

Numerical Methods for ODEs

Implicit Runge-Kutta-Methods

SDIRK methods

(a) **2-stage SDIRK** (order 2)

$$\begin{array}{c|cc} \omega & \omega & \\ 1-\omega & 1-2\omega & \omega \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

with $\omega = \frac{1}{2}(1 \pm \frac{\sqrt{3}}{3})$

Gauß methods (order $2r$)

(b) **1-stage Gauß method** (order 2)
(implicit midpoint rule)

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

(c) **2-stage Gauß method** (order 4)

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

(d) **3-stage Gauß method** (order 6)
(implicit midpoint rule)

$$\begin{array}{c|ccc} \frac{1}{2} - \frac{\sqrt{15}}{10} & \frac{5}{36} & \frac{2}{9} - \frac{\sqrt{15}}{15} & \frac{5}{36} - \frac{\sqrt{15}}{30} \\ \frac{1}{2} & \frac{5}{36} + \frac{\sqrt{15}}{24} & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} \\ \frac{1}{2} + \frac{\sqrt{15}}{10} & \frac{5}{36} + \frac{\sqrt{15}}{30} & \frac{2}{9} + \frac{\sqrt{15}}{15} & \frac{5}{36} \\ \hline & \frac{5}{18} & \frac{4}{9} & \frac{5}{18} \end{array}$$

Radau methods (order $2r - 1$)

(e) **1-stage Radau-I** (order 1)

$$\begin{array}{c|c} 0 & 1 \\ \hline & 1 \end{array}$$

(f) **2-stage Radau-I** (order 3)

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

(g) **3-stage Radau-I** (order 5)

$$\begin{array}{c|ccc} 0 & \frac{1}{9} & \frac{-1-\sqrt{6}}{18} & \frac{-1+\sqrt{6}}{18} \\ \frac{6-\sqrt{6}}{10} & \frac{1}{9} & \frac{88+7\sqrt{6}}{360} & \frac{88-43\sqrt{6}}{360} \\ \frac{6+\sqrt{6}}{10} & \frac{1}{9} & \frac{88+43\sqrt{6}}{360} & \frac{88-7\sqrt{6}}{360} \\ \hline & \frac{1}{9} & \frac{16+\sqrt{6}}{36} & \frac{16-\sqrt{6}}{36} \end{array}$$

(h) **1-stage Radau-II** (order 1)

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

(i) **2-stage Radau-II** (order 3)

$\frac{1}{3}$	$\frac{5}{12}$	$-\frac{1}{12}$
1	$\frac{3}{4}$	$\frac{1}{4}$
	$\frac{3}{4}$	$\frac{1}{4}$

(j) **3-stage Radau-II** (order 5)

$\frac{4-\sqrt{6}}{10}$	$\frac{88-7\sqrt{6}}{360}$	$\frac{296-169\sqrt{6}}{1800}$	$\frac{-2+3\sqrt{6}}{225}$
$\frac{4+\sqrt{6}}{10}$	$\frac{296+169\sqrt{6}}{1800}$	$\frac{88+7\sqrt{6}}{360}$	$\frac{-2-3\sqrt{6}}{225}$
1	$\frac{16-\sqrt{6}}{36}$	$\frac{16+\sqrt{6}}{36}$	$\frac{1}{9}$
	$\frac{16-\sqrt{6}}{36}$	$\frac{16+\sqrt{6}}{36}$	$\frac{1}{9}$

Lobatto methods

(order $2r - 2$)

(k) **Lobatto-IIIA** (order 4)

0	0	0	0
$\frac{1}{2}$	$\frac{5}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
1	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

(l) **Lobatto-IIIB** (order 4)

0	$\frac{1}{6}$	$-\frac{1}{6}$	0
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	0
1	$\frac{1}{6}$	$\frac{5}{6}$	0
	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$