Technische Universität Chemnitz
Dr. M. Winkler

Chemnitz, 29th May 2019
hand in on 12th June 2019

Numerical Methods for Partial Differential Equations
Sheet 8

Exercise 18: Derivative of a function in barycentric coordinates

Let $K \subset \mathbb{R}^d$ denote the unit simplex and $p : \mathbb{R}^{d+1} \to \mathbb{R}$ a function which is given in barycentric coordinates. Compute the derivative of p w.r.t. x_i, $i = 1, \ldots, d$, i.e., the partial derivatives of the function $\tilde{p} : \mathbb{R}^d \to \mathbb{R}$ given by $\tilde{p}(x) = p(\lambda(x))$.

Exercise 19: Assembly of the stiffness matrix for $\mathbb{P}_1$-elements

Let (K, P_K, Σ_K) be a first-order Lagrange element with $\Sigma_K = \{\sigma_{K,1}, \ldots, \sigma_{K,d+1}\}$. Denote by $p_{K,m}$, $m = 1, \ldots, d+1$, the local shape functions fulfilling $\sigma_{K,l}(p_{K,m}) = \delta_{l,m}$ for $l = 1, \ldots, d+1$.

Derive an exact formula for

(a) the local stiffness matrix

$$(A_K)_{m,n} = \int_K \nabla p_{K,n} \cdot \nabla p_{K,m} \,\mathrm{d}x, \quad m, n = 1, \ldots, d+1,$$

(b) the local mass matrix

$$(M_K)_{m,n} = \int_K p_{K,n}\, p_{K,m} \,\mathrm{d}x, \quad m, n = 1, \ldots, d+1,$$

without numerical integration.

Exercise 20: Assembly routines in case of Robin boundary conditions

Let $\{(K, P_K, \Sigma_K)\}_{K \in \mathcal{T}_h}$ be an affine family of Lagrange elements $\mathbb{P}_k$ of arbitrary order $k \in \mathbb{N}$. We consider a problem with Robin-boundary conditions, i. e.,

$$\partial_n u + \alpha\, u = g \text{ on } \Gamma = \partial\Omega.$$

In § 9.3 we have already derived the weak formulation. Derive a formula for the contributions coming from the terms

$$\int_\Gamma \alpha\, u\, v \,\mathrm{d}x \quad \text{and} \quad \int_\Gamma g\, v \,\mathrm{d}s$$

entering in the finite element system matrix and in the load vector.

Exercise 21: Relation between determinant and element volume

Show the relation

$$|\det(B_K)| = \frac{|K|}{|\widehat{K}|}$$

used in Section § 13.1.

Exercise 22: Norms of FE functions

Given is a global finite element space $V_{\mathcal{T}} \subset H^1(\Omega)$ and the global shape functions corresponding to the global DOFs $\{\sigma_i\}_{i=1}^S$ are denoted by $\{\varphi_i\}_{i=1}^S$.

Show: For each function $u_h \in V_{\mathcal{T}}$ the $L^2(\Omega)$-norm and the $H^1(\Omega)$-seminorm can be represented by

$$\|u_h\|_{L^2(\Omega)} = \sqrt{U^\top M U}, \quad \text{and} \quad |u_h|_{H^1(\Omega)} = \sqrt{U^\top A U},$$

where $M \in \mathbb{R}^{S\times S}$, $M_{i,j} = \int_\Omega \varphi_i \, \varphi_j \, \mathrm{d}x$, is the finite element mass matrix, $A \in \mathbb{R}^{S\times S}$, $A_{i,j} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, \mathrm{d}x$, is the finite element stiffness matrix, and $U \in \mathbb{R}^S$ is the coefficient vector $U_i = \sigma_i(u_h)$.

Homework 16: Relation between numbers of mesh entities

Let $\mathcal{T}$ be a geometrically conforming mesh consisting of triangles ($d = 2$) or tetrahedra ($d = 3$), respectively, on the simply connected domain $\Omega \subset \mathbb{R}^d$.

Give a "simple" relation (without rigorous proof) in case $d = 2$ between

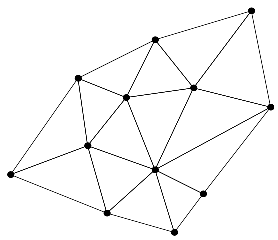
(a) N_{cells}, N_{edges} and N_{vertices};

(b) N_{cells}, N_{edges} and $N^\partial_{\text{edges}}$

and for $d = 3$ between

(a) N_{cells}, N_{faces}, N_{edges} and N_{vertices};

(b) N_{cells}, N_{faces} and $N^\partial_{\text{faces}}$.

