Technische Universität Chemnitz
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Numerical Methods for ODEs

Sheet 2

Exercise 4: Lipschitz continuity of two increment functions

Let Assumption 2.3 (a) be satisfied. Show that the increment functions of

(a) the implicit Euler method (under the assumption that h is sufficiently small), and

(b) the Heun-method

fulfill the Lipschitz condition

$$\|\Phi(t, y, h) - \Phi(t, z, h)\| \leq K \|y - z\|$$

from Lemma 3.15.

Exercise 5: Runge-Kutta methods for problems with affine right-hand side

Consider the initial value problem

$$\begin{aligned} y'(t) &= f(t, y(t)) & \qquad t \in [0, T] \\ y(0) &= y_a \end{aligned}$$

for the case that the right-hand side is affine linear, i.e., $f(t, y(t)) := A(t)\, y(t) + b(t)$, with a coefficient matrix $A(t) \in \mathbb{R}^{n\times n}$ and a vector $b(t) \in \mathbb{R}^n$.

In this case, the realization of a single step $y_k \rightsquigarrow y_{k+1}$ of an implicit Runge-Kutta method requires the solution of one (huge) linear equation system. Derive this system! How can we improve the computation in case of a simple diagonal Runge-Kutta method (SDIRK)? In this case there holds $\beta_{ij} = 0$ for $j \geq i + 1$.

Exercise 6: Interpretation of RKMs as quadrature formula

In this exercise we consider initial value problems with a right-hand side f depending only on the time variable t, i.e., $f = f(t)$.

(a) Show: If the OSM of the form

$$y_{k+1} = y_k + h \sum_{j=1}^{r} \gamma_j \, f(t_k + \alpha_j \, h)$$

is of order p, the quadrature formula

$$Q[g] = \sum_{j=1}^{r} \gamma_j \, g(\alpha_j) \approx \int_0^1 g(x) \, \mathrm{d}x$$

is exact for polynomials up to order $p-1$ (compare Remark 3.28)

(b) What is the relation between the number of stages of an RKM and the maximum possible order?

Hint: It suffices to show the result merely for the monomials t^n. To this end, perform one step of the OSM applied to the corresponding initial value problem

$$y'(t) = t^n, \quad y(0) = 0, \quad n = 0, 1, \dots, p-1.$$

Therefore, prove an estimate for $|y(h) - y_1|$?

Exercise 7: Experimental order of convergence

Frequently, the norm of the error $E_h := \|e_h\|$ and the discretization parameter h are related to each other by means of $E_h \leq Ch^p$. In order to determine the convergence order p experimentally, one uses the ansatz $E_h \approx Ch^p$ where C and p are unknown.

(a) How can we estimate p from two known error values E_{h_1} and E_{h_2}?

(b) The order p can be interpreted as the slope of a “linear” function through the points (h_1, E_{h_1}) and (h_2, E_{h_2}) in a plot with logarithmic axes. We can extend this idea for the case that multiple error values (h_i, E_{h_i}), $i = 1, \dots, n$, are given. In this case we determine the values of C and p of the “linear” function Ch^p such that it minimizes the distance to the error curve in the “least-squares”-sense. Derive the corresponding least-squares-problem and the its normal equations.

(c) Estimate (as explained in (b)) the constant C and the convergence rate p for the following measured values.

h	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
E_h	6.91e-01	1.78e-01	4.33e-02	1.78e-02	3.07e-03

Draw these measurements and the function Ch^p into a double-logarithmic plot (MATLAB: `loglog(...)`).

Homework 3: Consistency order for some 1- and 2-stage RKM

Show the following results using the order conditions from Theorem 3.27:

(a) The implicit Euler method is of order 1.

(b) The improved Euler method is of order 2.

(c) The Heun method is of order 2.

Homework 4: An ODE with non-smooth right-hand side

(a) Derive the exact solution of the initial value problem

$$y'(t) = f(y(t)) \text{ almost everywhere in } [0,1], \quad y(0) = 1, \tag{1}$$

with

$$f(y) := \begin{cases} y & \text{for } y \le y_s \\ 2y & \text{for } y > y_s \end{cases}, \quad y_s > 1.$$

For the example we expect discontinuities in $f(y(t))$. Thus, we expect only continuity of $y(t)$ in these points, but not differentiability, so that y fulfills the ODE only "almost everywhere" in $[0,1]$.[1]

Hint: For a constant $\lambda \in \mathbb{R}$ the ODE $y' = \lambda y$ can be solved with the ansatz $y(t) = c\exp(\lambda t)$ with $c \in \mathbb{R}$.

(b) Implement the improved Euler method (see Sheet 1, Homework 1 or Example 3.11). Solve the problem (1) for

(i) $y_s = 4$ (kink outside of the time interval $[0,1]$),

(ii) $y_s = 1.5$,

(iii) $y_s = \exp(71/128)$ (kink in a grid point for $n \ge 7$)

with the explicit Euler method and the improved Euler method on a sequence of equidistant grids with mesh size

$$h_n = 2^{-n}, \quad n = 1, 2, \ldots, 12.$$

(c) Draw both error curves into a double-logarithmic plot and interpret the results.

[1] This requirement can be justified theoretically when using a generalized derivative.