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Numerical Methods for Partial Differential Equations
Sheet 4

Exercise 7: The Lax-Friedrichs-Method

Considered is the transport equation

$$\partial_t u(t,x) + a\,\partial_x u(t,x) = 0 \qquad \text{for } t > 0, x \in \mathbb{R},$$

with a constant velocity $a > 0$ and initial state $u(0,x) = u_0(x)$, $x \in \mathbb{R}$. To get a numerical approximation one can use the **Lax-Friedrichs** method. The corresponding finite-difference scheme has the form

$$u_k^{n+1} = \frac{1}{2}\left((1+\gamma)\,u_{k-1}^n + (1-\gamma)\,u_{k+1}^n\right),$$

where $u_k^n = u_{\tau h}(t_n, x_k)$ and $\gamma = \frac{a\,\tau}{h}$. The points $(t_n, x_k) = (n\,\tau, k\,h)$, $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$, form a grid $Q_{\tau h}$ of the space-time domain $\mathbb{R}^+ \times \mathbb{R}$.

(a) Show, that the Lax-Friedrichs scheme is stable, i. e., that the inequality

$$\max_{k\in\mathbb{Z}} |u_k^{n+1}| \le \max_{k\in\mathbb{Z}} |u_k^n|$$

holds. Do we need a stability condition? Are the results also valid in case of $a < 0$?

(b) Show, that the Lax-Friedrichs scheme is consistent with consistency order 1 w. r. t. the space and the time grid size.

Exercise 8: Outlook to the Burgers equation

We consider the **Burgers equation**

$$\partial_t u(t,x) + u(t,x)\,\partial_x u(t,x) \quad \text{for } t > 0, x \in \mathbb{R}$$

with an initial state $u(0,x) = u_0(x)$.

(a) Derive the characteristics $\gamma(t)$ for the Burgers equation using the ansatz

$$u(t, \gamma(t)) = \text{const}.$$

(b) Draw the characteristics for the following cases:
- u_0 is Lipschitz-continuous and monotonically increasing,
- u_0 is discontinuous and monotonically increasing,
- u_0 is monotonically decreasing.

(c) Derive an explicit representation of the solution u in case of

$$u_0(x) = x \quad \text{and} \quad u_0(x) = -x.$$

Homework 5: The BTBS-Method

Discuss the questions from Exercise 7 for the "**B**ackward in **T**ime, **B**ackward in **S**pace"-scheme (BTBS). Is a stability condition required?

Homework 6: Implementation of finite differences for the transport equation

Given is an initial state u_0 as illustrated in Figure 1. This function is already imple-

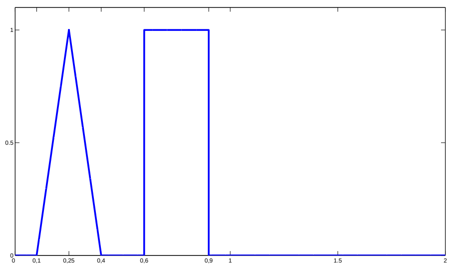

Figure 1: Initial state u_0.

mented in the file `u0.m`. Solve the transport equation

$$\partial_t u + a\,\partial_x u = 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}$$

with initial condition $u(0,\cdot) = u_0$ for the constant velocity $a = 1$ with the methods FTBS, FTCS, Lax-Friedrichs and BTBS. Implement for each algorithm a new MATLAB function with a header of the form:

```
function [u,x,t] = ftbs(a,h,tau,tstart,tend, ...
                        xstart,xend,u0)
```

Here `a` is the velocity, `h` and `tau` are the discretization parameters, `tstart` and `tend` is the time horizon, `xstart` and `xend` are the bounds of the domain and `u0` is a handle to the function u_0.

The return values should be a matrix `u` containing the entries $(u_{\tau h}(t_n, x_k))_{n,k}$, a vector `x` containing the space grid, and a vector `t` containing the time grid. One can assume homogeneous boundary conditions at the inflow boundary $x = 0$, i. e., $u(t, 0) = 0$.

The implementation can be tested with the script `FDM_transport_equation.m`.

Interpret the results of the computation. In which cases are stability conditions required? How good do the methods approximate the discontinuities in the exact solution?