Technische Universität Chemnitz
Dr. M. Winkler

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Numerical Methods for Partial Differential Equations

Sheet 7

Exercise 14: Barycentric coordinates

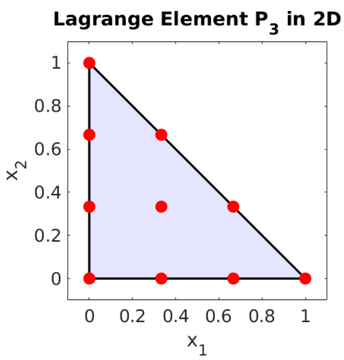
Let $\{a_0, \dots, a_d\} \subset \mathbb{R}^d$ be affine independent points, i. e., $\{a_1 - a_0, \dots, a_d - a_0\}$ are linearly independent. The domain

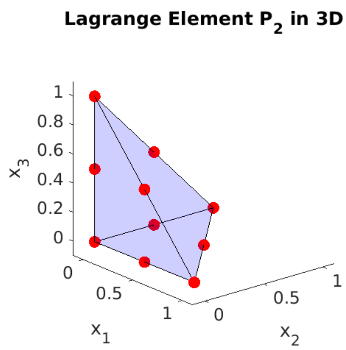
$$K := \operatorname{conv}\{a_0, \dots, a_d\} := \left\{ x \in \mathbb{R}^d : x = \sum_{i=0}^{d} \lambda_i \, a_i \; : \; \lambda_i \geq 0 \text{ und } \sum_{i=0}^{d} \lambda_i = 1 \right\}$$

is a **simplex** with $\operatorname{int}(K) \neq \emptyset$. The numbers $(\lambda_0(x), \dots, \lambda_d(x))$ are the **barycentric coordinates** of a point $x \in \mathbb{R}^d$.

(a) Determine the barycentric coordinates of an arbitrary point x in the unit simplex $\operatorname{conv}\{0, e_1, \dots, e_d\}$.

(b) Compute the barycentric coordinates of the red points:





(c) Which sets can be described by $\lambda_i = \text{const}$, $\sum_{j=0}^{d} \lambda_j = 1$

(d) Confirm, that λ_j are affine functions.

Exercise 15: Shape functions of Lagrange elements

(a) Derive the shape functions of the 2D Lagrange elements $\mathbb{P}_k$ for $k = 2$ and $k = 3$. **Hint:** Use barycentric coordinates.

(b) Derive the shape functions of the 2D Lagrange elements $\mathbb{Q}_k$ for $k = 2$ and $k = 3$. **Hint:** Use the shape functions known from the 1D case.

Exercise 16: P_k-unisolvency for Lagrange elements

We consider the Lagrange element $\mathbb{P}_k(K)$, $k \geq 1$ on a simplex $K = \mathrm{conv}\{v_0, \ldots, v_d\}$. The degrees of freedom are point evaluations in the Lagrange points

$$a_\alpha = \sum_{j=0}^{d} \frac{\alpha_j}{k} v_j.$$

Here, $\alpha \in \mathbb{N}^{d+1}$, is a multi-index with $\sum_{j=0}^{d} \alpha_j = k$.

The associated barycentric coordinates are given by

$$\lambda_j(a_\alpha) = \frac{\alpha_j}{k} \qquad j = 0, \ldots, d.$$

(a) Show that the Lagrangian interpolation problem is uniquely solvable for all dimensions $d \geq 1$ and polynomial degrees $k \geq 1$, i.e., functions $p_\alpha \in P_k(K)$ satisfying

$$\sigma_\beta(p_\alpha) := p_\alpha(a_\beta) = \delta_{\beta\alpha} = \begin{cases} 1 & \text{if } \alpha_j = \beta_j \text{ for all } j = 0, \ldots, d \\ 0 & \text{else} \end{cases}$$

exist.

(b) Conclude that σ_β are linearly independent.

Exercise 17: Invariance of barycentric coordinates under affine mappings

Let $K = \mathrm{conv}\{v_0, \ldots, v_d\} \subset \mathbb{R}^d$ a simplex and $\lambda : \mathbb{R}^d \to \mathbb{R}^{d+1}$ the function mapping a point $x \in K$ to its barycentric coordinates w.r.t. K. Further, let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine mapping, i.e., $T(x) = Bx + b$ with a regular $B \in \mathbb{R}^{d \times d}$ and a vector $b \in \mathbb{R}^d$. Then

$$\widetilde{K} := T(K) = \mathrm{conv}\{T(v_0), \ldots, T(v_d)\},$$

is a simplex. Show that the barycentric coordinates $\widetilde{\lambda}$ w.r.t. $\widetilde{K}$ satisfy

$$\widetilde{\lambda}(T(x)) = \lambda(x),$$

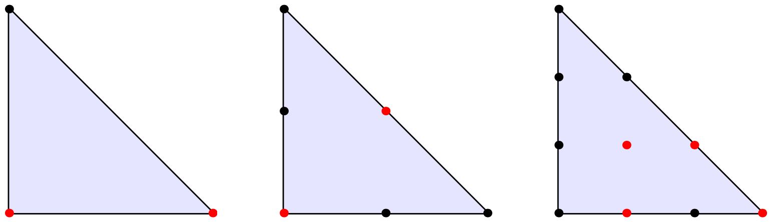
i.e., barycentric coordinates are invariant under affine mappings.

Homework 13: Plot some shape functions of Lagrange elements

Plot (in Matlab) the shape functions of the following Lagrange elements associated with the highlighted degrees of freedom.

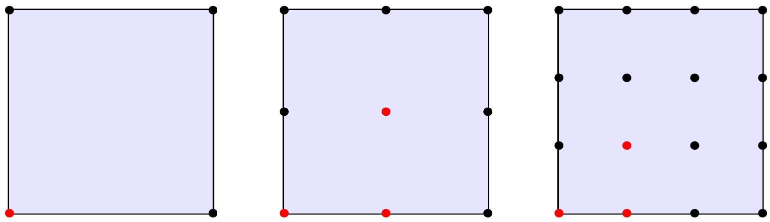
(a) $\mathbb{P}_k$ (cf. Example 11.4 and Exercise 15) $(d = 2)$

for $k = 1$ for $k = 2$ for $k = 3$



(b) $\mathbb{Q}_k$ (cf. Example 11.5) $(d = 2)$

for $k = 1$ for $k = 2$ for $k = 3$



Homework 14: Hermite form functions in 1D

Analogous to Example 11.7 the Hermite elements in 1D on are defined by $K = [0, 1]$, $P = P_3(K)$ and

$$\sigma_1(p) = p(0), \qquad \sigma_2(p) = p'(0), \qquad \sigma_3(p) = p(1), \qquad \sigma_4(p) = -p'(1).$$

Compute and draw the form functions.

Homework 15: The Hilbert matrix

Verify that the Hilbert matrix A with entries

$$A_{i,j} = \frac{1}{i + j - 1}, \quad i, j = 1, \dots, n,$$

is obtained as the stiffness matrix for the problem

$$-u'' = f \quad \text{in } (0, 1), \quad u(0) = u'(1) = 0$$

by using a Galerkin approach with

$$V_h = \operatorname{span}\Big\{ \underbrace{\frac{x^i}{i}}_{=:\varphi_i} \Big\}_{i=1}^{n},$$

i.e., polynomials of degree at most n.

Check numerically, up to which degree n, the linear system $A\,x = b$ can be solved reliably in MATLAB.

Hint: The inverse of the Hilbert matrix can be constructed explicitly. In MATLAB you can use the functions `hilb` and `invhilb`.