for the functions I_j and J:

$$\begin{aligned} I_j(x) &= [a_j e^{\mathrm{i}x\xi^j} + \tilde{a}_j e^{-\mathrm{i}x\xi^j}]\, |x|^{-(n-1)/2} + O(|x|^{-(n+1)/2}), \\ J(x) &= \sum_{j=1}^{m} [b_j e^{\mathrm{i}x\xi^j} + \tilde{b}_j e^{-\mathrm{i}x\xi^j}]\, |x|^{-(n-1)/2} + O(|x|^{-(n+1)/2}), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} a_j = a_j(\xi^j) &= (2\pi)^{(n-1)/2} \frac{1}{[\kappa(\xi^j)]^{1/2}} \frac{f(\xi^j)}{|\nabla\Phi_m(\xi^j)|}\, e^{-\mathrm{i}(n-1)\pi/4}, \\ \tilde{a}_j = \tilde{a}_j(-\xi) &= (2\pi)^{(n-1)/2} \frac{1}{[\kappa(-\xi^j)]^{1/2}} \frac{f(-\xi^j)}{|\nabla\Phi_m(-\xi^j)|}\, e^{\mathrm{i}(n-1)\pi/4}, \\ b_j &= \mathrm{i}\pi a_j \operatorname{sgn}(\eta \cdot \nabla\Phi_m(\xi^j)) = \mathrm{i}\pi(-1)^j a_j, \\ \tilde{b}_j &= \mathrm{i}\pi\tilde{a}_j \operatorname{sgn}(\eta \cdot \nabla\Phi_m(-\xi^j)) = -\mathrm{i}\pi(-1)^j \tilde{a}_j, \end{aligned} \tag{2.16}$$

$\kappa(\xi)$ is the full curvature at the point $\xi \in S_j$.

The asymptotic formulas (2.15) can be differentiated any times with respect to x.

It is easy to see that the symmetry properties of S_j imply

$$\kappa(\xi) = \kappa(-\xi), \quad \nabla\Phi_m(-\xi) = -\nabla\Phi_m(\xi), \tag{2.17}$$

for any $\xi \in S_j$ $j = 1, ..., m$.

By virtue of (2.12), (2.13) and (2.15) we get

$$\begin{aligned} &J(x) + \lambda \sum_{j=1}^{m} \mathrm{i}\pi(-1)^j I_j(x) = \\ &= \sum_{j=1}^{m} \mathrm{i}\pi(-1)^j [(1+\lambda)a_j e^{\mathrm{i}x\xi^j} - (1-\lambda)\tilde{a}_j e^{-\mathrm{i}x\xi^j}]\, |x|^{-(n-1)/2} + O(|x|^{-(n+1)/2}) \end{aligned} \tag{2.18}$$

with a_j and $\tilde{a}_j$ defined by (2.16) and an arbitrary λ.

Now we can prove the following

LEMMA 2.2 *Entries of matrices* (2.11) *belong to* $C^\infty(I\!R^3)$ *and for sufficiently large* $|x|$ *the asymptotic formulas*

$$\Gamma^{(2)}_{\pm}(x,-i\omega)=\sum_{j=1}^{m} c^{(j)}_{\pm} e^{\pm i x\xi^j}|x|^{-1}+O(|x|^{-2}) \tag{2.19}$$

hold, where the point ξ^j *corresponds to* x (*i.e.,* $n(\xi^j)=x/|x|$) *and*

$$c^{(j)}_{+}\equiv c^{(j)}_{1}(\xi^j,-i\omega)=(-1)^j\frac{1}{2\pi[\kappa(\xi^j)]^{1/2}}\frac{N(i\xi^j,-i\omega)}{|\nabla\Phi_m(\xi^j,-i\omega)|\,\Psi_m(\xi^j,-i\omega)},$$
$$c^{(j)}_{-}\equiv c^{(j)}_{2}(\xi^j,-i\omega)=(-1)^j\frac{1}{2\pi[\kappa(\xi^j)]^{1/2}}\frac{N(-i\xi^j,-i\omega)}{|\nabla\Phi_m(\xi^j,-i\omega)|\,\Psi_m(\xi^j,-i\omega)}; \tag{2.20}$$

(2.19) *can be differentiated any times with respect to* x.

Proof. The first part of the lemma is evident due to (2.2) and I^0-II^0. To prove the asymptotic formulas (2.19) we first perform a change of variable ξ by $-\xi$ in (2.11) and afterwards rewrite it as follows

$$\Gamma^{(2)}_{\pm}(x,-i\omega)=(2\pi)^{-3}[J(x)\pm\sum_{j=1}^{m} i\pi(-1)^j I_j(x)], \tag{2.21}$$

where $I_j(x)$ and $J(x)$ are given by (2.12) and (2.13), respectively, with $n=3$,

$$f(\xi)=\frac{h(\xi)N(i\xi,-i\omega)}{\Psi_m(\xi,-i\omega)}, \tag{2.22}$$

$h(\xi)$ defined by (2.2), $\Phi_m(\xi,-i\omega)$ and $\Psi_m(\xi,-i\omega)$ defined by (1.52) and (1.53); here we have used the fact that h, Φ_m and Ψ_m are even functions in ξ.

Now (2.19) follows from (2.21), (2.18), (2.17), (2.22) and (2.16). ∎

Thus we have proved that there exist one sided limits of the matrix (2.1) as $\mathrm{Re}\tau=\sigma\to 0\pm$.

Let us set

$$\sigma\omega>0: \quad \lim_{\sigma\to 0}\Gamma(x,\sigma-i\omega)=\Gamma^{(1)}(x,-i\omega)+\Gamma^{(2)}_{+}(x,-i\omega)\equiv\Gamma(x,\omega,1), \tag{2.23}$$

$$\sigma\omega<0: \quad \lim_{\sigma\to 0}\Gamma(x,\sigma-i\omega)=\Gamma^{(1)}(x,-i\omega)+\Gamma^{(2)}_{-}(x,-i\omega)\equiv\Gamma(x,\omega,2), \tag{2.24}$$

where $\Gamma^{(1)}$, $\Gamma^{(2)}_{+}$ and $\Gamma^{(2)}_{-}$ are given by (2.3), (2.7) and (2.8), respectively.

Uniting the two latter formulas we have

$$\begin{aligned}\Gamma(x,\omega,r)&=\mathcal{F}^{-1}_{\xi\to x}[(1-h(\xi))A^{-1}(-i\xi,-i\omega)]+\\ &+(2\pi)^{-3}\,\mathrm{V.P.}\int_{R^3} h(\xi)A^{-1}(-i\xi,-i\omega)e^{-ix\xi}\,d\xi+\\ &+(-1)^{r+1}\frac{i\pi}{(2\pi)^3}\sum_{j=1}^{m}(-1)^j\int_{S_j}\frac{N(-i\xi,-i\omega)e^{-ix\xi}}{|\nabla\Phi_m(\xi,-i\omega)|\Psi_m(\xi,-i\omega)}\,dS_j,\quad r=1,2.\end{aligned} \tag{2.25}$$

Now we will formulate the main result of this section.

THEOREM 2.3 *The matrix–functions $\Gamma(x,\omega,r)$, $r=1,2$, defined by* (2.25)*, are fundamental matrices of the operator $A(D,-\mathrm{i}\omega)$ and satisfy the following conditions:*

i) $\Gamma(x,\omega,r)\in \mathrm{C}^\infty(I\!R^3\setminus\{0\})$ *and in a neighbourhood of the origin* ($|x|<1/2$)

$$|D_x^\beta \Gamma_{kj}(x,\omega,r) - D_x^\beta \Gamma_{kj}(x)| \le c\,\varphi_{|\beta|}^{(kj)}(x),\ \ c=const>0,\ \ k,j=1,...,4,$$

where $\Gamma_{kj}(x)$, $\varphi_{|\beta|}^{(kj)}$, $c=const>0$ and β are the same as in Lemma 2.1;

ii) *for sufficiently large $|x|$*

$$\Gamma(x-y,\omega,r)=\sum_{j=1}^{m} c_r^{(j)}(\xi^j,-\mathrm{i}\omega)\ e^{(-1)^{r+1}\mathrm{i}(x-y)\xi^j}\ |x|^{-1}+O(|x|^{-2}), \tag{2.26}$$

where $c_r^{(j)}$ are defined by (2.20)*, $\xi^j\in S_j$ corresponds to the vector x and the variable y varies in a bounded subset of $I\!R^3$; the equation* (2.26) *can be differentiated any times with respect to x and y.*

Proof. It follows directly from Lemmas 2.1 and 2.2. ∎

REMARK 2.4 *Note that, if in* (2.26) *the vector $(x-y)$ is replaced by $-(x-y)$, then the point ξ^j is to be changed by $-\xi^j$, simultaneously, since to the vector $-x$ there corresponds the point $-\xi^j\in S_j$ $(-x/|x|=n(-\xi^j))$. As a result the exponential factor in* (2.26) *will not be changed.*

REMARK 2.5 *The fundamental matrix of the adjoint operator $A^*(D,\tau)$, clearly, has the form*

$$\Gamma^*(x,\tau)=\mathcal{F}^{-1}_{\xi\to x}[\{A^*(-\mathrm{i}\xi,\tau)\}^{-1}]=\mathcal{F}^{-1}_{\xi\to x}[\{A^\top(\mathrm{i}\xi,\overline{\tau})\}^{-1}]=$$
$$=\mathcal{F}^{-1}_{\xi\to x}[\{\overline{A^\top(-\mathrm{i}\xi,\tau)}\}^{-1}]=(2\pi)^{-3}\overline{\int\limits_{R^3}[A^\top(-\mathrm{i}\xi,\tau)]^{-1}\,e^{\mathrm{i}x\xi}\,d\xi}=$$
$$=\overline{\Gamma^\top(-x,\tau)},\ \ \tau=\sigma-\mathrm{i}\omega,\ \ \sigma\ne 0, \tag{2.27}$$

where $\Gamma(x,\tau)$ is given by (2.1).

Therefore there exist limits similar to (2.23) *and* (2.24)

$$\Gamma^*(x,\omega,r)=\lim_{\sigma\to 0}\Gamma^*(x,\tau)=\lim_{\sigma\to 0}\overline{\Gamma^\top(-x,\tau)}=\overline{\Gamma^\top(-x,\omega,r)},\ \ r=1,2, \tag{2.28}$$

where $(-1)^{r+1}\sigma\omega>0$ is assumed.

The entries of matrix (2.27) *and their derivatives decrease more rapidly then any negative power of $|x|$ as $|x|\to+\infty$ if $0<|\sigma|<\varepsilon_0$ (see Remark* 1.2 *).*

Concerning to the asymptotic formulas for $\Gamma^*(x,\omega,r)$, from (2.28) *and Theorem* 2.3*, we get*

$$\Gamma^*(x,\omega,r)=\sum_{j=1}^{m}\tilde{c}_r^{(j)}e^{(-1)^r\mathrm{i}x\xi^j}\,|x|^{-1}+O(|x|^{-2}),$$

where $|x|$ is sufficiently large, $\xi^j\in S_j$ corresponds to x, and

$$\tilde{c}_r^{(j)}=[\overline{c_r^{(j)}(-\xi^j,-\mathrm{i}\omega)}]^\top$$

with $c_r^{(j)}$ *defined by* (2.20).

From Lemmas 2.1, 2.2 *and Theorem* 2.3 *together with the equations* (2.27) *and* (2.28) *it follows that the matrices* $\Gamma(x,\tau)$, $\Gamma(x,\omega,r)$, $\Gamma^*(x,\tau)$ *and* $\Gamma^*(x,\omega,r)$ *have the same matrix* $\Gamma(x)$ *as the principal singular part in a neighbourhood of the origin, since* $\Gamma(x)$ *is a real, symmetric matrix with entries which are homogeneous* (*of order* -1) *and even functions in* x :

$$\Gamma(x)=\overline{\Gamma(x)}=\Gamma^\top(x)=\Gamma(-x),\ \Gamma(tx)=t^{-1}\Gamma(x),\ t>0.$$

REMARK 2.6 *Equation* (2.26) *implies the following representation*

$$\Gamma(x-y,\omega,r)=\sum_{j=1}^{m}\overset{(j)}{\Gamma}(x-y,\omega,r),$$

where for sufficiently large $|x|$

$$\overset{(j)}{\Gamma}(x-y,\omega,r)=c_r^{(j)}\,e^{(-1)^{r+1}\mathrm{i}(x-y)\xi^j}\,|x|^{-1}+O(|x|^{-2}),$$

$$D_{x_p}\overset{(j)}{\Gamma}(x-y,\omega,r)+\mathrm{i}(-1)^r\,\xi_p^j\,\overset{(j)}{\Gamma}(x-y,\omega,r)=O(|x|^{-2}),$$

$$j=1,...,m,\ \ p=1,2,3,\ \ r=1,2,$$

$\xi^j\in S_j$ *corresponds to* x *and* y *varies again in a bounded subset of* $I\!R^3$.

