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Numerical Methods for ODEs
Sheet 3

Exercise 8: Stability functions of certain one-step methods

Compute the stability functions of

(a) the improved Euler method,

(b) the Heun method,

(c) the implicit Euler method,

(d) the implicit trapezoidal rule.

Which assumptions on the grid size h are required to guarantee A-stability? Consider the easier case $\lambda \in \mathbb{R}^-$.

Exercise 9: Representation for the stability function of an RKM

Prove the representation (3.41b) for the stability function from Theorem 3.45.

Exercise 10: Stability regions of the θ-method

Show: The stability region of the one-step-θ-method (compare Example 3.49)

$$y_{i+1} = y_i + h\big[(1-\theta)\, f(t_i, y_i) + \theta\, f(t_{i+1}, y_{i+1})\big],$$

is given by

$$S = \begin{cases} \{z \in \mathbb{C} : |z - \frac{1}{2\theta-1}| \le \frac{1}{1-2\theta}\} & \text{for } \theta < 1/2, \\ \{z \in \mathbb{C} : \mathrm{Re}(z) \le 0\} & \text{für } \theta = 1/2, \\ \{z \in \mathbb{C} : |z - \frac{1}{2\theta-1}| \ge \frac{1}{2\theta-1}\} & \text{for } \theta > 1/2. \end{cases}$$

What is the geometrical interpretation of S?

Exercise 11: Construction of the Radau-Methods

Show:

(a) The Radau-I-Method of order 3 is given by the Butcher table from Example 3.40b).

(b) The Radau-II-Method of order 3 is given by the Butcher table from Example 3.40e).

Homework 5: Implementation of RKM

(a) Implement a general explicit Runge-Kutta method (compare Algorithm 3.29) for the solution of systems of initial value problems. The header of the method could be

```
function [t,y] = erkm(f, time, y0, N, a, c, B).
```

Here, `f` is a function handle to the right-hand side (compare e. g. the definition in `convergence_test.m`), `time` is a vector containing the time horizon, `y0` the column vector containing the initial values and `N` the number of grid points that shall be used. The remaining parameters `a`, `c`, `B` are the coefficients from the Butcher table of the RKM. The output parameters are the time grid represented by the vector `t` and the function values of y_h on the grid, represented by `y`.
Hint: If possible try to avoid `for`-loops for the computation of the slopes and the new iterate y_{k+1} and use matrix-vector products instead. This will make the implementation more efficient.

(b) Check your implementation using the test script `erkm_test.m`. Therein, the coefficients for several explicit RKMs are already defined. The test script is solving the equation

$$y'(t) = -5\,y\,\sin(5\,t), \quad y(0) = e, \tag{1}$$

in the interval $[0, 3]$. Therefore, each method is applied for several equidistant grids with grid size

$$h_i = 0.1 \cdot 2^{-i} \quad (\Rightarrow N_i = 1/h_i) \quad i = 0, 1, \ldots, 10.$$

Confirm that the function $y(t) = e^{\cos(5t)}$ is the exact solution of (1). The test script computes the error in the norm $\|\cdot\|_{\infty,h}$. Afterwards, a curve through the points $(N_i, \|y - y_{h_i}\|_{\infty,h})$, $i = 0, \ldots, 10$, is plotted in a diagram with logarithmic axes. Are the convergence rates as predicted by the theory?

(c) Add the following functionality to the script `erkm_test.m`: Determine the computational time using

```
tic;
[y,t] = erkm(..., N(l), ...);
times(l,method)=toc;
```

for each method and each grid size. Plot the error against the computational time in a logarithmic plot. Which method is the most efficient one?

Homework 6: Gauß-Method of order 4

Show, using Theorem 3.27 (calculate by hand or with a computer algebra system), that the Gauß method of order 4 from Example 3.39 (b) is really of order 4.

$$
\begin{array}{c|cc}
\frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\
\frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Homework 7: Stiffness for nonlinear ODE systems

Discuss the definition of "stiffness" for nonlinear ODE systems by means of the Brusselator model (for more information see https://en.wikipedia.org/wiki/Brusselator)

$$
\begin{aligned}
\dot{y}_1 &= A - (B+1)\, y_1 + y_1^2\, y_2 \\
\dot{y}_2 &= B\, y_1 - y_1^2\, y_2
\end{aligned}
$$

with initial values $y_1(0) = y_{a,1}$ and $y_2(0) = y_{a,2}$. Investigate the influence of the model parameters A, B to the stiffness quotient. Solve the model numerically using the MATLAB-functions `ode23` and `ode32s`. Reasonable parameter choices are $A = 1$, $B \in \{1.7, 3, 5, 10, 20\}$, $T \in \{40, 200\}$, $y_a = (1\ 1)^\top$. Compute also the stiffness quotient in each grid point and illustrate it in a plot.