Technische Universität Chemnitz
Dr. M. Winkler

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Numerical Methods for Partial Differential Equations

Sheet 6

Exercise 12: Elliptic equations with convection

Prove Theorem 9.13.

Hints: For Theorem 9.13 (a), you should verify

$$(\beta \cdot \nabla u,\, u)_{L^2(\Omega)} = -\frac{1}{2}\,(u,\, (\nabla \cdot \beta)\, u)_{L^2(\Omega)}\,,$$

which can be shown with the integration-by-parts formula from Homework 11.

Exercise 13: Solvability of the pure Neumann problem

To obtain unique solvability of the Poisson equation with pure Neumann conditions (9.12) we have to "extract" constants from the solution space. To this end we use the quotient space $H^1(\Omega)/\mathbb{R}$. An element of this quotient space is an equivalence class

$$[v] = v + \mathbb{R} := \{v + c\colon c \in \mathbb{R}\}.$$

A norm in $H^1(\Omega)/\mathbb{R}$ is defined by

$$\|[v]\|_{H^1(\Omega)/\mathbb{R}} = \inf_{w \in [v]} \|w\|_{H^1(\Omega)}.$$

(a) Show that the norm of the quotient space $H^1(\Omega)/\mathbb{R}$ is equivalent to

$$|||\,[v]\,|||_{H^1(\Omega)/\mathbb{R}} := \left\| v - \frac{1}{|\Omega|} \int_\Omega v \; \mathrm{d}x \right\|_{H^1(\Omega)}.$$

(b) Show that the space $H^1(\Omega)/\mathbb{R}$ with the norm introduced above is isomorphic to

$$V = \left\{ v \in H^1(\Omega) : \int_\Omega v \; \mathrm{d}x = 0 \right\}.$$

(c) Prove using the Lax-Milgram-Lemma that the pure Neumann problem possesses a unique solution in the space V which depends continuously on the data $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

(d) We could have also used the space $H^1(\Omega)/\mathbb{R}$ which is isomorphic to V. To obtain well-posedness of the functional $\tilde{F}([v]) = \int_\Omega f\, v \;\mathrm{d}x + \int_\Gamma g\, v \;\mathrm{d}s$ in this space, we require the compatibility condition

$$\int_\Omega f \;\mathrm{d}x + \int_\Gamma g \;\mathrm{d}s = 0.$$

Why is this condition not needed in (c)?

Hint: Use the Poincaré inequality shown in Sheet 5, Exercise 11.

Homework 10: An elliptic problem with diffusion matrix

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with Lipschitz boundary. Considered is the boundary value problem

$$\begin{aligned} -\mathrm{div}(A\,\nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma \end{aligned}$$

with $f \in L^2(\Omega)$ and a symmetric matrix $A \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ which is uniformly positive definite. Derive the weak formulation of this problem and show existence of a weak solution.

Hint: Prove the identity $\int_\Omega \mathrm{div}(\vec{w})\, v \,\mathrm{d}x = -\int_\Omega \vec{w} \cdot \nabla v \,\mathrm{d}x + \int_\Gamma \vec{w} \cdot \vec{n}\, v\, ds$ for $\vec{w} \in H^1(\Omega; \mathbb{R}^d)$ and $v \in H^1(\Omega)$, similar to Sheet 5, Homework 8.

Homework 11: Integration by parts for Sobolev functions

Let $p \in (1, \infty)$ be given and denote by p' its conjugate exponent.

(a) Show that

$$\int_\Omega D_i u\, v + u\, D_i\, v \,\mathrm{d}x = 0$$

holds for all $u \in W_0^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$.

(b) Assume that Ω possesses a Lipschitz boundary. Prove that

$$\int_\Omega D_i u\, v + u\, D_i\, v \,\mathrm{d}x = \int_{\partial\Omega} u\, v\, n_i \,\mathrm{d}s$$

holds for all $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$.

Hints: Show the result first for smooth functions u and v (Definition 8.16) and use density results.

Homework 12: Generalized Poincaré-Friedrich inequality

Prove Lemma 9.10.

Hint: Argue by contradiction.