REMARK 2.7 *If the system of equations* (1.44) *is inconsistent in* $I\!R^3$ *for some* $\omega>0$, *then* $M(\xi,-\mathrm{i}\omega)=\det A(-\mathrm{i}\xi,-\mathrm{i}\omega)\neq 0$ *for arbitrary* $\xi\in I\!R^3$ *and* $\omega\in I\!R$, *and*

$$\Gamma(x,-\mathrm{i}\omega)=\mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-\mathrm{i}\xi,-\mathrm{i}\omega)]\in \mathrm{C}^\infty(I\!R^3\setminus\{0\}) \tag{2.29}$$

is a fundamental matrix of the operator $A(D,-\mathrm{i}\omega)$ *whose entries together with all derivatives decrease more rapidly than any negative power of* $|x|$ *as* $|x|\to+\infty$.

The main singular part of (2.29) *in a neighbourhood of the origin is again the matrix* $\Gamma(x)$. *Therefore this case is very similar to the pseudo–oscillation one* [16].

3 RADIATION CONDITIONS AND INTEGRAL REPRESENTATIONS

3.1. Let us introduce the classes SK_r^m on an unbounded domain of type Ω^- (which is the complement to a compact set $\overline{\Omega^+}$ in $I\!R^3$).

A function (vector, matrix) u belongs to the class $SK_r^m(\Omega^-)$, $r=1,2$, if it is C^1-smooth in Ω^- and for sufficiently large $|x|$ the following relations hold (no summation over j in the last equation)

$$u(x)=\sum_{j=1}^{m}\overset{(j)}{u}(x),\qquad \overset{(j)}{u}(x)=O(|x|^{-1}),$$

$$D_p\overset{(j)}{u}(x)+\mathrm{i}(-1)^r\,\xi_p^j\,\overset{(j)}{u}(x)=O(|x|^{-2}),\ p=1,2,3,\ j=1,...,m, \tag{3.1}$$

where $\xi^j\in S_j$ corresponds to the vector x.

Clearly, this definition is essentially related to the operator $A(D, -\mathrm{i}\omega)$ and its characteristic equation (1.43). The conditions (3.1) will be referred to as *generalized Sommerfeld-Kupradze type radiation conditions* in anisotropic thermoelasticity (cf. [12]).

A four-dimensional vector $U = (u_1, ..., u_4)^\top$, satisfying conditions (3.1), will be called $(m, r)-$*thermo-radiating vector.*

Remark 2.6 implies that $\Gamma(x, \omega, r) \in SK_r^m(I\!R^3 \setminus \{0\})$.

In the isotropic case $m = 1$ and S_1 is defined by the equation $\varrho^2 = k_1^2$ with $k_1^2 = \omega^2\mu^{-1}$ (μ is the Lamé constant and ω is the oscillation parameter). Therefore the point $\xi^1 \in S_1$, which corresponds to the given direction (vector) x, is given by $\xi^1 = k_1\eta$, $\eta = x/|x|$, and conditions (3.1) are equivalent to the well–known thermoelastic radiation conditions (see, e.g., [12], Ch. III).

3.2. Let U be a regular vector in $\Omega^\pm$, i.e., $U \in \mathrm{C}^2(\Omega^\pm) \cap \mathrm{C}^1(\overline{\Omega^\pm})$.

In addition let $A(D, \tau)U \in \mathrm{L}_1(\Omega^\pm)$ and conditions (1.25) be satisfied (in the case of the domain Ω^-). If we assume that either $0 < |\mathrm{Re}\tau| = |\sigma| < \varepsilon_0$ or $\sigma > 0$ and use the identity (1.26), by standard arguments we obtain the following integral representation formulas

$$\int\limits_{\Omega^\pm} \Gamma(x-y, \tau)\, A(D, \tau)U(y)\, dy \;\pm\; \int\limits_S \{[Q(D_y, n(y), \tau)\Gamma^\top(x-y, \tau)]^\top [U(y)]^\pm -$$

$$-\Gamma(x-y, \tau)[B(D_y, n(y))U(y)]^\pm\}\, dS_y = \begin{cases} U(x), & x \in \Omega^\pm, \\ 0, & x \in \Omega^\mp, \end{cases} \tag{3.2}$$

where boundary operators B and Q are given by (1.23) and (1.28), respectively, and the fundamental matrix $\Gamma(x, \tau)$ is defined by (2.1) (see [3], [14]); $n(y)$ is the outward unit normal vector of S at the point $y \in S$ and S is a C^2-smooth surface.

Due to Theorem 2.3 and equalities (2.23), (2.24) analogous representation formulas can be written by means of the fundamental matrices $\Gamma(x, \omega, r)$ in the case of the domain Ω^+. One needs only to replace $A(D, \tau)$ and $\Gamma(x, \tau)$ by $A(D, -\mathrm{i}\omega)$ and $\Gamma(x, \omega, r)$, respectively. Concerning the domain Ω^- we will prove the following

THEOREM 3.1 *Let $\partial\Omega^- = S$ be C^2-smooth boundary and U be a regular $(m, r)-$thermo-radiating vector in Ω^- : $U \in \mathrm{C}^2(\Omega^-) \cap \mathrm{C}^1(\overline{\Omega^-}) \cap SK_r^m(\Omega^-)$. Let, in addition, $A(D, -\mathrm{i}\omega)U \in \mathrm{L}_1(\Omega^-)$ and have a compact support. Then*

$$U(x) = \int\limits_{\Omega^-} \Gamma(x-y, \omega, r)\, A(D_y, -\mathrm{i}\omega)U(y)\, dy \;+$$

$$+\int\limits_S \{\Gamma(x-y, \omega, r)[B(D_y, n(y))U(y)]^- -$$

$$-[Q(D_y, n(y), -\mathrm{i}\omega)\Gamma^\top(x-y, \omega, r)]^\top [U(y)]^-\}\, dS_y, \;\; x \in \Omega^-; \tag{3.3}$$

here B, Q and n are the same as in (3.2).

Proof. Let R be a sufficiently large positive number and $\overline{\Omega^+} \subset B_R = \{x \in I\!R^3 : |x| < R\}$. We assume also that $\mathrm{supp}\, A(D, -\mathrm{i}\omega)U \subset B_R$. Denote $\Omega_R^- = \Omega^- \cap B_R$ and $\partial B_R = \Sigma_R$. Then for the regular vector U in Ω_R^-, we have the following integral representation (cf. (3.2))

$$U(x)=\int\limits_{\Omega_R^-}\Gamma(x-y,\omega,r)\,A(D_y,-\mathrm{i}\omega)U(y)\,dy\;+$$

$$+\left\{\int\limits_{\Sigma_R}-\int\limits_{S}\right\}\{[Q(D_y,n(y),-\mathrm{i}\omega)\Gamma^\top(x-y,\omega,r)]^\top[U(y)]-$$

$$-\Gamma(x-y,\omega,r)[B(D_y,n(y))U(y)]\}\,dS_y,\;\;x\in\Omega_R^-, \tag{3.4}$$

where $n(y)$ is the exterior normal on the both surfaces S and Σ_R; clearly, $n(y)=y/R$ for $y\in\Sigma_R$. Note that in the first integral Ω_R^- can be replaced by Ω^-.

Our goal is to show that the integral over Σ_R tends to zero as $R\to+\infty$.

To this end denote the right-hand side expression in (3.3) by $\mathcal{T}[U]$. Then by integrating of (3.4) from ν to 2ν with respect to R and deviding the result by ν, we get

$$U(x)=\mathcal{T}[U](x)+X(\nu),$$

where

$$X(\nu)=\frac{1}{\nu}\int\limits_{\nu}^{2\nu}dR\int\limits_{\Sigma_R}\{[Q(D_y,\eta,-\mathrm{i}\omega)\Gamma^\top(x-y,\omega,r)]^\top[U(y)]-$$

$$-\Gamma(x-y,\omega,r)[B(D_y,\eta)U(y)]\}\,d\Sigma_R,\quad \eta=n(y)=y/R.$$

Let us prove that $X(\nu)\to 0$ as $\nu\to+\infty$.

It can be done by applying arguments similar to that of [19]. In fact, for definiteness, let $r=1$. Then due to (3.1)

$$B(D_y,\eta)U(y)=\sum_{j=1}^{m}B(\mathrm{i}\xi^j,\eta)\overset{(j)}{U}(y)+O(R^{-2}),$$

where $\xi^j\in S_j$ corresponds to the vector η.

According to Remarks 2.4, 2.6 and Theorem 2.3 analogous formulas hold also for $[Q(D_y,\eta,-\mathrm{i}\omega)\Gamma^\top(x-y,\omega,1)]^\top$ and $\Gamma(x-y,\omega,1)$ (note that x is some fixed point in Ω_R^-). Terms, corresponding to $O(R^{-2})$ in the expression of $X(\nu)$, decay as $O(\nu^{-1})$ and therefore there remain only terms of the type

$$v_{st}(\nu)=\frac{1}{\nu}\int\limits_{\nu}^{2\nu}dR\int\limits_{\Sigma_1}\psi(\eta)\,g_s(R\eta)\,h_t(R\eta)\,R^2\,d\Sigma_1,$$

where $\psi\in\mathrm{C}^\infty(\Sigma_1)$, $\eta\in\Sigma_1$, g_s and h_t $(1\le s,t\le m)$ are smooth functions satisfying the following inequalities

$$|g_s(R\eta)|<c\,R^{-1},\qquad |\frac{\partial}{\partial R}g_s(R\eta)-\mathrm{i}\mu_s(\eta)g_s(R\eta)|<c\,R^{-2},$$

$$|h_t(R\eta)|<c\,R^{-1},\qquad |\frac{\partial}{\partial R}h_t(R\eta)-\mathrm{i}\mu_t(\eta)h_t(R\eta)|<c\,R^{-2},$$

$$c=const>0,\quad \mu_j(\eta)=(\eta\cdot\xi^j)>0,$$

due to (3.1).

The latter inequality is a consequence of (2.10), since

$$(\eta \cdot \xi^j) = (n(\xi^j) \cdot \xi^j) = (-1)^j \left(\frac{\nabla \Phi_m(\xi^j, -\mathrm{i}\omega)}{|\nabla \Phi_m(\xi^j, -\mathrm{i}\omega)|} \cdot \xi^j \right) =$$

$$= (-1)^j \frac{|\xi^j|}{|\nabla \Phi_m(\xi^j, -\mathrm{i}\omega)|} \left[\frac{\partial}{\partial |\xi|} \Phi_m(\xi^j, -\mathrm{i}\omega) \right]_{\xi = \xi^j} > 0.$$

Now we proceed as follows

$$v_{st}(\nu) = \frac{1}{\mathrm{i}\nu} \int_{\nu}^{2\nu} dR \int_{\Sigma_1} \frac{\psi(\eta)}{\mu_s(\eta) + \mu_t(\eta)} [\mathrm{i}\mu_s(\eta)\, g_s(R\eta)\, h_t(R\eta) +$$

$$+ g_s(R\eta)\, \mathrm{i}\mu_t(\eta)\, h_t(R\eta)]\, R^2\, d\Sigma_1 =$$

$$= \frac{1}{\mathrm{i}\nu} \int_{\Sigma_1} d\Sigma_1 \int_{\nu}^{2\nu} \left\{ \frac{\psi(\eta)}{\mu_s(\eta) + \mu_t(\eta)} \frac{\partial}{\partial R} [g_s(R\eta)\, h_t(R\eta)] + O(R^{-3}) \right\} R^2\, dR =$$

$$= \frac{1}{\mathrm{i}\nu} \int_{\Sigma_1} \frac{\psi(\eta)}{\mu_s(\eta) + \mu_t(\eta)} \{ (2\nu)^2 g_s(2\nu\eta)\, h_t(2\nu\eta) - \nu^2 g_s(\nu\eta)\, h_t(\nu\eta) -$$

$$- \int_{\nu}^{2\nu} g_s(R\eta)\, h_t(R\eta)\, 2R\, dR \}\, d\Sigma_1 + O(\nu^{-1}) = O(\nu^{-1}).$$

Thus $X(\nu) \to 0$ as $\nu \to +\infty$ which completes the proof. ∎

REMARK 3.2 *From the above proof it follows: if U satisfies the conditions of Theorem* 3.1 *and R is a sufficiently large positive number such that* $\operatorname{supp} A(D, -\mathrm{i}\omega)U \subset B_R$, *then*

$$\int_{\Sigma_R} \{ [Q(D_y, n(y), -\mathrm{i}\omega) \Gamma^\top (x - y, \omega, r)]^\top [U(y)] -$$

$$- \Gamma(x - y, \omega, r)[B(D_y, n(y))U(y)] \}\, d\Sigma_R = 0$$

for an arbitrary $x \in B_R \cap \Omega^-$.

