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# Numerical Methods for Partial Differential Equations
# Sheet 1

### Exercise 1: Boundary conditions of the heat equation

Let $\Omega := (0, L)$ be a one-dimensional domain. In this domain we consider the stationary (i. e. time independent) heat equation

$$\begin{aligned} -u''(x) &= 1, \quad x \in \Omega, \\ u(0) &= 0, \\ u'(L) + \alpha(u(L) - u_{\text{room}}) &= 0, \end{aligned}$$

whose solution $u(x)$ is the temperature of the shape $\Omega$ in the point $x$.

The physical interpretation of the boundary condition is, that the temperature at the left boundary is fixed to zero, and that the heat flux at the right boundary is proportional to the difference to the room temperature, i. e., to $u - r_{\text{room}}$. The parameter $\alpha \in \mathbb{R}$ is called the *heat exchange coefficient*.

Derive the analytical solution of this boundary value probelm. What happens in the special cases $\alpha = 0$ and $\alpha \to \infty$?

### Exercise 2: Classification of differential equations

A partial differential equation $F(u, \nabla u, \ldots, \nabla^k u) = 0$ with $u\colon \mathbb{R}^n \to \mathbb{R}$ is called *linear*, if it possesses the representation

$$\sum_{|\alpha|\le k} a_\alpha(x)\, D^\alpha u(x) = f(x), \quad D^\alpha := \prod_{i=1}^n \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

where $a_\alpha$ are arbitrary coefficient functions independent of $u$. Here, $\alpha \in \mathbb{N}_0^n$ are multi-indices, i. e., $\alpha = (\alpha_1, \ldots, \alpha_n)$, of order $|\alpha| := \sum_{i=0}^n \alpha_i$. To simplify notation we abbreviate

$$u_{x_1} = \frac{\partial}{\partial x_1} u, \quad u_{x_1x_1} = \frac{\partial^2}{\partial x_1^2} u, \quad \ldots.$$

Investigate whether the following PDEs are linear and determine the order:

(a) $u_x + u_y = 0$

(b) $u_x + u_y = \sin(\pi x)\sin(\pi y)$

(c) $u_y + u u_x = 1$

(d) $\sqrt{1 + x^2} u_{xx} + \cos(y) u_{yy} = \sqrt{x^2 + y^2}$

(e) $u_{xx} + u_{yy}^2 = 1$

(f) $u_t + \mathrm{i} u_x = 0$

**Exercise 3: Laplace operator in polar coordinates**

(a) Show, that the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ for some function $u(r,\,\varphi)$ in polar coordinates $(r,\,\varphi) \in (0,\,\infty) \times [0,\,2\pi)$, i. e., $x = r\,\cos\varphi$, $y = r\,\sin\varphi$, possesses the representation

$$\Delta u = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \varphi^2}.$$

**Hint:** Define the function $u$ in Cartesian coordinates as $\tilde{u}$, i. e., there holds $\tilde{u}(x(r,\varphi), y(r,\varphi)) = u(r,\varphi)$ for all $(r,\varphi) \in (0,\infty) \times [0, 2\,\pi)$. Then, differentiate for $r$ and $\varphi$ using the chain rule.

(b) For some angle $\omega \in (0, 2\pi]$ let $\Omega := \{(r,\,\varphi) : 0 < r < 1,\ \varphi \in (0,\,\omega)\}$ be the sector of the unit circle. Show that the function

$$u(r,\,\varphi) := r^{\pi/\omega}\,\sin\left(\frac{\varphi\,\pi}{\omega}\right)$$

is a solution of the PDE

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ u(r,\,0) &= 0 && \text{for } 0 \le r \le 1 \\ u(r,\,\omega) &= 0 && \text{for } 0 \le r \le 1 \\ u(1,\,\varphi) &= \sin\left(\frac{\varphi\,\pi}{\omega}\right) && \text{for } 0 \le \varphi \le \omega \end{aligned}$$

and, that the first derivatives can not be extended continuously to the boundary, i. e., $u \notin C^1(\overline{\Omega})$.

(c) Compute for $R \in (0, 1)$ a rotationally symmetric solution $u$ of the equation

$$\begin{aligned} -\Delta u &= 0 && \text{in } \{(x, y) \in \mathbb{R}^2 \mid R < x^2 + y^2 < 1\}, \\ u &= 1 && \text{for } x^2 + y^2 = R, \\ u &= 0 && \text{for } x^2 + y^2 = 1. \end{aligned}$$