Technische Universität Chemnitz
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Numerical Methods for ODEs

Sheet 4

Exercise 12: Construction of Adams formulas and BDF methods

(a) Derive the explicit r-step Adams formulas (Adams-Bashford) for $r = 2, 3$.

(b) Derive the implicit r-step Adams formulas (Adams-Moulton) for $r = 2, 3$.

(c) Derive the implicit r-step BDF methods for $r = 2, 3$.

Moreover, think about a construction of these methods for $r > 3$, for instance in Matlab or Maple.

Exercise 13: Consistency of explicit Adams formulas

Show: The explicit Adams formula (Adams-Bashforth) with $r = 2$ has consistency order 2.

Exercise 14: Construction of Nyström- and Milne-Simpson-Methods

In Example 4.5 we discussed further generalizations of multistep methods that can be constructed with the same arguments as in Example 4.3 and Example 4.4.

(a) Derive a general formula for the construction of Nyström methods. The idea is to integrate the differential equation $y'(t) = f(t, y(t))$ over the interval $[t_{k+r-2}, t_{k+r}]$ of length $2h$. Approximate the right-hand side function by an polynomial that interpolates the function in the points

$$(t_j, f_j) \in \mathbb{R} \times \mathbb{R}^n, \quad j = k,\, k+1\, \dots,\, \boldsymbol{k+r-1}.$$

(b) Use the same ideas to construct the Milne-Simpson methods. In this case, we have to interpolate the right-hand side function in the points

$$(t_j, f_j) \in \mathbb{R} \times \mathbb{R}^n, \quad j = k,\, k+1\, \dots,\, \boldsymbol{k+r}.$$

Compute the coefficients of these methods in case of $r = 1, 2, 3$.

Exercise 15: Proof of Theorem 4.11 (c)

Given is a linear MSM with associated polynomials $\varrho(\mu)$ and $\sigma(\mu)$. Complete the proof of Theorem 4.11 (c), i.e., show that the condition

$$\varphi(\mu) := \varrho(\mu) - \sigma(\mu)\ln\mu \;\text{ has the root } \mu = 1 \text{ with multiplicity } p+1.$$

is equivalent to:

The method has at least consisitency order p.

Hint: One can substitute $\mu := e^z$ in φ.

Homework 8: Implementation of grid control

(a) Implement the adaptive grid control strategy from Algorithm 3.58 using the embedded Runge-Kutta method RKF4(5). Use the method of order $p = 4$ for Φ. The method $\widehat{\Phi}$ of order $p = 5$ shall be used only for the error estimation. Measure all errors in the Euclidean vector norm.

The proposed syntax of our method is:

```
function [t,y,n_rejected] = RKF45(f,time,y0,h0,options)
```

Here, `n_rejected` is the number of rejected steps, `t` is the time grid, `y` is the approximate solution on the grid points `t`. As usual, `f` is a function handle to the right-hand side, `time` a vector containing the time horizon, `y0` the initial value (column vector) and `h0` the initial grid size. The parameter `options` is an array containing further input parameters:

Key	**Symbol**	**Meaning**
`alpha_min`	$\underline{\alpha}$	lower step size bound
`alpha_max`	$\overline{\alpha}$	upper step size bound
`tol`	TOL	error tolreance in each step
`rho`	ρ	safety factor

Hint: The parameters contained in the Butcher table of RKF4(5) can be generated by the function

```
[a, B, c, chat] = ButcherDiagramm45;
```

which is available on the web page of this lecture.

(b) Test your implementation using the examples we learned in the lecture (simple pendulum, predator-prey-model, Brusselator model, van-der-Pol oscillator). In the template `RKF45.m`, the following default values

$$\underline{\alpha} = 0.2, \quad \overline{\alpha} = 5, \quad \text{TOL} = 10^{-6}, \quad \rho = 0.9$$

and the initial grid size $h = 10^{-1}$ are already set. To check your implementation plot the solution as well as the local grid size using:

```
plot(t,t(2:end)-t(1:end-1))
```

Does the adaptive algorithm work as expected?

(c) Use your implementation to compute the trajectory of a satellite which is influenced by the gravity of the earth and the moon. The model reads as follows. The earth and the moon move on a circular orbit around the joint center of gravity $(0,0)$. The initial position of the earth is $(-\mu,0)$ and of the moon $(1-\mu,0)$. Here, $\mu = 0.012277471$ is the ratio between the mass of the moon and of the earth. The trajectory $t \mapsto (u(t),v(t))$ of the satellite can be computed by means of the differential equation

$$\begin{pmatrix} u \\ v \end{pmatrix}'' = \begin{pmatrix} 2v' - \frac{\partial}{\partial u} V(u,v) \\ -2u' - \frac{\partial}{\partial v} V(u,v) \end{pmatrix},$$

where V is a potential defined by

$$V(u,v) = -\frac{1}{2}(u^2+v^2) + \frac{1-\mu}{\sqrt{(u+\mu)^2+v^2}} + \frac{\mu}{\sqrt{(u-\mu+1)^2+v^2}}.$$

Note that the potential grows unboundedly when (u,v) tends to the earth or moon. Transform the second-order ODE system to a first-order system. Solve this model with the method implemented in task a). As input data, the values

$$T = 17.0652166, \quad (u,v)(0) = (0.994, 0), \quad (u',v')(0) = (0, -2.001585106)$$

can be used. The corresponding solution is even periodic! A test script (`satellite.m`) is also available on the homepage of this lecture. As an output of this script, you should see a movie illustrating the trajectory of the satellite.

(d) Try to solve this problem with other ODE solvers that you have implemented before (implicit/explicit Euler, Crank-Nicolson, explicit Runge-Kutta, ...) on equidistant grids. You will probably need a very fine grid.

(e) Extend the script `satellite.m` by the following functionality. Due to the periodicity of the exact solution, the value $E_h = \|(u_h(0), v_h(0)) - (u_h(T), v_h(T))\|$ can be used to measure the error of the numerical method. Compute and plot the errors for several computations with $TOL \in \{10^{-4}, 10^{-6}, \dots, 10^{-10}\}$. Determine also the experimental convergence rate. What do you observe?

Homework 9: Consistency of multistep methods

Investigate the multistep methods

a) $$y_{k+2} - \frac{1}{2}\, y_{k+1} + \frac{1}{4}\, y_k = \frac{h}{3}\,(2f_{k+1} + f_k)$$

b) $$y_{k+2} - y_{k+1} = h\,(2f_{k+1} - f_k)$$

c) $$y_{k+1} - y_k = \frac{h}{2}\,(f_{k+1} + f_k)$$

regarding consistency and determine the consistency order (e.g. using Theorem 4.11). Furthermore, check if these methods are null-stable.