Technische Universität Chemnitz
Dr. M. Winkler Chemnitz, 13th December 2018

Numerical Methods for ODEs
Sheet 5

Exercise 16: Stable MSM with $r = 2$

Given is a linear multi-step method of the form

$$y_{k+2} + \alpha_1 \, y_{k+1} + \alpha_0 \, y_k = h \, (\beta_1 \, f_{k+1} + \beta_0 \, f_k).$$

We have already seen in Example 4.15 that the method with highest-possible consistency order ($p = 3$) is unstable.

(a) Derive conditions on α_1 such that the method is zero-stable and fulfills the first consistency condition $\alpha_0 + \alpha_1 = -1$.

(b) Construct a one-parametric family of zero-stable methods with highest-possible consistency order. Are some methods we already know part of this family?

Exercise 17: Realization of an Adams-Moulton method

We want to apply the 2-step Adams-Moulton method

$$y_{k+2} - y_{k+1} = \frac{h}{12} \left(5 f_{k+2} + 8 f_{k+1} - f_k\right), \quad f_k := f(t_k, y_k),$$

to the initial value problem

$$\begin{aligned} y''(t) + 20 \, y'(t) + 19 \, y(t) &= 0 \qquad t \in (0, T], \\ y(0) = 1 \qquad y'(0) &= -10. \end{aligned}$$

First, transform this second order ODE into a first-order system. Then, we want to realize the (implicit) Adams-Moulton method with a fixed-point iteration. Determine an upper bound for the grid size h such that convergence of the fixed-point iteration is guaranteed.

Homework 10: Interpretation of MSM as OSM

Mention in each case one example for an OSM which – interpreted as a MSM – possesses the properties

	explicit	implicit
linear		
non-linear		

Homework 11: Implementation of multi-step methods

(a) Implement MATLAB-routines for the

(i) explicit midpoint rule

$$y_{k+2} = y_k + 2\,h\,f_{k+1},$$

(ii) 2-step Adams-Bashforth method

$$y_{k+2} = y_{k+1} + \frac{h}{2}\,(3\,f_{k+1} - f_k),$$

(iii) 2-step Adams-Moulton method

$$y_{k+2} - y_{k+1} = \frac{h}{12}\,(5f_{k+2} + 8f_{k+1} - f_k)\,,$$

(iv) 2-step BDF method

$$\frac{1}{2}(3y_{k+2} - 4y_{k+1} + y_k) = h\,f_{k+2}.$$

The function header can be chosen as

```
function [t,y] = midpoint_expl(f,time,y0,N)
```

with the usual arguments, compare for instance Homework Sheet 1, Homework 1. The only difference is, that `y0` is a matrix containing the initial approximations y_0 und y_1 column-wise. The implicit methods should be realized with a fixed-point iteration.

(b) Apply these methods to solve the initial value problem

$$\dot{y} = \lambda\,y, \quad y(0) = 1$$

in the interval $[0, 8]$ with parameter $\lambda = -1$. Use exact initial data $y_0 = 1$ und $y_1 = \exp(-1/N)$, and solve the problem on a sequence of grids with

$$N_k = 2^k, \quad k = 1, \dots, 15,$$

grid points. Measure the global discretization error and compute the experimental convergence orders. Display the results in a table and a (double-logarithmic) plot.

(c) Derive the difference equations characterizing the computed approximations. As an example, for the explicit midpoint rule we get

$$y_{k+2} - 2\,h\,f_{k+1} - y_k \overset{f(t,y)=-y}{=} y_{k+2} + 2\,h\,y_{k+1} - y_{k+1} = 0.$$

Determine the roots of this polynomial in the range $h \in [0, 1]$. Then, try to explain the behavior of the computed approximations that you probably observe in the plot from task b).