COROLLARY 3.3 *Let U be the same as in Theorem* 3.1. *Then $D^\beta U$ is a $(m, r)-$thermo-radiating vector for an arbitrary multi-index β and the asymptotic representation of $D^\beta U$ at infinity can be obtained from the asymptotic formula of U by the direct differentiation.*

COROLLARY 3.4 *Let $A(D, -\mathrm{i}\omega)U(x) = 0$ in $I\!R^3$ and $U \in SK_r^m(I\!R^3)$. Then $U = 0$ in $I\!R^3$.*

COROLLARY 3.5 *Let $F = (F_1, ..., F_4)^\top \in \mathrm{C}^1(I\!R^3)$ and* $\operatorname{diam} \operatorname{supp} F < +\infty$. *Then the equation*

$$A(D, -\mathrm{i}\omega)U(x) = F(x), \quad x \in I\!R^3$$

has a unique solution in the class $\mathrm{C}^2(I\!R^3)\cap SK_r^m(I\!R^3)$ *and it is representable by a convolution type integral*

$$U(x)=\int\limits_{\boldsymbol{R}^3}\Gamma(x-y,\omega,r)\,F(y)\,dy,\quad x\in I\!R^3.$$

4 UNIQUENESS THEOREMS

4.1. First we will establish some auxiliary results concerning the coefficients of asymptotic formulas (2.26) and ascertain the structure of the matrix functions (2.20).

We recall that

$$N(\mathrm{i}\xi,-\mathrm{i}\omega)=||N_{kj}(\mathrm{i}\xi,-\mathrm{i}\omega)||_{4\times 4} \tag{4.1}$$

is the adjoint matrix to

$$A(\mathrm{i}\xi,-\mathrm{i}\omega)=\left\|\begin{matrix}[\omega^2\,I_3-C(\xi)]_{3\times 3} & [-\mathrm{i}\beta_{kj}\xi_j]_{3\times 1}\\ [-\omega T_0\beta_{kj}\xi_j]_{1\times 3} & -\Lambda(\xi)+\mathrm{i}\omega c_0\end{matrix}\right\|_{4\times 4}, \tag{4.2}$$

where $C(\xi)$ and $\Lambda(\xi)$ are defined by (1.5) and (1.6), respectively, while $N_{kj}(\mathrm{i}\xi,-\mathrm{i}\omega)$ denotes the cofactor of the element $A_{jk}(\mathrm{i}\xi,-\mathrm{i}\omega)$ of the matrix (4.2) (cf. (1.30), (1.31)).

Let us set

$$C(\xi,\omega)=\omega^2\,I_3-C(\xi),\;\;\tilde{C}(\xi,\omega)=\omega^2\,I_3-\tilde{C}(\xi), \tag{4.3}$$

where $\tilde{C}(\xi)$ is given by (1.33). Denote by $C^*(\xi,\omega)$ and $\tilde{C}^*(\xi,\omega)$ the corresponding adjoint matrices.

Due to (1.41) and (1.42) we have

$$C(\xi,\omega)\,C^*(\xi,\omega)=-\Phi(\xi,\omega)\,I_3,\quad \tilde{C}(\xi,\omega)\,\tilde{C}^*(\xi,\omega)=-\tilde{\Phi}(\xi,\omega)\,I_3. \tag{4.4}$$

From the condition I^0 (see Subsection 1.7) it follows that $\mathrm{rank}C(\xi,\omega)=2$ and, consequently, $\mathrm{rank}C^*(\xi,\omega)=1$ for an arbitrary $\xi\in S_l^0$. Moreover (for the same $\xi\in S_l^0$) there exists an orthogonal real matrix $G(\xi,\omega)$ such that

$$G^\top(\xi,\omega)\,C^*(\xi,\omega)G(\xi,\omega)=\left\|\begin{matrix}\lambda_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix}\right\|=\lambda_1\mathcal{I}_0,\quad \mathcal{I}_0=\left\|\begin{matrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix}\right\|, \tag{4.5}$$

where the real value $\lambda_1=\lambda_1(\xi,\omega)\neq 0$ is an eigenvalue of the matrix $C^*(\xi,\omega)$ (two other eigenvalues are equal to zero; for details see [15]).

Further let $d(\xi,\omega)=-\omega c_0[\Lambda(\xi)]^{-1}$ and

$$d(\xi,\omega)G^\top(\xi,\omega)\,\tilde{C}^*(\xi,\omega)G(\xi,\omega)=||b_{kj}(\xi,\omega)||_{3\times 3}. \tag{4.6}$$

LEMMA 4.1 *Let $\xi \in S_j$, $j = 1, ..., m$, where S_j are characteristic surfaces defined in Subsection* 1.7. *Then the matrix N has the following structure*

$$N(\pm i\xi, -i\omega) = \left\| \begin{matrix} [\mathcal{N}(\xi,\omega)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{matrix} \right\|_{4\times 4},$$

where $\mathcal{N}(\xi,\omega) = -\Lambda(\xi)[1 + ib_{11}(\xi,\omega)\lambda_1^{-1}(\xi,\omega)]C^*(\xi,\omega)$.

Proof. Let $\xi \in S_j$ be an arbitrary point ($1 \leq j \leq m$). Clearly, ξ belongs to some surface S_l^0, $1 \leq l \leq 3$, as well (see Subsection 1.7). Therefore

$$N_{44}(\pm i\xi, -i\omega) = -\Phi(\xi,\omega) = 0, \tag{4.7}$$

due to (1.44).

By direct calculations we get

$$N_{k4}(i\xi, -i\omega) = -i\omega T_0 N_{4k}(i\xi, -i\omega), \quad k = 1, 2, 3, \tag{4.8}$$

$$N_{pq}(i\xi, -i\omega) = -\Lambda(\xi)C_{pq}^*(\xi,\omega) + i\omega c_0 \tilde{C}_{pq}^*(\xi,\omega), \ \ 1 \leq p, q \leq 3. \tag{4.9}$$

The condition I^0 implies

$$\nabla M(\xi, -i\omega) = \Lambda(\xi)\nabla\Phi(\xi,\omega) - i\omega c_0 \nabla\tilde{\Phi}(\xi,\omega) \neq 0,$$

since $\Lambda(\xi) \neq 0$ on S_j.

The latter relation together with the equations (1.29), (1.30), (1.32) and

$$\det A(-i\xi', -i\omega) = \det A(i\xi', -i\omega) = M(-\xi', -i\omega) = M(\xi', -i\omega), \ \ \xi' \in I\!R^3,$$

yield

$$\mathrm{rank} A(i\xi, -i\omega) = 3, \quad \mathrm{rank} N(i\xi, -i\omega) = 1, \tag{4.10}$$

i.e., any two columns (rows) of the matrix (4.1) are linearly dependent.

Taking into account the symmetry property (4.8) and equation (4.7) it can be easily proved that

$$N_{k4}(i\xi, -i\omega) = 0, \quad N_{4k}(i\xi, -i\omega) = 0, \quad k = 1, 2, 3.$$

Thus we have obtained the following representation

$$N(\pm i\xi, -i\omega) = \left\| \begin{matrix} [N^{(0)}(\xi,\omega)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{matrix} \right\|_{4\times 4}$$

with

$$N^{(0)}(\xi,\omega) = \|N_{pq}(i\xi, -i\omega)\|_{3\times 3}, \tag{4.11}$$

where $N_{pq}(i\xi, -i\omega) = N_{qp}(i\xi, -i\omega)$ are defined by (4.9).

Now from (4.9) and (4.11) together with (4.5) and (4.6) it follows

$$N^{(0)}(\xi,\omega) = -\Lambda(\xi)C^*(\xi,\omega) + \mathrm{i}\omega c_0 \tilde{C}^*(\xi,\omega)\,,$$

$$\begin{aligned} G^\top(\xi,\omega)\, N^{(0)}(\xi,\omega)G(\xi,\omega) &= -\Lambda(\xi)\lambda_1(\xi,\omega)\mathcal{I}_0 + \\ &\quad + \mathrm{i}\omega c_0 G^\top(\xi,\omega)\,\tilde{C}^*(\xi,\omega)G(\xi,\omega) = \\ &= -\Lambda(\xi) \left\| \begin{matrix} \lambda_1(\xi,\omega)+\mathrm{i}b_{11} & \mathrm{i}b_{12} & \mathrm{i}b_{13} \\ \mathrm{i}b_{12} & \mathrm{i}b_{22} & \mathrm{i}b_{23} \\ \mathrm{i}b_{13} & \mathrm{i}b_{23} & \mathrm{i}b_{33} \end{matrix} \right\|, \end{aligned} \tag{4.12}$$

where b_{pq} are real functions defined by (4.6).

By virtue of (4.10) $\mathrm{rank} N^{(0)}(\xi,-\mathrm{i}\omega) = 1$, and, consequently,

$$\mathrm{rank}\,[G^\top(\xi,\omega)N^{(0)}(\xi,\omega)G(\xi,\omega)] = 1,$$

since G is an orthogonal matrix. This in turn implies that the matrix (4.12) has only one linearly independent column (row). Inasmuch as $\lambda_1 \neq 0$, we conclude: there exist complex numbers $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ and $\beta = \beta_1 + \mathrm{i}\beta_2$, such that

$$\begin{pmatrix} \mathrm{i}b_{12} \\ \mathrm{i}b_{22} \\ \mathrm{i}b_{23} \end{pmatrix} = \alpha \begin{pmatrix} \lambda_1 + \mathrm{i}b_{11} \\ \mathrm{i}b_{12} \\ \mathrm{i}b_{13} \end{pmatrix}, \quad \begin{pmatrix} \mathrm{i}b_{13} \\ \mathrm{i}b_{23} \\ \mathrm{i}b_{33} \end{pmatrix} = \beta \begin{pmatrix} \lambda_1 + \mathrm{i}b_{11} \\ \mathrm{i}b_{12} \\ \mathrm{i}b_{13} \end{pmatrix}. \tag{4.13}$$

Equating the corresponding elements and separating the real and imaginary parts lead to the equations

$$(\alpha_1^2 + \alpha_2^2)\lambda_1 = 0, \;\; (\beta_1^2 + \beta_2^2)\lambda_1 = 0,$$

i.e., $\alpha = \beta = 0$. But then from (4.13), (4.12) and (4.5) it follows

$$\begin{aligned} N^{(0)}(\xi,\omega) &= -\Lambda(\xi)\{\lambda_1(\xi,\omega)G(\xi,\omega)\mathcal{I}_0 G^\top(\xi,\omega) + \mathrm{i}b_{11}(\xi,\omega)G(\xi,\omega)\mathcal{I}_0 G^\top(\xi,\omega)\} = \\ &= -\Lambda(\xi)[\lambda_1(\xi,\omega) + \mathrm{i}b_{11}(\xi,\omega)]G(\xi,\omega)\mathcal{I}_0 G^\top(\xi,\omega) = \\ &= -\Lambda(\xi)[1 + \mathrm{i}\lambda_1^{-1}(\xi,\omega)b_{11}(\xi,\omega)]C^*(\xi,\omega), \end{aligned}$$

which completes the proof. ∎

REMARK 4.2 *Due to equation* (2.20) *and Lemma* 4.1 *we get* (*for arbitrary* $\xi \in S_j$, $j = 1, ..., m$, *and* $r = 1, 2$)

$$c_r^{(j)}(\xi,-\mathrm{i}\omega) = d_j(\xi,-\mathrm{i}\omega) \left\| \begin{matrix} [C^*(\xi,\omega)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{matrix} \right\|_{4\times 4} \tag{4.14}$$

with

$$d_j(\xi,-\mathrm{i}\omega) = (-1)^{j+1} \frac{\Lambda(\xi)[1 + \mathrm{i}\lambda_1^{-1}(\xi,\omega)b_{11}(\xi,\omega)]}{[2\pi(\kappa(\xi))^{1/2}|\nabla\Phi_m(\xi,-\mathrm{i}\omega)|\Psi_m(\xi,-\mathrm{i}\omega)]}\,.$$

LEMMA 4.3 *Let $U = (u, u_4)^\top$ be a regular vector in Ω^- of the class $SK_r^m(\Omega^-)$ and let $A(D, -\mathrm{i}\omega)U$ have a compact support.*

Then for sufficiently large $|x|$

$$u(x) = \sum_{j=1}^{m} |x|^{-1} d_j(\xi^j, -\mathrm{i}\omega) e^{(-1)^{r+1}\mathrm{i}x\xi^j} C^*(\xi^j, \omega) b(\xi^j) + O(|x|^{-2}), \tag{4.15}$$

$$u_4(x) = O(|x|^{-2}), \tag{4.16}$$

with the same d_j as in Remark 4.2; here $C^*(\xi, \omega)$ is the adjoint matrix to $C(\xi, \omega)$, $b = (b_1, b_2, b_3)^\top$ (see (4.18)), and the point $\xi^j \in S_j$ corresponds to the vector $x/|x|$.