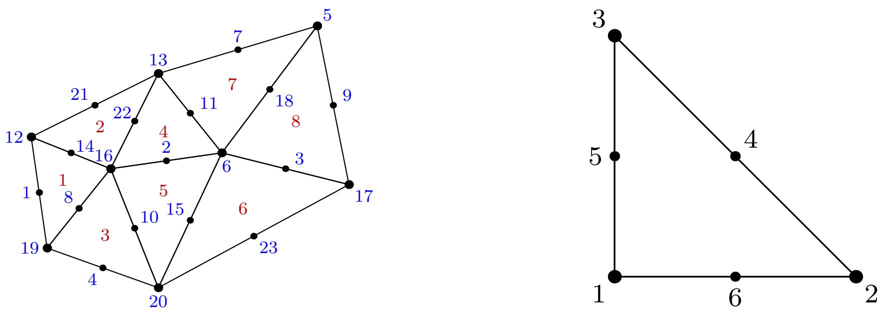


We use the following notations for the components of 1-, 2-, and 3-dimensional meshes:

notation	Number of ...	dimension	$d = \ldots$	in the above example
N_{cells}	cells K	d	1,2,3	14
N_{faces}	faces	2	3	n.a.
$N^\partial_{\text{faces}}$	boundary faces	2	3	n.a.
N_{edges}	edges	1	2,3	25
$N^\partial_{\text{edges}}$	boundary edges	1	2,3	8
N_{vertices}	vertices,	0	1,2,3	12
$N^\partial_{\text{vertices}}$	boundary vertices	0	1,2,3	8

Homework 17: Example of a connectivity matrix

Build up the connectivity matrices C_K for the triangulation from Figure 1(a). The points nummerated in blue represent the global DOFs. The local numbering on a single element should correspond to Figure 1(b).



(a) Global numbering of DOFs (b) Local numbering of DOFs

Figure 1: Example grid and numbering of global and local DOFs

Homework 18: Quadrature formulas on triangles

Show that the quadrature formulas from Handout Quadrature Formulas of second order are really of second order, i. e. polynomials $p \in P_2(K)$ are integrated exactly (for an arbitrary triangle K).

Homework 19: FEM in one spatial dimension

For given $L > 0$ and $f \in C([0, L])$, we consider the differential equation

$$\begin{aligned} -u''(x) &= f(x) \qquad \text{for } x \in (0, L), \\ u(0) = u'(L) &= 0. \end{aligned}$$

In this exercise, we are going to write a MATLAB implementation for a solution with finite elements. Therefore, we introduce an equidistant grid

$$\mathcal{T} := \{[x_i, x_{i+1}] : i = 0, \ldots, N-1\} \quad \text{with} \quad x_i = h\,i, \; i = 0, \ldots, N,$$

with $h = L/N$ the discretization parameter. The ansatz and test space $V_{\mathcal{T}}$ is the space of piecewise polynomials of degree $p \in \{1, 2\}$.

(a) Write a function `fem_1d.m` using the following header.

```
function [U] = fem_1d( x, p, f )
```

Here, $x = (x_0, \ldots, x_N)^\top$ are the nodes of the grid, $p \in \{1, 2\}$ is the polynomial degree and f is a function handle for the right-hand side. The return value is the coefficient vector U which solves the system (13.7). You can call your function, e.g., via

```
U = fem_1d( linspace(0,1,21), 1, @(x)(sin(3*pi*x)) )
```

(b) Solve the problem for $L = 1$ and $f(x) = \sin(3\,\pi\,x)$ and different values of the parameter N and plot the computed solutions.

Compute the errors $\|\mathcal{I}_\mathcal{T}(u) - u_h\|_{L^2(\Omega)}$ and $|\mathcal{I}_\mathcal{T}(u) - u_h|_{H^1(\Omega)}$, for $f(x) = \sin(3\,\pi\,x)$ using the formulas from Exercise 22. For this, it is beneficial to return also the stiffness matrix (without the boundary modifications) and the mass matrix, i.e.,

```
function [U,K,M] = fem_1d( x, p, f )
```

Estimate the order of convergence and interpret the results. Finally, approximate the error $|u - u_h|_{H^1(\Omega)}$ by using the Simpson rule (for $p = 1$) and some higher order quadrature rule (for $p = 2$) on the intervals $[x_{i-1}, x_i]$ and compare the results.