Technische Universität Chemnitz — Chemnitz, 5th April 2017
Dr. M. Winkler — hand in on 24th April 2017

Numerical Methods for Partial Differential Equations
Sheet 2

Exercise 4: Numerical verification of the rate 2 for the 5-point-difference-stencil

Verify numerically that 5-point-difference-stencil is of order 2 (compare Theorem 4.3). As starting point, use the template `FDM_Convergence_Test.m`.

(a) Implement assembly routines for the matrix L_h and the vector f_h as explained in (3.4). Modify the scheme such that inhomogeneous Dirichlet boundary conditions can be realized.

(b) Compute the consistency rate

$$d_h := \left\| -\Delta_h^{(5)} R_h u - f_h \right\|_{L^\infty,h}.$$

for different grid sizes h. Draw the relation between h and d_h into a diagram with logarithmic axes. As exact solution use the function

$$u(x) = e^{x_1 \cdot x_2} \cdot \sin(x_1^2)$$

on the unit square The consistency order can determined experimentally with the formula

$$\operatorname{eoc}(d_h; h_1, h_2) = \frac{\log(d_{h_1}/d_{h_2})}{\log(h_1/h_2)},$$

where h_1, h_2 are 2 different grid sizes.

(c) Compute the numerical solution u_h. Analogously to (b), determine the order of the error

$$e_h := \left\| R_h u - u_h \right\|_{L^\infty,h}.$$

Hint: For the assembly of L_h the functions `spdiags` and `kron` are helpful.

Exercise 5: Structure of the system matrix for different numberings

The system matrix obtained from (3.3) depends on the global numbering of the unknown function values $u_{i,j}$. Draw systematically the structure of the system matrix if the unknowns are numbered

(a) lexicographically,

(b) diagonally,

(c) in a checkerboard pattern.

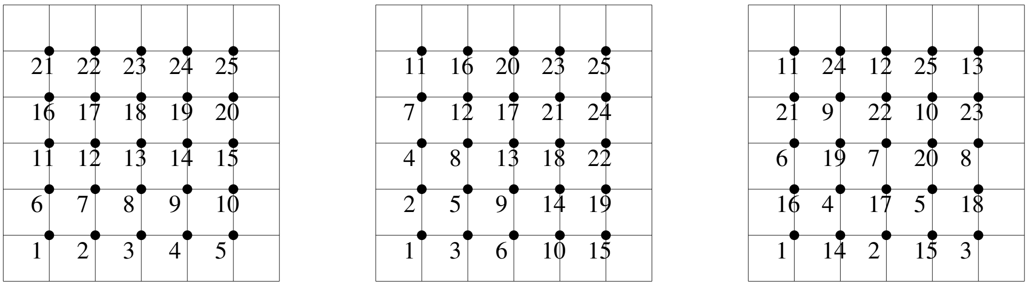

Figure 1: lexicographic, diagonal and checkerboard-ordering

Homework 1: Eigenvalues of the discrete Laplace operator

(a) Confirm that the eigenvalues and eigenfunctions of the Laplace operator on the unit square $\Omega := (0,1)^2$ are

$$\lambda^{i,j} = (i^2 + j^2)\pi^2 \quad \text{and} \quad w^{i,j} = \sin(i\,\pi\,x)\sin(j\,\pi\,y), \qquad i,j = 1,2,\ldots,$$

i. e., the pair $(\lambda^{i,j}, w^{i,j})$ solves the boundary value problem

$$-\Delta w^{i,j} = \lambda^{i,j} w^{i,j} \quad \text{in} \quad \Omega, \qquad w^{i,j} = 0 \quad \text{on} \quad \partial\Omega.$$

(b) We consider the matrix $L_h \in \mathbb{R}^{(n-1)^2 \times (n-1)^2}$ corresponding to the 5-point difference stencil defined in (3.4).

Show that L_h has the eigenvalues

$$\begin{aligned} \lambda^{i,j} &= \frac{1}{h^2}\Big(4 - 2\left(\cos(i\,h\,\pi) + \cos(j\,h\,\pi)\right)\Big) \\ &= \frac{4}{h^2}\Big(\sin^2\big(\frac{i\,h\,\pi}{2}\big) + \sin^2\big(\frac{j\,h\,\pi}{2}\big)\Big) \qquad i,j = 1,\,\ldots,\,n-1 \end{aligned}$$

and the corresponding eigenvectors

$$w^{i,j}(x_{k,m}) = \sin(i\,\pi\,k\,h)\,\sin(j\,\pi\,m\,h) \qquad k,m = 1,\,\ldots,\,n-1.$$

(c) Show: The condition number of L_h w.r.t. the spectral norm is equal to

$$\kappa_2(L_h) = \frac{\lambda_{\max}(L_h)}{\lambda_{\min}(L_h)} = \frac{4}{\pi^2}\cdot\frac{1}{h^2} + \mathcal{O}(1).$$

Homework 2: Consistency of the 5-point difference stencil

(a) Prove Remark 4.4 (a).

(b) Show that the 5-point difference stencil possesses only the consistency order 1 if $u \in C^3(\overline{\Omega})$.