Proof. Denote by Ω the support of $A(D, -\mathrm{i}\omega)U$. Then by Theorems 2.3, 3.1 and Remark 2.6 we have (for sufficiently large $|x|$)

$$\begin{aligned} U(x) = \sum_{j=1}^{m} \Bigg\{ \int_{\Omega} |x|^{-1} e^{(-1)^{r+1}\mathrm{i}(x-y)\xi^j} c_r^{(j)}(\xi^j, -\mathrm{i}\omega)[A(D_y, -\mathrm{i}\omega)U(y)]\, dy + \\ + \int_S |x|^{-1} e^{(-1)^{r+1}\mathrm{i}(x-y)\xi^j} c_r^{(j)}(\xi^j, -\mathrm{i}\omega)[B(D_y, n(y))U(y)]^-\, dS_y - \\ - \int_S |x|^{-1} e^{(-1)^{r+1}\mathrm{i}(x-y)\xi^j} \{Q((-1)^r\mathrm{i}\xi^j, n(y), -\mathrm{i}\omega)[c_r^{(j)}(\xi^j, -\mathrm{i}\omega)]^\top\}^\top [U(y)]^-\, dS_y \Bigg\} + \\ + O(|x|^{-2}) = \sum_{j=1}^{m} |x|^{-1} e^{(-1)^{r+1}\mathrm{i}x\xi^j} c_r^{(j)}(\xi^j, -\mathrm{i}\omega)\tilde{b}(\xi^j) + O(|x|^{-2}), \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \tilde{b}(\xi^j) = (b(\xi^j), b_4(\xi^j))^\top = \int_{\Omega} e^{(-1)^r\mathrm{i}y\xi^j}[A(D_y, -\mathrm{i}\omega)U(y)]\, dy + \\ + \int_S e^{(-1)^r\mathrm{i}y\xi^j}[B(D_y, n(y))U(y)]^-\, dS_y - \\ - \int_S e^{(-1)^r\mathrm{i}y\xi^j} Q^\top((-1)^r\mathrm{i}\xi^j, n(y), -\mathrm{i}\omega)[U(y)]^-\, dS_y, \end{aligned} \tag{4.18}$$

ξ^j corresponds to the vector $x/|x|$.

Now (4.15) and (4.16) follow immediately from (4.17) and (4.14). Note that the vector $b(\xi^j)$ is represented explicitly by (4.18). ∎

REMARK 4.4 *From* (4.15) *with the help of equation* (4.5) *we get the following equivalent asymptotic formula for u*

$$u(x) = \sum_{j=1}^{m} |x|^{-1} e^{(-1)^{r+1}\mathrm{i}x\xi^j} \lambda_1(\xi^j, \omega) G(\xi^j, \omega)\mathcal{I}_0 G^\top(\xi^j, \omega) a^{(j)}(\xi^j, \omega) + O(|x|^{-2}), \tag{4.19}$$

where

$$a^{(j)}(\xi^j, \omega) = d_j(\xi^j, -\mathrm{i}\omega) b(\xi^j), \tag{4.20}$$

d_j *and* b *are the same as in Lemma* 4.3.

Note that due to (4.5)

$$\mathcal{I}_0\, G^\top\, a^{(j)} = ([G^\top\, a^{(j)}]_1, 0, 0)^\top. \tag{4.21}$$

4.2. In this subsection we assume $S = \partial\Omega^-$ to be a connected C^1-regular surface and prove the following uniqueness theorem.

THEOREM 4.5 *Let* U *be a regular solution to the homogeneous exterior Problem* $(\overset{\omega}{P}_k)^-$ $(k = 1, ..., 4)$ *and* $U \in SK^m_r(\Omega^-)$ *with* $r = 1$ *for* $\omega > 0$ *and* $r = 2$ *for* $\omega < 0$.

Then $U = 0$ *in* Ω^-.

Proof. Let R, B_R, Σ_R and Ω^-_R be the same as in the proof of Theorem 3.1. Since U satisfies the homogeneous conditions of Problem $(\overset{\omega}{P}_k)^-$, from (1.27) (with $\Omega^+ = \Omega^-_R$ and $\mu = -\mathrm{i}\omega$) it follows that

$$\int\limits_{\Omega^-_R} \{c_{kjpq}\, D_p u_q\, D_k \overline{u}_j - \omega^2\, |u|^2 - \mathrm{i}(\omega T_0)^{-1} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4 + c_0 (T_0)^{-1} |u_4|^2\}\, dx =$$

$$= \int\limits_{\Sigma_R} \{[B(D,n)U]_k\, [\overline{u}_k] - \frac{\mathrm{i}}{\omega T_0} [u_4]\, [\partial_n \overline{u}_4]\}\, d\Sigma_R,$$

where $B(D,n)$ and ∂_n are defined by (1.23) and (1.18).

Owing the fact that $c_{kjpq}\, D_p u_q\, D_k \overline{u}_j$ and $\lambda_{kj}\, D_k u_4\, D_j \overline{u}_4$ are non–negative real quantities, from the latter equation (by separating the imaginary part) we get

$$\mathrm{Im} \left\{ \int\limits_{\Sigma_R} \{[B(D_x, \eta) U(x)]_k\, [\overline{u}_k(x)] - \frac{\mathrm{i}}{\omega T_0} [u_4(x)]\, [\partial_\eta \overline{u}_4(x)]\}\, d\Sigma_R \right\} +$$

$$+ \frac{1}{\omega T_0} \int\limits_{\Omega^-_R} \lambda_{kj}\, D_k u_4(x)\, D_j \overline{u}_4(x)\, dx = 0, \tag{4.22}$$

where $\eta = x/|x|$ is the unit outward normal at the point $x \in \Sigma_R$.

Due to Lemma 4.3 it is easily seen that

$$\int\limits_{\Omega^-_R} \lambda_{kj}\, D_k u_4(x)\, D_j \overline{u}_4(x)\, dx = \int\limits_{\Omega^-} \lambda_{kj}\, D_k u_4(x)\, D_j \overline{u}_4(x)\, dx + O(R^{-1}),$$

$$\int\limits_{\Sigma_R} |u_4(x)\, \partial_\eta \overline{u}_4(x)|\, d\Sigma_R = O(R^{-2}), \quad \int\limits_{\Sigma_R} |u_4(x)\, \overline{u}_k(x)|\, d\Sigma_R = O(R^{-1}),$$

as $R \to +\infty$ $(k = 1, 2, 3)$.

Taking into account (1.23) and applying the above relations to (4.22) we obtain

$$\mathrm{Im} \left\{ \int\limits_{\Sigma_R} [T(D_x, \eta) u]_k\, [\overline{u}_k]\, d\Sigma_R \right\} + \frac{1}{\omega T_0} \int\limits_{\Omega^-} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4\, dx = O(R^{-1}), \tag{4.23}$$

where $T(D,\eta)$ is the stress operator of elastostatics defined by (1.10).

By the same way as in the proof of Theorem 3.1 (by integrating with respect to R from ν to 2ν and deviding the result by ν) from (4.23) we arrive to

$$\mathrm{Im}\left\{\frac{1}{\nu}\int\limits_{\nu}^{2\nu}\int\limits_{\Sigma_R}[T(D_x,\eta)u]_k\,[\overline{u}_k]\,d\Sigma_R\,dR\right\}+\frac{1}{\omega T_0}\int\limits_{\Omega^-}\lambda_{kj}\,D_k u_4\,D_j\overline{u}_4\,dx=O(\nu^{-1}), \tag{4.24}$$

where ν is large enough.

Now by Lemma 4.3 the first summand in the left–hand side of (4.24) can be transformed as follows

$$F(\nu)=\mathrm{Im}\left\{\frac{1}{\nu}\int\limits_{\nu}^{2\nu}\int\limits_{\Sigma_R}[T(D,\eta)u]_k\,[\overline{u}_k]\,d\Sigma_R\,dR\right\}=$$

$$=\mathrm{Im}\left\{\frac{1}{\nu}\int\limits_{\nu}^{2\nu}\int\limits_{\Sigma_R}\sum_{j=1}^{m}[\mathrm{i}(-1)^{r+1}R^{-1}d_j(\xi^j,-\mathrm{i}\omega)e^{(-1)^{r+1}\mathrm{i}x\xi^j}T(\xi^j,\eta)C^*(\xi^j,\omega)b(\xi^j)]_k\times\right.$$

$$\left.\times\sum_{l=1}^{m}[R^{-1}\overline{d_l(\xi^l,-\mathrm{i}\omega)}e^{(-1)^r\mathrm{i}x\xi^l}C^*(\xi^l,\omega)\overline{b(\xi^l)}]_k\,d\Sigma_R\,dR+O(\nu^{-1})\right\}=$$

$$=\mathrm{Re}\left\{\frac{(-1)^{r+1}}{\nu}\int\limits_{\Sigma_1}\sum_{j,l=1}^{m}d_j(\xi^j,-\mathrm{i}\omega)\overline{d_l(\xi^l,-\mathrm{i}\omega)}[T(\xi^j,\eta)C^*(\xi^j,\omega)b(\xi^j)]_k\times\right.$$

$$\left.\times[C^*(\xi^l,\omega)\overline{b(\xi^l)}]_k\left(\int\limits_{\nu}^{2\nu}e^{(-1)^{r+1}\mathrm{i}R[\mu_j(\eta)-\mu_l(\eta)]}\,dR\right)\,d\Sigma_1\right\}+O(\nu^{-1}), \tag{4.25}$$

where $\mu_j(\eta)=(\eta\cdot\xi^j)$ and ξ^j corresponds to the vector $x/|x|$.

It can be easily proved that $\mu_j(\eta)\neq\mu_l(\eta)$ if $j\neq l$ (see Subsection 1.7). Therefore, if $j\neq l$, clearly,

$$\int\limits_{\nu}^{2\nu}e^{\pm\mathrm{i}R[\mu_j(\eta)-\mu_l(\eta)]}\,dR=O(1),$$

and (4.25) implies

$$F(\nu)=\mathrm{Re}\left\{(-1)^{r+1}\sum_{j=1}^{m}\int\limits_{\Sigma_1}T(\xi^j,\eta)C^*(\xi^j,\omega)a^{(j)}\,\cdot\,C^*(\xi^j,\omega)a^{(j)}\,d\Sigma_1\right\}+O(\nu^{-1}) \tag{4.26}$$

with $a^{(j)}$ defined by (4.20).

In view of the symmetry property of $C^*(\xi,\eta)$ and equality $T^\top(\xi,\eta)=T(\eta,\xi)$ we have from (4.26)

$$F(\nu)=\frac{(-1)^{r+1}}{2}\sum_{j=1}^{m}\int\limits_{\Sigma_1}C^*(\xi^j,\omega)[T(\xi^j,\eta)+T(\eta,\xi^j)]C^*(\xi^j,\omega)a^{(j)}\cdot a^{(j)}\,d\Sigma_1+$$

$$+O(\nu^{-1}). \tag{4.27}$$

Now passing to the limit in (4.24) as $\nu \to +\infty$ and bearing in mind (4.25), (4.27) we get

$$\frac{1}{\omega T_0} \int\limits_{\Omega^-} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4\, dx + \frac{(-1)^{r+1}}{2} \sum_{j=1}^{m} \int\limits_{\Sigma_1} E_j(\xi^j, \omega)\, d\Sigma_1 = 0 \tag{4.28}$$

with

$$E_j(\xi^j, \omega) = C^*(\xi^j, \omega)[T(\xi^j, \eta) + T(\eta, \xi^j)]C^*(\xi^j, \omega)a^{(j)} \cdot a^{(j)}, \tag{4.29}$$

where $\xi^j \in S_j$ corresponds to η, i.e., $n(\xi^j) = \eta$.

In what follows we claim that the integral in the second term of (4.28) is a non–negative function for all $\xi^j \in S_j$.

To this end let us note that

$$T(\xi, \eta) + T(\eta, \xi) = \frac{\partial}{\partial n(\xi)} C(\xi) = -\frac{\partial}{\partial n(\xi)} C(\xi, \omega),$$

where $\eta = n(\xi)$, $\partial/\partial n(\xi) = n_k(\xi)\, D_k$ is a directional derivative, $C(\xi)$ and $C(\xi, \omega)$ are defined by (1.5) and (4.3), respectively.

We recall that in Subsection 1.7 we introduced the two sets of surfaces $\{S_j\}_{j=1}^m$ and $\{S_p^0\}_{p=1}^3$ defined by equations (1.44) and by the first equation of the same system, respectively. Therefore each S_j coincides with some S_p^0 for some $p = p(j)$. Let us fix this correspondence, i.e., $S_j = S_p^0$.

Further we proceed as follows. It is obvious that

$$\begin{aligned} -\left[C^*(\xi, \omega)\left(\frac{\partial}{\partial n(\xi^j)} C(\xi, \omega)\right) C^*(\xi, \omega)\right] &= -\frac{\partial}{\partial n(\xi^j)}[C^*(\xi, \omega) C(\xi, \omega) C^*(\xi, \omega)] = \\ &= \left[\frac{\partial}{\partial n(\xi^j)} \Phi(\xi, \omega)\right] C^*(\xi, \omega) \end{aligned} \tag{4.30}$$

for all $\xi = \xi^j \in S_j$ (see (4.4)).

