Numerical Methods for ODEs

General Runge-Kutta-Methods and order conditions

Definition of Runge-Kutta-Methods

Runge-Kutta methods are one-step methods ($y_{k+1} = y_k + h_k\Phi(t_k, h_k, y_k)$) with an increment function of the form

$$\Phi(t_k, h_k, y_k) = \sum_{j=1}^{r} \gamma_j f_{jk},$$

where the "slopes" are computed by means of

$$f_{jk} = f\left(t_k + \alpha_j h_k, y_k + h_k \sum_{i=1}^{r} \beta_{ji}\, f_{ik}\right), \quad j = 1, \ldots, r.$$

The coefficients can be collected in vectors and matrices and the RK-method can be represented by a Butcher-Tableau:

$$\begin{aligned} a &= (\alpha_1 \;\; \ldots \;\; \alpha_r)^\top \\ c &= (\gamma_1 \;\; \ldots \;\; \gamma_r)^\top \end{aligned} \qquad B = \begin{pmatrix} \beta_{11} & \ldots & \beta_{1r} \\ \vdots & \ddots & \vdots \\ \beta_{r1} & \ldots & \beta_{rr} \end{pmatrix} \qquad \Longrightarrow \qquad \begin{array}{c|ccc} \alpha_1 & \beta_{11} & \ldots & \beta_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r & \beta_{r1} & \ldots & \beta_{rr} \\ \hline & \gamma_1 & \ldots & \gamma_r \end{array}$$

Order conditions

- **Autonomy invariance:**

$$1 = \sum_{j=1}^{r} \gamma_j \qquad \alpha_j = \sum_{i=1}^{r} \beta_{ji}, \quad j = 1, \ldots, r$$

- **High (consistency) order:**
  - ▷ Order **1**: $$1 = \sum_{j=1}^{r} \gamma_j$$
  - ▷ Order **2**: $$\sum_{j=1}^{r} \gamma_j\, \alpha_j = \frac{1}{2}$$
  - ▷ Order **3**: $$\sum_{j,i=1}^{r} \gamma_j\, \alpha_i\, \beta_{ji} = \frac{1}{6} \qquad \sum_{j=1}^{r} \gamma_j\, \alpha_j^2 = \frac{1}{3}$$
  - ▷ Order **4**: $$\sum_{j=1}^{r} \gamma_j\, \alpha_j^3 = \frac{1}{4} \qquad \sum_{j,i=1}^{r} \gamma_j\, \alpha_j\, \beta_{ji}\, \alpha_i = \frac{1}{8}$$ $$\sum_{j,i=1}^{r} \gamma_j\, \beta_{ji}\, \alpha_i^2 = \frac{1}{12} \qquad \sum_{i,j,k=1}^{r} \gamma_k\, \beta_{kj}\, \beta_{ji}\, \alpha_i = \frac{1}{24}$$