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Numerical Methods for Partial Differential Equations
Sheet 5

Exercise 9: A piecewise smooth function

On $\Omega = B_1(0) = \{x \in \mathbb{R}^d : |x| < 1\}$, $d \geq 1$, we consider the function f_α defined via $f_\alpha(x) := |x|^\alpha$ with $\alpha \in \mathbb{R}$.

(a) For which values of $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$ do we have $f_\alpha \in L^p(\Omega)$?

(b) For which values of $\alpha \in \mathbb{R}$ does f_α possess weak derivatives of first order?

(c) For which values of $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$ do we have $f_\alpha \in W^{1,p}(\Omega)$?

Exercise 10: Chain rule and Stampacchia's lemma

Let Ω be a bounded domain.

(a) Let $f \in C^1(\mathbb{R})$ be given such that a constant $M > 0$ exists with $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Show that $f(u) \in W^{1,p}(\Omega)$ for all $u \in W^{1,p}(\Omega)$ and $D_i(f(u)) = f'(u)\, D_i u$ for $i = 1, \dots, d$.

(b) Let $u \in W^{1,p}(\Omega)$ be given. Show that u^+, which is defined via $u^+(x) = \max(u(x), 0)$, belongs to $W^{1,p}(\Omega)$ with

$$D_i\, u^+(x) = \begin{cases} D_i\, u(x) & \text{if } u(x) > 0, \\ 0 & \text{else} \end{cases}$$

for a.a. $x \in \Omega$ and $i = 1, \dots, d$.

(c) Let $u \in W^{1,p}(\Omega)$ be given. Show that $D_i\, u = 0$ a.e. on $\{x \in \Omega : u(x) = 0\}$.

(d) Show that $u^+, u^- \in W_0^{1,p}(\Omega)$ for $u \in W_0^{1,p}(\Omega)$.

Exercise 11: Poincaré inequality for $H^1(\Omega)$-functions

Let $\Omega := (0,1)^2$ be the unit square. For an arbitrary function $v \in L^2(\Omega)$ we define the projection $\bar{v} = \frac{1}{|\Omega|} \int_\Omega v(x)\, \mathrm{d}x$ which is constant on Ω. Show that a constant $C = C(\Omega) > 0$ exists such that the inequality

$$\|v - \bar{v}\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}$$

holds for all $v \in H^1(\Omega)$. Conclude that the following Poincaré inequality holds:

$$\|v\|_{L^2(\Omega)} \leq C \left(\|\nabla v\|^2_{L^2(\Omega)} + \left(\int_\Omega v(x) \mathrm{d}x \right)^2 \right)^{1/2}.$$

Homework 7: A piecewise smooth function

On $\Omega = B_1(0) = \{x \in \mathbb{R}^d : |x| < 1/2\}$, $d \geq 1$, we consider the function f_α defined via $f_\alpha(x) := \big|\log|x|\big|^\alpha$ with $\alpha \in \mathbb{R}$.

(a) For which values of $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$ do we have $f_\alpha \in L^p(\Omega)$?

(b) For which values of $\alpha \in \mathbb{R}$ does f_α possess weak derivatives of first order?

(c) For which values of $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$ do we have $f_\alpha \in W^{1,p}(\Omega)$?

Homework 8: Integral theorems

It is well-known that each function $u \in H^1(\Omega)$ defined in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with piecewise smooth boundary Γ fulfills for each $i = 1, \dots, d$ the integration-by-parts formula

$$\int_\Omega (D_i u)\, v \,\mathrm{d}x = -\int_\Omega u\,(D_i v)\,\mathrm{d}x + \int_\Gamma u\, v\, n_i \,\mathrm{d}s_x \quad \forall v \in H^1(\Omega).$$

Conclude the following results

(a) The Gauss integral theorem: for a vector field $u \in H^1(\Omega; \mathbb{R}^d)$ there holds

$$\int_\Omega \operatorname{div}(u)\,\mathrm{d}x = \int_\Gamma u \cdot n \,\mathrm{d}s_x.$$

(b) Green's identity: For $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ and $v \in H^1(\Omega)$ there holds

$$\int_\Omega \Delta u\, v\,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x + \int_\Gamma \nabla u \cdot n\, v\,\mathrm{d}s_x$$

(c) A harmonic function $u \in H^1(\Omega)$ (i. e. $\Delta u = 0$) fulfills the property $\int_\Gamma \nabla u \cdot n\,\mathrm{d}s_x = 0$.

Homework 9: Poincaré inequality for $H^1(\Omega)$-functions

Let $\Omega \in \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with boundary Γ. Show that a constant $C = C(\Omega) > 0$ exists such that the inequality

$$\|u\|_{L^2(\Omega)} \leq C \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma)} \right)^{1/2}$$

is fulfilled for each function $u \in H^1(\Omega)$.

Hint: Consider the expression $\int_\Omega (u^2 \Delta\phi)$ with $\phi = (2d)^{-1}|x|^2$, use the property $\Delta\phi = 1$ and apply the Green's identity from Homework 8.

(Punkte)