With the help of (4.5), (4.30) and (4.29) we deduce

$$\begin{aligned} E_j(\xi^j, \omega) &= \left\{\left[\frac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] C^*(\xi, \omega) a^{(j)} \cdot a^{(j)}\right\}_{\xi = \xi^j} = \\ &= \left\{\left[\frac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] \lambda_1(\xi, \omega) \mathcal{I}_0 G^\top(\xi, \omega) a^{(j)} \cdot G^\top(\xi, \omega) a^{(j)}\right\}_{\xi = \xi^j} = \\ &= \left\{\left[\frac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] \lambda_1(\xi, \omega) \left|\left[G^\top(\xi, \omega) a^{(j)}\right]_1\right|^2\right\}_{\xi = \xi^j}. \end{aligned} \tag{4.31}$$

Now we show that the function

$$\psi(\xi) = \left[\frac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] \lambda_1(\xi, \omega), \;\; \xi \in S_j, \tag{4.32}$$

is strictly positive.

Since $\lambda_1(\xi,\omega)$ is the only non–zero eigenvalue of the matrix $C^*(\xi,\omega)$ for $\xi \in S_j = S_p^0$, we have

$$\{\lambda_1(\xi,\omega)\}_{\xi\in S_j} = \{\operatorname{Sp} C^*(\xi,\omega)\}_{\xi\in S_j} = \{C_{11}^*(\xi,\omega) + C_{22}^*(\xi,\omega) + C_{33}^*(\xi,\omega)\}_{\xi\in S_j} =$$
$$= \frac{1}{2\omega}\left\{\frac{\partial}{\partial\omega}\begin{vmatrix} \omega^2 - C_{11}(\xi) & -C_{12}(\xi) & -C_{13}(\xi) \\ -C_{12}(\xi) & \omega^2 - C_{22}(\xi) & -C_{23}(\xi) \\ -C_{13}(\xi) & -C_{23}(\xi) & \omega^2 - C_{33}(\xi) \end{vmatrix}\right\}_{\xi\in S_j} =$$
$$= -\frac{1}{2\omega}\left\{\frac{\partial}{\partial\omega}\Phi(\xi,\omega)\right\}_{\xi\in S_j} = -\frac{1}{2\omega}\left\{\frac{\partial}{\partial\omega}\Phi(\xi,\omega)\right\}_{\xi\in S_p^0} =$$
$$= \Phi(\zeta,0)\omega^4\{\varrho_1^2(\varrho_p^2-\varrho_2^2)(\varrho_p^2-\varrho_3^2) + \varrho_2^2(\varrho_p^2-\varrho_1^2)(\varrho_p^2-\varrho_3^2) +$$
$$+\varrho_3^2(\varrho_p^2-\varrho_1^2)(\varrho_p^2-\varrho_2^2)\} = (-1)^{p+1}\left|\left\{\frac{\varrho}{2\omega^2}\frac{\partial}{\partial\varrho}\Phi(\xi,\omega)\right\}_{\varrho=|\omega|\varrho_p}\right| =$$
$$= (-1)^{p+1}\left|\{\lambda_1(\xi,\omega)\}_{\xi\in S_p^0}\right|, \tag{4.33}$$

where $\zeta = \xi/|\xi|$, $\Phi(\zeta,0) > 0$; here we used the representation (1.45).

It is easily checked that the exterior unit normal vector on S_p^0 is calculated by the following equality

$$n(\xi) = (-1)^{p+1}\frac{\nabla\Phi(\xi,\omega)}{|\nabla\Phi(\xi,\omega)|}, \quad \xi \in S_p^0.$$

Therefore

$$\left\{\frac{\partial}{\partial n(\xi)}\Phi(\xi,\omega)\right\}_{\xi\in S_j} = \left\{(-1)^{p+1}\frac{\nabla\Phi(\xi,\omega)}{|\nabla\Phi(\xi,\omega)|}\cdot\nabla\Phi(\xi,\omega)\right\}_{\xi\in S_p^0} =$$
$$= \left\{(-1)^{p+1}|\nabla\Phi(\xi,\omega)|\right\}_{\xi\in S_p^0}, \tag{4.34}$$

which together with (4.33) implies

$$\psi(\xi) = |\nabla\Phi(\xi,\omega)|\;|\lambda_1(\xi,\omega)| > 0 \quad \text{for} \quad \xi \in S_p^0 = S_j. \tag{4.35}$$

Hence by virtue of (4.31)–(4.35) we get

$$E_j(\xi^j,\omega) = \left\{|\nabla\Phi(\xi,\omega)|\;|\lambda_1(\xi,\omega)|\;\left|\left[G^\top(\xi,\omega)a^{(j)}\right]_1\right|^2\right\}_{\xi=\xi^j} \ge 0. \tag{4.36}$$

Now from (4.28) it follows that

$$\lambda_{kj}\,D_k u_4(x)\,D_j\overline{u_4(x)} = 0, \quad x \in \Omega^-, \quad E_j(\xi^j,\omega) = 0, \quad \xi \in S_j,$$

if $(-1)^{r+1}\omega > 0$.

Applying (1.14), (4.35), (4.36) and (4.19)–(4.21) we conclude that $u_4(x) = 0$ in Ω^- and $[G^\top(\xi^j,\omega)a^{(j)}(\xi^j,\omega)]_1 = 0$, i.e.,

$$D^\beta u(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to +\infty \tag{4.37}$$

for an arbitrary multi–index β.

Thus we have obtained that u is a solution to the steady state oscillation equations of elasticity theory

$$C(D)u(x) + \omega^2 u(x) = 0, \quad x \in \Omega^-,$$

satisfying a homogeneous boundary condition on S (either $[u]^- = 0$ or $[Tu]^- = 0$; see (1.15)–(1.22)) and the decay condition (4.37) at infinity.

Then due to the results of [10] (Lemma 3.4), [15] (Section 4) we have $u(x) = 0$ in Ω^-, which completes the proof. ∎

5 PROPERTIES OF POTENTIALS AND BOUNDARY OPERATORS

5.1. Now we introduce the following generalized single- and double layer potentials constructed by the fundamental solution (2.25)

$$V(g)(x) = \int_S \Gamma(x-y,\omega,r)g(y)\,dS_y, \quad x \in I\!R^3 \setminus S, \tag{5.1}$$

$$W(g)(x) = \int_S [Q(D_y,n(y),-\mathrm{i}\omega)\Gamma^\top(x-y,\omega,r)]^\top g(y)\,dS_y, \quad x \in I\!R^3 \setminus S, \tag{5.2}$$

where $S = \partial\Omega^\pm$, $g = (g_1,...,g_4)^\top = (\tilde g, g_4)^\top$, $\tilde g = (g_1,g_2,g_3)^\top$; the operator Q is defined by (1.28) with $\mu = -\mathrm{i}\omega$.

To investigate the existence of solutions to the non–homogeneous BVPs posed in Subsection 1.4 we need special mapping properties of the above potentials and boundary integral (pseudodifferential) operators generated by them.

Let

$$\mathcal{H}\,g(z) = \int_S \Gamma(z-y,\omega,r)\,g(y)\,dS_y, \quad x \in S, \tag{5.3}$$

$$\mathcal{K}_1\,g(z) = \int_S [B(D_z,n(z))\Gamma(z-y,\omega,r)]\,g(y)\,dS_y, \quad z \in S, \tag{5.4}$$

$$\mathcal{K}_2\,g(z) = \int_S [Q(D_y,n(y),-\mathrm{i}\omega)\Gamma^\top(z-y,\omega,r)]^\top\,g(y)\,dS_y, \quad z \in S, \tag{5.5}$$

$$\mathcal{L}^\pm\,g(z) = \lim_{\Omega^\pm \ni x \to z \in S} B(D_x,n(z))W(g)(x), \quad z \in S. \tag{5.6}$$

In the sequel the two positive numbers γ and γ' are chosen as follows $0 < \gamma < \gamma' < 1$.

LEMMA 5.1 *Let $k \geq 0$ be an integer and $S \in \mathrm{C}^{k+1+\gamma'}$.*

Then for an arbitrary summable g the potentials $V(g)$ and $W(g)$ are $\mathrm{C}^\infty(\Omega^\pm)$–smooth solutions to the equation (1.8) in $\Omega^\pm$ and belong to the class $SK_r^m(\Omega^-)$.

The following formulas

$$[V(g)(z)]^+ = [V(g)(z)]^- = \mathcal{H}g(z), \;\; g \in \mathrm{C}(S), \tag{5.7}$$

$$[B(D,n)V(g)(z)]^{\pm} = (\mp 2^{-1} I_4 + \mathcal{K}_1)g(z), \;\; g \in \mathrm{C}^{\gamma}(S), \tag{5.8}$$

$$[W(g)(z)]^{\pm} = (\pm 2^{-1} I_4 + \mathcal{K}_2)g(z), \;\; g \in \mathrm{C}^{\gamma}(S), \tag{5.9}$$

hold and the operators

$$\mathcal{H} \;:\; \mathrm{C}^{k+\gamma}(S) \to \mathrm{C}^{k+1+\gamma}(S), \tag{5.10}$$

$$\mathcal{K}_1, \; \mathcal{K}_2 \;:\; \mathrm{C}^{k+\gamma}(S) \to \mathrm{C}^{k+\gamma}(S), \tag{5.11}$$

$$V \;:\; \mathrm{C}^{k+\gamma}(S) \to \mathrm{C}^{k+1+\gamma}(\overline{\Omega^{\pm}}), \tag{5.12}$$

$$W \;:\; \mathrm{C}^{k+\gamma}(S) \to \mathrm{C}^{k+\gamma}(\overline{\Omega^{\pm}}), \tag{5.13}$$

are bounded.

Proof. The first part of the lemma follows immediately from the properties of the fundamental matrix $\Gamma(x-y,\omega,r)$ and is trivial.

To prove the second part we proceed as follows.

From equations (1.23), (1.28) and Theorem 2.3 we have

$$\Gamma(x-y,\omega,r) = \Gamma(x-y) + \tilde{\Gamma}(x-y,\omega,r), \tag{5.14}$$

$$B(D,n) = B_0(D,n) - \tilde{B}(n), \tag{5.15}$$

$$Q(D,n,-\mathrm{i}\omega) = B_0(D,n) - \mathrm{i}\omega T_0 \tilde{B}(n), \tag{5.16}$$

where $|D^{\beta}\tilde{\Gamma}_{kj}(x,\omega,r)| < c\,\varphi^{(kj)}_{|\beta|}(x), \;\; k,j = 1,...,4,$

$$B_0(D,n) = \left\| \begin{array}{ll} [T(D,n)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \partial_n \end{array} \right\|_{4\times 4}, \quad \tilde{B}(n) = \left\| \begin{array}{ll} [0]_{3\times 3} & [\beta_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{array} \right\|_{4\times 4}$$

with the same $\Gamma(x)$, β, c and $\varphi^{(kj)}_{|\beta|}$ as in Lemma 2.1.

Therefore we can separate the principal singular terms in the above potentials and represent them in the form

$$V(g)(x) = V_0(g)(x) + \tilde{V}(g)(x), \tag{5.17}$$

$$W(g)(x) = W_0(g)(x) + \tilde{W}(g)(x), \tag{5.18}$$

$$B(D,n)V(g)(x) = B_0(D,n)V_0(g)(x) + R(g)(x),$$

where

$$V_0(g)(x) = \int_S \Gamma(x-y)\, g(y)\, dS_y,$$

$$W_0(g)(x) = \int\limits_S [B_0(D_y, n(y))\Gamma(x-y)]^\top g(y)\, dS_y.$$

The kernels of the potentials $\tilde{V}(g)$, $\tilde{W}(g)$ and $R(g)$ have singularities of type $O(|x-y|^{-1})$ as $|x-y| \to 0$. Therefore $\tilde{V}$, $\tilde{W}$ and R are continuous vectors in $I\!R^3$ provided $g \in \mathrm{C}(S)$.

It is easy to see that

$$V_0(g) = (v^{(0)}(\tilde{g}), v_4^{(0)}(g_4))^\top, \quad W_0(g) = (w^{(0)}(\tilde{g}), w_4^{(0)}(g_4))^\top,$$

$$B_0(D,n)V_0(g) = (T(D,n)v^{(0)}(\tilde{g}), \partial_n v_4^{(0)}(g_4))^\top,$$

where $v^{(0)}(\tilde{g})$ and $w^{(0)}(\tilde{g})$ are single- and double layer potentials of elastostatics (corresponding to the operator $C(D)$) constructed by the fundamental matrix $\Gamma^{(0)}(x)$:

$$v^{(0)}(\tilde{g})(x) = \int\limits_S \Gamma^{(0)}(x-y)\, \tilde{g}(y)\, dS_y,$$

$$w^{(0)}(\tilde{g})(x) = \int\limits_S [T(D_y, n(y))\Gamma^{(0)}(y-x)]^\top\, \tilde{g}(y)\, dS_y, \tag{5.19}$$

while $v_4^{(0)}(g_4)$ and $w_4^{(0)}(g_4)$ are potentials of the same type (corresponding to the homogeneous operator $\Lambda(D)$) constructed by the fundamental function $\gamma^{(0)}(x)$:

$$v_4^{(0)}(g_4)(x) = \int\limits_S \gamma^{(0)}(x-y)\, g_4(y)\, dS_y,$$

$$w_4^{(0)}(g_4)(x) = \int\limits_S \partial_{n(y)}\gamma^{(0)}(y-x)\, g_4(y)\, dS_y, \tag{5.20}$$

(see Lemma 2.1).

The properties of the latter potentials and boundary integral operators on S, generated by them, are studied in detail for regular function spaces in [2], [13], [14], [16], [17]. The results mentioned together with the representation formulas (5.17), (5.18), yield equations (5.7)–(5.9) and mapping properties (5.10)–(5.13). ∎

For a pseudodifferential operator (ΨDO) $\mathcal{P}$ on S we denote by $(\mathcal{P})_0$ and $\sigma(\mathcal{P})(x,\tilde{\xi})$ ($x \in S$, $\tilde{\xi} \in I\!R^2$) the principal singular part and the principal homogeneous symbol, respectively.

LEMMA 5.2 *Operators $\mathcal{H}$, $\pm 2^{-1}I_4 + \mathcal{K}_1$ and $\pm 2^{-1}I_4 + \mathcal{K}_2$ are elliptic ΨDOs of order -1, 0 and 0, respectively, with index equal to zero.*

Proof. From equations (5.14), (5.15), (5.16) together with (5.3), (5.4), (5.5) it follows that

$$(\mathcal{H})_0 = \left\| \begin{matrix} [\mathcal{H}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{H}_4^{(0)} \end{matrix} \right\|_{4\times 4}, \tag{5.21}$$

$$
(\pm 2^{-1}I_4+\mathcal{K}_1)_0=\left\|\begin{matrix}[\pm 2^{-1}I_3+\mathcal{K}^{(0)}]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \pm 2^{-1}I_1+\mathcal{K}_4^{(0)}\end{matrix}\right\|_{4\times 4}, \tag{5.22}
$$

$$
(\pm 2^{-1}I_4+\mathcal{K}_2)_0=\left\|\begin{matrix}[\pm 2^{-1}I_3+\overset{*}{\mathcal{K}}{}^{(0)}]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \pm 2^{-1}I_1+\overset{*}{\mathcal{K}}{}_4^{(0)}\end{matrix}\right\|_{4\times 4}, \tag{5.23}
$$

where

$$
\mathcal{H}^{(0)}\,\tilde g(z)=\int\limits_S \Gamma^{(0)}(z-y)\,\tilde g(y)\,dS_y,\quad \mathcal{H}_4^{(0)}\,g_4(z)=\int\limits_S \gamma^{(0)}(z-y)\,g_4(y)\,dS_y,
$$

$$
\mathcal{K}^{(0)}\,\tilde g(z)=\int\limits_S [T(D_z,n(z))\Gamma^{(0)}(z-y)]\,\tilde g(y)\,dS_y,
$$

$$
\overset{*}{\mathcal{K}}{}^{(0)}\,\tilde g(z)=\int\limits_S [T(D_y,n(y))\Gamma^{(0)}(y-z)]^\top\,\tilde g(y)\,dS_y,
$$

$$
\mathcal{K}_4^{(0)}\,g_4(z)=\int\limits_S \partial_{n(z)}\gamma^{(0)}(z-y)\,g_4(y)\,dS_y,
$$

$$
\overset{*}{\mathcal{K}}{}_4^{(0)}\,g_4(z)=\int\limits_S \partial_{n(y)}\gamma^{(0)}(y-z)\,g_4(y)\,dS_y.
$$

Due to the general theory of ΨDOs (see, e.g., [4]) we have to show that the principal symbol matrices of the operators (5.21), (5.22) and (5.23) are non–singular and that the indices of these operators are equal to zero.

It is evident that $\mathcal{K}^{(0)}$ $[\mathcal{K}_4^{(0)}]$ and $\overset{*}{\mathcal{K}}{}^{(0)}$ $[\overset{*}{\mathcal{K}}{}_4^{(0)}]$ are mutually adjoint singular integral operators while $\mathcal{H}^{(0)}$ $[\mathcal{H}_4^{(0)}]$ is a formally self–adjoint integral operator with a weakly singular kernel.

For the principal symbols we have (see [14], [17])

$$
\sigma(\mathcal{H}^{(0)})=-\frac{1}{2\pi}\int\limits_{l^\mp} C^{-1}(a\xi)\,d\xi_3=-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} C^{-1}(a\xi)\,d\xi_3, \tag{5.24}
$$

$$
\sigma(\pm 2^{-1}I_3+\mathcal{K}^{(0)})=\frac{\mathrm{i}}{2\pi}\int\limits_{l^\mp} T(a\xi,n)C^{-1}(a\xi)\,d\xi_3=\overline{[\sigma(\pm 2^{-1}I_3+\overset{*}{\mathcal{K}}{}^{(0)})]}^\top, \tag{5.25}
$$

$$
\sigma(\mathcal{H}_4^{(0)})=-\frac{1}{2\pi}\int\limits_{l^\mp} \Lambda^{-1}(a\xi)\,d\xi_3=-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \Lambda^{-1}(a\xi)\,d\xi_3, \tag{5.26}
$$

$$
\sigma(\pm 2^{-1}I_1+\mathcal{K}_4^{(0)})=\frac{\mathrm{i}}{2\pi}\int\limits_{l^\mp} \lambda(a\xi,n)\Lambda^{-1}(a\xi)\,d\xi_3=\overline{\sigma(\pm 2^{-1}I_1+\overset{*}{\mathcal{K}}{}_4^{(0)})}, \tag{5.27}
$$

where $\xi = (\tilde{\xi}, \xi_3)$, $\tilde{\xi} = (\xi_1, \xi_2) \in I\!R^2$, $\lambda(\xi, n)$ is defined by (1.18), $a(x)$ is an orthogonal matrix

$$a(x) = \left\| \begin{array}{ccc} l_1(x) & m_1(x) & n_1(x) \\ l_2(x) & m_2(x) & n_2(x) \\ l_3(x) & m_3(x) & n_3(x) \end{array} \right\| ;$$

$l = (l_1, l_2, l_3)^\top$, $m = (m_1, m_2, m_3)^\top$ and $n = (n_1, n_2, n_3)^\top$ is a triple of orthogonal vectors at $x \in S$ (l and m lie in the tangent plane at $x \in S$ and n is the exterior unit normal), l^- (l^+) is a closed clockwise (counter–clockwise) oriented contour in the lower (upper) complex half–plane $\xi_3 = \xi_3' + \mathrm{i}\xi_3''$ enclosing all roots of the equations

$$\det C(a\xi) = 0, \quad \Lambda(a\xi) = 0,$$

with respect to ξ_3 with negative (positive) imaginary parts.

The entries of the matrices (5.25) [(5.24)] and functions (5.27) [(5.26)] are homogeneous of order 0 [-1] in $\tilde{\xi} \in I\!R^2$. Moreover all the above principal symbols are non–singular for $|\tilde{\xi}| = 1$, the corresponding integral operators are elliptic ΨDOs of order 0 and -1, respectively, and their indices are equal to zero (for details see [3], [10], [14], [17]).

Now (5.21), (5.22) and (5.23) imply

$$\sigma(\mathcal{H}) = \left\| \begin{array}{cc} [\sigma(\mathcal{H}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{H}_4^{(0)}) \end{array} \right\|_{4\times 4},$$

$$\sigma(\pm 2^{-1} I_4 + \mathcal{K}_1) = \left\| \begin{array}{cc} [\sigma(\pm 2^{-1} I_3 + \mathcal{K}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\pm 2^{-1} I_1 + \mathcal{K}_4^{(0)}) \end{array} \right\|_{4\times 4} = \\ = [\overline{\sigma(\pm 2^{-1} I_4 + \mathcal{K}_2)}]^\top,$$

which together with equations (5.21), (5.22), (5.23) and the above mentioned results completes the proof. ■

REMARK 5.3 *More subtle analyse of the fundamental solution $\Gamma(x, \omega, r)$ shows that in a vicinity of the origin the following representation*

$$\Gamma(x, \omega, r) = \Gamma(x) + \mathrm{i}\tilde{\Gamma}'(x) - \omega T_0 [\tilde{\Gamma}'(x)]^\top + \tilde{\Gamma}''(x, \omega, r), \tag{5.28}$$

$$\tilde{\Gamma}'(x) = \left\| \begin{array}{cc} [0]_{3\times 3} & [\tilde{\Gamma}_{k4}'(x)]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{array} \right\|_{4\times 4},$$

holds, where $\Gamma(x)$ is the same as in Lemma 2.1 and $\tilde{\Gamma}_{k4}'(x)$ is independent of ω; first order derivatives of $\tilde{\Gamma}_{k4}'(x)$ are homogeneous functions of order -1 and

$$|D^\beta \tilde{\Gamma}_{k4}'(x)| < c\,\varphi_{|\beta|}^{(k4)}(x)$$

with the same $\varphi_{|\beta|}^{(k4)}(x)$ as in Lemma 2.1, the second order derivatives of entries of the matrix $\tilde{\Gamma}''(x, \omega, r)$ have singularities of the type $O(|x|^{-1})$.

REMARK 5.4 *Note that the operator* $-\mathcal{H}^{(0)}$ $[-\mathcal{H}_4^{(0)}]$ *is a positive operator which implies that the corresponding principal symbol is a positive definite matrix [is a positive function] (see* [14]*).*

5.2. Now we turn our attention to the equation (5.6). To prove the existence of limits (5.6) and to study properties of the operators $\mathcal{L}^{\pm}$ we need some auxiliary results which are now presented.

LEMMA 5.5 *Let* $U = (u, u_4)^\top$ *be a regular solution of the homogeneous interior Problem* $(\overset{\omega}{P}_1)^+$. *Then* $u_4(x) = 0$ *in* Ω^+ *and* u *is a solution to the following interior homogeneous BVP of steady state oscillations of the elasticity theory*

$$C(D)u(x) + \omega^2\, u(x) = 0 \quad \text{in} \quad \Omega^+, \tag{5.29}$$

$$[u(z)]^+ = 0 \quad \text{on} \quad S. \tag{5.30}$$

Proof. The equation $u_4(x) = 0$ in Ω^+ follows from the identity (1.27) if we look at the imaginary part. Then we obtain the BVP (5.29)–(5.30) for the vector u due to the conditions (1.8), (1.15) and (1.16) (where $f_k = 0$, $k = 1, ..., 4$, are provided). ∎

By $\Sigma[(\overset{\omega}{P}_1)^+]$ we denote the spectral set corresponding to Problem $(\overset{\omega}{P}_1)^+$ (i.e., the set of values of parameter ω for which the homogeneous Problem $(\overset{\omega}{P}_1)^+$ possesses a non–trivial solution). Now Lemma 5.5 implies (see [14])

COROLLARY 5.6 *The set* $\Sigma[(\overset{\omega}{P}_1)^+]$ *is either finite or countable (with the only possible accumulation point at infinity).*

LEMMA 5.7 *Let* $S \in \mathrm{C}^{2+\gamma'}$ *and* $g \in \mathrm{C}^{1+\gamma}(S)$. *Then limits* (5.6) *exist and*

$$\mathcal{L}^+ g(z) = \mathcal{L}^- g(z) \equiv \mathcal{L}\, g(z), \quad z \in S. \tag{5.31}$$

Moreover the operator

$$\mathcal{L} \,:\, \mathrm{C}^{k+1+\gamma}(S) \to \mathrm{C}^{k+\gamma}(S), \quad S \in \mathrm{C}^{k+2+\gamma'}, \tag{5.32}$$

is a bounded singular integro–differential operator with non–singular (positive definite) principal symbol matrix and index equal to zero.

Proof. First we prove the existence of limits (5.6). With the help of equations (5.15), (5.16) and (5.28) we deduce

$$\begin{aligned} B(D_x, n(x))[Q(D_y, n(y), -\mathrm{i}\omega)\Gamma^\top(x-y,\omega,r)]^\top &= \tilde{K}_3(x,y,x-y) + \\ +[\tilde{K}_2'(x,y,x-y) + \omega T_0 \tilde{K}_2''(x,y,x-y)] &+ \tilde{K}_1(x,y,x-y;\omega), \end{aligned} \tag{5.33}$$

where

$$\tilde{K}_3(x,y,x-y) = B_0(D_x,n(x))[B_0(D_y,n(y))\Gamma(y-x)]^\top =$$

$$= \left\| \begin{matrix} [T(D_x,n(x))[T(D_y,n(y))\Gamma(y-x)]^\top]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \partial_{n(x)}\partial_{n(y)}\gamma^{(0)}(y-x) \end{matrix} \right\|_{4\times 4}$$

is a hypersingular kernel with entries of type $O(|x-y|^{-3})$ as $|x-y|\to 0$,

$$\tilde{K}_2'(x,y,x-y)=\mathrm{i}B_0(D_x,n(x))\{B_0(D_y,n(y))[\tilde{\Gamma}'(x-y)]^\top\}^\top-\\ -\tilde{B}(n(x))[B_0(D_y,n(y))\Gamma(x-y)]^\top$$

and

$$\tilde{K}_2''(x,y,x-y)=-B_0(D_x,n(x))[B_0(D_y,n(y))\tilde{\Gamma}'(x-y)]^\top-\\ -\mathrm{i}[B_0(D_x,n(x))\Gamma(x-y)]\tilde{B}^\top(n(y))$$

are singular kernels on S with entries of type $O(|x-y|^{-2})$ as $|x-y|\to 0$, and entries of the matrix $\tilde{K}_1(x,y,x-y;\omega)$ have singularities of type $O(|x-y|^{-1})$; here either $x\in\Omega^+$ or $x\in\Omega^-$.

In turn (5.33) implies

$$B(D_x,n(x))W(g)(x)=(T(D_x,n(x))w^{(0)}(\tilde{g})(x),\partial_{n(x)}w_4^{(0)}(g_4)(x))^\top+\\ +\int\limits_S[\tilde{K}_2'(x,y,x-y)+\omega T_0\tilde{K}_2''(x,y,x-y)]\,g(y)\,dS_y+\\ +\int\limits_S\tilde{K}_1(x,y,x-y;\omega)\,g(y)\,dS_y,\tag{5.34}$$

where $w^{(0)}(\tilde{g})$ and $w_4^{(0)}(g_4)$ are defined by (5.19) and (5.20), respectively. In [3], [14], [17] it is proved that the limits

$$\lim_{\Omega^\pm\ni x\to z\in S}T(D_x,n(x))w^{(0)}(\tilde{g})(x)=\mathcal{L}^{(0)}\tilde{g}(z),\tag{5.35}$$

$$\lim_{\Omega^\pm\ni x\to z\in S}\partial_{n(x)}w_4^{(0)}(g_4)(x)=\mathcal{L}_4^{(0)}g_4(z),\tag{5.36}$$

exist for any $g_k\in\mathrm{C}^{1+\gamma}(S)$, $k=1,...,4$, and the operators $\mathcal{L}^{(0)}$ and $\mathcal{L}_4^{(0)}$ are non–negative, formally self–adjoint singular integro–differential operators with positive definite principal symbols

$$\sigma(\mathcal{L}^{(0)})=-\frac{1}{2\pi}\int\limits_{l^\mp}T(a\xi,n)C^{-1}(a\xi)T^\top(a\xi,n)\,d\xi_3,\tag{5.37}$$

$$\sigma(\mathcal{L}_4^{(0)})=-\frac{1}{2\pi}\int\limits_{l^\mp}\lambda^2(a\xi,n)\Lambda^{-1}(a\xi)\,d\xi_3.\tag{5.38}$$

Here the contours $l^\mp$ are the same as in formulas (5.24)–(5.27).

The operators $\mathcal{L}^{(0)}$ and $\mathcal{L}_4^{(0)}$ are elliptic ΨDOs of order 1 with index equal to zero and they possess mapping property (5.32) (for details see [3]).

Further, Remark 5.3 yields that there exist limits on S from $\Omega^\pm$ of the second term in the right–hand side expression of (5.34)

$$\lim_{\Omega^\pm\ni x\to z\in S}\int\limits_S[\tilde{K}_2'(x,y,x-y)+\omega T_0\tilde{K}_2''(x,y,x-y)]\,g(y)\,dS_y=\\ =[\alpha_\pm'(z)+\omega T_0\alpha_\pm''(z)]g(z)+\tilde{\mathcal{K}}_2'\,g(z)+\omega T_0\,\tilde{\mathcal{K}}_2''\,g(z),$$

where $\tilde{\mathcal{K}}_2'$ and $\tilde{\mathcal{K}}_2''$ are singular integral operators with singular kernels $\tilde{K}_2'$ and $\tilde{K}_2''$, respectively; $\alpha_\pm'$ and $\alpha_\pm'$ are some smooth matrices independent of ω.

The existence of the limits on S (from $\Omega^\pm$) of the third term in the right–hand side of (5.34) is evident. It is also obvious that these limits are the same and the boundary operator $\tilde{\mathcal{K}}_1$ generated by this term is a weakly singular integral operator (ΨDO of order $s \leq -1$).

Thus the existence of the operators $\mathcal{L}^\pm$ is proved in the space $\mathrm{C}^{1+\gamma}(S)$ and we have

$$\mathcal{L}^\pm g(z) = \left\| \begin{array}{ll} [\mathcal{L}^{(0)}\tilde{g}(z)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)} g_4(z) \end{array} \right\|_{4\times 4} + $$
$$+[\alpha_\pm'(z) + \omega T_0\, \alpha_\pm''(z)]g(z) + \tilde{\mathcal{K}}_2' g(z) + \omega T_0 \tilde{\mathcal{K}}_2'' g(z) + \tilde{\mathcal{K}}_1 g(z). \tag{5.39}$$

We also see that operators (5.39) possess the mapping property (5.32).

It remains to show $\mathcal{L}^+ = \mathcal{L}^-$.

The integral representation formulas (3.2) and (3.3) of a regular vector U we rewrite as follows

$$U(x) = \pm\{W([U]^\pm)(x) - V([BU]^\pm)(x)\}, \quad x \in \Omega^\pm, \tag{5.40}$$

provided $A(D, -\mathrm{i}\omega)U(x) = 0$ in $\Omega^\pm$ and $U \in SK_r^m(\Omega^-)$; here W and V are double- and single layer potentials (see (5.1) and (5.2)).

Due to Lemma 5.1 from (5.40) we have

$$(-2^{-1}I_4 + \mathcal{K}_2)[U]^+ = \mathcal{H}[BU]^+, \quad (2^{-1}I_4 + \mathcal{K}_2)[U]^- = \mathcal{H}[BU]^-,$$

where the operators $\mathcal{H}$ and $\mathcal{K}_2$ are defined by (5.3) and (5.5), respectively.

If in the latter equations we substitute $U(x) = W(g)(x)$ with an arbitrary $g \in \mathrm{C}^{1+\gamma}(S)$, apply the same Lemma 5.1 and the above results concerning the limits (5.6), we arrive to the following relations

$$(-2^{-1}I_4 + \mathcal{K}_2)(2^{-1}I_4 + \mathcal{K}_2)\, g = \mathcal{H}\mathcal{L}^+ g,$$
$$(2^{-1}I_4 + \mathcal{K}_2)(-2^{-1}I_4 + \mathcal{K}_2)\, g = \mathcal{H}\mathcal{L}^- g.$$

Whence

$$\mathcal{H}(\mathcal{L}^+ g - \mathcal{L}^- g) = 0. \tag{5.41}$$

By (5.39) we have $\mathcal{L}^+ g - \mathcal{L}^- g \equiv h \in \mathrm{C}^\gamma(S)$ and therefore $V(h)$ is a regular vector in $\Omega^\pm$.

Now, from one side, (5.41) yields that $V(h)$ is a regular solution of the homogeneous Problem $(\overset{\omega}{P}_1)^-$ and we conclude $V(h)(x) = 0, \quad x \in \Omega^-$, due to Theorem 4.5.

On the other side, the same equation (5.41) implies that $V(h)$ is a regular solution of the homogeneous Problem $(\overset{\omega}{P}_1)^+$ as well and by Corollary 5.6 we get $V(h)(x) = 0, \quad x \in \Omega^+$, provided $\omega \overline{\in} \ \Sigma[(\overset{\omega}{P}_1)^+]$.

The above equations imply $h = [BV(h)]^- - [BV(h)]^+ = 0$.

Thus we have proved that $\mathcal{L}^+ g = \mathcal{L}^- g$ for all $g \in \mathrm{C}^{1+\gamma}(S)$, if $\omega \overline{\in} \ \Sigma[(\overset{\omega}{P}_1)^+]$, which according to (5.39) leads to the equation

$$[\alpha_+'(z) - \alpha_-'(z)]g(z) + \omega T_0[\alpha_+''(z) - \alpha_-''(z)]g(z) = 0.$$

Consequently, $\alpha'_+(z) = \alpha'_-(z)$, $\quad \alpha''_+(z) = \alpha''_-(z)$, and (5.31) holds for an arbitrary value of the parameter ω.

It is also evident that the principal singular part $(\mathcal{L})_0$ of the operator $\mathcal{L}$ and the corresponding principal symbol matrix read

$$(\mathcal{L})_0 = \left\| \begin{matrix} [\mathcal{L}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)} \end{matrix} \right\|_{4\times 4}, \tag{5.42}$$

$$\sigma(\mathcal{L}) = \left\| \begin{matrix} [\sigma(\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{L}_4^{(0)}) \end{matrix} \right\|_{4\times 4}, \tag{5.43}$$

(see (5.35), (5.36), (5.37), (5.38)) from which positive definiteness of the matrix (5.43) and formally self–adjointness of the operator (5.42) follow immediately. ∎

6 EXISTENCE THEOREMS

6.1. First we present two lemmas which will essentially be used in the proof of existence theorems.

LEMMA 6.1 *Let* $g \in \mathrm{C}^{1+\gamma}(S)$, $\; S \in \mathrm{C}^{2+\gamma'}$ *and*

$$U(x) = W(g)(x) + p_0\, V(g)(x), \quad x \in I\!R^3 \setminus S, \;\; S = \partial\Omega^{\pm}, \tag{6.1}$$

$$p_0 = p_1 + \mathrm{i}p_2, \quad p_1 \geq 0, \;\; p_2 \operatorname{sgn} \omega < 0, \tag{6.2}$$

where V *and* W *are single- and double layer potentials defined by* (5.1) *and* (5.2), *respectively, while* ω *is the oscillation (frequency) parameter.*

If the vector U *vanishes in* Ω^-, *then the density* $g = 0$ *on* S.

Proof. Due to Lemmas 5.1 and 5.7, we clearly have

$$g = [U]^+ - [U]^- = [U]^+, \quad -p_0\, g = [BU]^+ - [BU]^- = [BU]^+, \tag{6.3}$$

whence

$$[BU]^+ = -p_0\, [U]^+ \quad \text{on} \quad S \tag{6.4}$$

follows.

Since U is a regular vector in Ω^+ we can apply the identity (1.27). Taking into account (6.4) and separating the imaginary part, we arrive to the equation

$$\frac{1}{\omega T_0} \int\limits_{\Omega^+} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4\, dx - p_2 \int\limits_S |[u]^+|^2\, dS + \frac{p_1}{\omega T_0} \int\limits_S |[u_4]^+|^2\, dS = 0.$$

In view of (1.14), (1.23), (6.2) and (6.4) from the latter equality it follows that $[U]^+ = 0$ and by (6.3) we get $g = 0$. ∎

In the sequel we fix the complex number p_0 as follows

$$p_0 = 1 - \mathrm{i}\omega. \tag{6.5}$$

The next lemma is well–known from the theory of harmonic functions (see, e.g., [4]).

LEMMA 6.2 *The scalar operator*

$$\mathcal{R}\,h(z) = \frac{1}{2\pi}\int\limits_S |z-y|^{-1}\,h(y)\,dS_y, \quad z\in S, \quad S\in \mathrm{C}^{1+\gamma'}, \tag{6.6}$$

generated by the harmonic single layer potential, is a formally self–adjoint, equivalent lifting ΨDO *of order* -1 ($\mathcal{R}\,h=0$ *implies* $h=0$) *with the principal symbol equal to* 1 *on the unit circle* (*i.e.*, $\boldsymbol{\sigma}(\mathcal{R})(x,\tilde{\xi})=1,\ x\in S,\ |\tilde{\xi}|=1$).

REMARK 6.3 *The latter lemma yields that* $\mathcal{L}\mathcal{R}$ *and* $\mathcal{R}\mathcal{L}$ *are singular integral operators of normal type with index equal to zero.*

6.2. Problem $(\overset{\omega}{P}_1)^-$**.** We look for a solution of the problem in the form (6.1) with p_0 defined by (6.5). By virtue of the boundary conditions (1.15), (1.16) and Lemma 5.1, we get the following ΨDE on S for the unknown density vector g

$$\mathcal{N}_1\,g \equiv (-2^{-1}I_4 + \mathcal{K}_2 + p_0\,\mathcal{H})\,g \;=\; f \tag{6.7}$$

with $f=(f_1,...,f_4)^\top$.

LEMMA 6.4 *Let* $k\geq 0$ *be an integer and* $S\in \mathrm{C}^{k+2+\gamma'}$.
Then the ΨDO

$$\mathcal{N}_1 \equiv -2^{-1}I_4 + \mathcal{K}_2 + p_0\,\mathcal{H} \,:\, \mathrm{C}^{l+\gamma}(S)\to \mathrm{C}^{l+\gamma}(S), \quad 0\leq l\leq k+1, \tag{6.8}$$

is an isomorphism.
The inverse operator to (6.8) *is a singular integral operator of normal type with index equal to zero.*

Proof. First let us note that the operator $\mathcal{N}_1$ is a singular integral operator of normal type with index equal to zero and possesses the mapping property (6.8) due to Lemmas 5.1 and 5.2. Therefore it remains to prove that

$$\mathcal{N}_1\,g = 0 \tag{6.9}$$

has only the trivial solution in $\mathrm{C}^{\gamma}(S)$.

Let g be some solution of (6.9) and construct the vector U by formula (6.1). Applying the emmbeding theorems for solutions to a singular integral equation (SIE) of normal type on closed smooth manifold we infer that $g\in \mathrm{C}^{k+1+\gamma}(S)$ (see, e.g., [12], Ch. 4). This implies that U is a regular vector in $\Omega^{\pm}$. Now the equation (6.9) yields that $[U]^-=0$ on S, and, consequently, $U(x)=0$ in Ω^- follows immediately by Theorem 4.5, since $U\in SK_r^m(\Omega^-)$. But then $g=0$ by Lemma 6.1. Therefore (6.8) is a one–to–one correspondence and due to the general theory of SIE the inverse operator possesses all properties stated in the above lemma. ∎

The material collected until now is enough to prove the existence theorem.

THEOREM 6.5 *Let* $S\in \mathrm{C}^{k+2+\gamma'}$, $k\geq 0$ *and* $f_j\in \mathrm{C}^{k+1+\gamma}(S)$ $(j=1,...,4)$.
Then Problem $(\overset{\omega}{P}_1)^-$ *has a unique regular solution of the class* $\mathrm{C}^{k+1+\gamma}(\overline{\Omega^-})\cap$ $\cap SK_r^m(\Omega^-)$ *and it is representable in the form* (6.1) *with the density* g *defined by the uniquely solvable SIE* (6.7).

Proof. It follows from Lemmas 5.1, 6.4 and Theorem 4.5. ∎

REMARK 6.6 *We note that the special representation* (6.1) *reduces the BVP* $(\overset{\omega}{P}_1)^-$ *to the equivalent boundary integral equation* (6.7) *for an arbitrary value of the frequency parameter* ω. *If one seeks the solution in the form of either single or double layer potentials then such equivalency will be violated in general* (*for details see* [15], *Remark* 5.7).

6.3. Problem $(\overset{\omega}{P}_2)^-$**.** We look for a solution again in the form (6.1). Then the boundary conditions (1.17) and (1.18) lead to the following system of ΨDEs on S for the unknown density $g = (\tilde{g}, g_4)^\top$

$$\mathcal{N}_2\, g \equiv \{B^{(2)}(D,n)[W(g) + p_0\, V(g)]\}^- = f,$$

i.e.,

$$\{[-2^{-1}I_4 + \mathcal{K}_2 + p_0\mathcal{H}]\, g\}_q = f_q, \quad q = 1,2,3, \tag{6.10}$$

$$\{[\mathcal{L} + p_0\,(2^{-1}I_4 + \mathcal{K}_1)]g\}_4 = f_4. \tag{6.11}$$

Therefore the operator $\mathcal{N}_2$ is represented as

$$\mathcal{N}_2 = \left\| \begin{matrix} [\{-2^{-1}I_4 + \mathcal{K}_2 + p_0\mathcal{H}\}_{ql}]_{3\times 4} \\ [\{\mathcal{L} + p_0\,(2^{-1}I_4 + \mathcal{K}_1)\}_{4l}]_{1\times 4} \end{matrix} \right\|_{4\times 4} = (\mathcal{N}_2)_0 + \tilde{\mathcal{N}}_2,$$

$$q = 1,2,3, \quad l = 1,\dots,4,$$

where $(\mathcal{N}_2)_0$ is the main singular part of $\mathcal{N}_2$. Due to (5.23), (5.42) and Lemma 5.1 we have

$$(\mathcal{N}_2)_0 = \left\| \begin{matrix} [-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)} \end{matrix} \right\|_{4\times 4}. \tag{6.12}$$

The entries of the first three rows of the matrix $\tilde{\mathcal{N}}_2$ are weakly singular integral operators (ΨDOs of order $s \le -1$) while the fourth row contains singular integral operators (ΨDOs of order $s \le 0$). It is easy to see that (6.12) is a ΨDO elliptic in the sense of Douglis–Nirenberg.

Now it is also evident that the operator

$$\mathcal{R}_2 = \left\| \begin{matrix} [I_3]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{R} \end{matrix} \right\|_{4\times 4}$$

with $\mathcal{R}$ defined by (6.6), is an equivalent lifting operator, which reduces the system (6.10)-(6.11) to the equivalent system of singular integral equations

$$\mathcal{R}_2\mathcal{N}_2\, g = (f_1, f_2, f_3, \mathcal{R}f_4)^\top.$$

For the principal symbol matrix (homogeneous of order 0) we have

$$\sigma(\mathcal{R}_2\mathcal{N}_2) = \left\| \begin{matrix} [\sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{R}\mathcal{L}_4^{(0)}) \end{matrix} \right\|_{4\times 4},$$

which is non–singular due to Lemmas 5.2, 5.7 and 6.2.

LEMMA 6.7 *Let $S \in \mathrm{C}^{k+2+\gamma'}$, $k \geq 0$.*
Then the ΨDO

$$\mathcal{N}_2 \,:\, [\mathrm{C}^{l+1+\gamma}(S)]^4 \to [\mathrm{C}^{l+1+\gamma}(S)]^3 \times \mathrm{C}^{l+\gamma}(S), \;\; 0 \leq l \leq k, \tag{6.13}$$

is an isomorphism.

Proof. The mapping property (6.13) of the operator $\mathcal{N}_2$ is an easy consequence of Lemmas 5.1 and 5.7. Clearly, the invertibility of the operator (6.13) is equivalent to the invertibility of the operator

$$\mathcal{R}_2\mathcal{N}_2 \,:\, \mathrm{C}^{l+1+\gamma}(S) \to \mathrm{C}^{l+1+\gamma}(S), \;\; 0 \leq l \leq k, \tag{6.14}$$

according to Lemma 6.2.

Now from Lemmas 5.2, 5.7 and 6.2 it follows that $\mathcal{R}_2\mathcal{N}_2$ is a singular integral operator of normal type with index equal to zero. By the arguments applied in the proof of Lemma 6.4 we can show that the homogeneous equation $\mathcal{N}_2 g = 0$, $g \in \mathrm{C}^{\gamma}(S)$, has only the trivial solution $g = 0$. Further by Lemma 6.2, we conclude that the same is also valid for the equation $\mathcal{R}_2\mathcal{N}_2 g = 0$, which completes the proof. ∎

THEOREM 6.8 *Let $S \in \mathrm{C}^{k+2+\gamma'}$, $k \geq 0$, and $f_q \in \mathrm{C}^{k+1+\gamma}(S)$, $q = 1,2,3$, $f_4 \in \mathrm{C}^{k+\gamma}(S)$.*

Then Problem $(\overset{\omega}{P}_2)^-$ has a unique regular solution of the class $\mathrm{C}^{k+1+\gamma}(\overline{\Omega^-}) \cap$ $\cap SK_r^m(\Omega^-)$ and it is representable in the form (6.1) *with the density g defined by the uniquely solvable ΨDEs* (6.10), (6.11).

Proof. It is a ready consequence of Lemmas 5.1, 6.7 and Theorem 4.5. ∎

6.4. Problem $(\overset{\omega}{P}_3)^-$. We use the same representation (6.1) of a solution. Then the boundary conditions (1.19) and (1.20) imply the following system of ΨDEs for the unknown density g on S:

$$\mathcal{N}_3\, g \equiv \{B^{(3)}(D,n)[W(g) + p_0\, V(g)]\}^- = f,$$

i.e.,

$$\{[\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)]\, g\}_q = f_q, \quad q = 1,2,3, \tag{6.15}$$

$$\{[-2^{-1}I_4 + \mathcal{K}_2 + p_0\mathcal{H}]g\}_4 = f_4. \tag{6.16}$$

Clearly, $\mathcal{N}_3$ is representable in the form

$$\mathcal{N}_3 = \left\| \begin{array}{c} [\{\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)\}_{ql}]_{3\times 4} \\ [\{-2^{-1}I_4 + \mathcal{K}_2) + p_0\mathcal{H}\}_{4l}]_{1\times 4} \end{array} \right\|_{4\times 4} = (\mathcal{N}_3)_0 + \tilde{\mathcal{N}}_3,$$

$$q = 1,2,3, \;\; l = 1,...,4,$$

where

$$(\mathcal{N}_3)_0 = \left\| \begin{matrix} [\mathcal{L}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & -2^{-1}I_1 + \overset{*}{\mathcal{K}}{}_4^{(0)} \end{matrix} \right\|_{4\times 4}$$

is the main singular part of $\mathcal{N}_3$ due to (5.23) and (5.42); as to the operator $\tilde{\mathcal{N}}_3$ it contains ΨDOs of order $s \le 0$ in the first three rows and ΨDOs of order $s \le -1$ in the fourth row. Obviously $\mathcal{N}_3$ is again an elliptic ΨDO in the sense of Douglis–Nirenberg.

The following diagonal operator

$$\mathcal{R}_3 = \left\| \begin{matrix} [I_3\mathcal{R}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & I_1 \end{matrix} \right\|_{4\times 4}$$

with $\mathcal{R}$ defined by (6.6), is an equivalent lifting operator, which reduces the system (6.15)-(6.16) to the equivalent system of singular integral equations

$$\mathcal{R}_3\mathcal{N}_3\, g = (\mathcal{R}f_1, \mathcal{R}f_2, \mathcal{R}f_3, f_4)^\top.$$

The principal symbol matrix (homogeneous of order 0) of the operator $\mathcal{R}_3\mathcal{N}_3$ reads

$$\sigma(\mathcal{R}_3\mathcal{N}_3) = \left\| \begin{matrix} [\sigma(\mathcal{R}\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(-2^{-1}I_1 + \overset{*}{\mathcal{K}}{}_4^{(0)}) \end{matrix} \right\|_{4\times 4},$$

and is non–singular according to the results of Section 5.

Now by the same way as in the previous subsection we can prove the following assertions.

LEMMA 6.9 *Let* $S \in \mathrm{C}^{k+2+\gamma'}$, $k \ge 0$.
Then the Ψ*DO*

$$\mathcal{N}_3 \,:\, [\mathrm{C}^{l+1+\gamma}(S)]^4 \to [\mathrm{C}^{l+\gamma}(S)]^3 \times \mathrm{C}^{l+1+\gamma}(S), \;\; 0 \le l \le k,$$

is an isomorphism.

THEOREM 6.10 *Let* $S \in \mathrm{C}^{k+2+\gamma'}$, $k \ge 0$, *and* $f_q \in \mathrm{C}^{k+\gamma}(S)$, $q = 1,2,3$, $f_4 \in \mathrm{C}^{k+1+\gamma}(S)$.

Then Problem $(\overset{\omega}{P}_3)^-$ *has a unique regular solution of the class* $\mathrm{C}^{k+1+\gamma}(\overline{\Omega^-}) \cap$ $\cap SK_r^m(\Omega^-)$ *and it is representable in the form* (6.1) *with the density* g *defined by the uniquely solvable* Ψ*DEs* (6.15), (6.16).

6.5. Problem $(\overset{\omega}{P}_4)^-$**.** The representation (6.1) of a solution and the boundary conditions (1.21), (1.22) reduce the BVP under consideration to the system of ΨDEs on S

$$\mathcal{N}_4\, g \equiv [\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)]g = f. \tag{6.17}$$

For the principal singular part we have the following elliptic ΨDO (of order 1) $(\mathcal{N}_4)_0 = (\mathcal{L})_0$, where $(\mathcal{L})_0$ is given by (5.42). It is easy to check that the diagonal operator $\mathcal{R}_4 = I_4\mathcal{R}$ with

$\mathcal{R}$ defined by (6.6), is a lifting operator, which reduces equivalently the equations (6.17) to the following system of singular integral equations of normal type with index equal to zero:

$$\mathcal{R}_4 \mathcal{N}_4 \, g = \mathcal{R}_4 f.$$

The proofs of the next lemma and theorem are quite similar to that proofs of Lemma 6.4 and Theorem 6.5. Therefore we confine ourselves by formulation of the final results.

LEMMA 6.11 *Let* $S \in \mathrm{C}^{k+2+\gamma'}$, $k \geq 0$.
Then the ΨDO

$$\mathcal{N}_4 \, : \, \mathrm{C}^{l+1+\gamma}(S) \to \mathrm{C}^{l+\gamma}(S), \;\; 0 \leq l \leq k,$$

is an isomorphism.

THEOREM 6.12 *Let* $S \in \mathrm{C}^{k+2+\gamma'}$, $k \geq 0$, *and* $f_j \in \mathrm{C}^{k+\gamma}(S)$, $j = 1, ..., 4$.
Then Problem $(\overset{\omega}{P}_4)^-$ *has a unique regular solution of the class* $\mathrm{C}^{k+1+\gamma}(\overline{\Omega^-}) \cap$ $\cap SK_r^m(\Omega^-)$ *and it is representable in the form* (6.1) *with the density* g *defined by the uniquely solvable* ΨDE (6.17).

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