# Numerical Methods for Partial Differential Equations

## Quadrature formulas

| weak forumlation | local contributions of the cell $K$ |
|---|---|
| $\displaystyle\int_\Omega \nabla\varphi_j^\top A\,\nabla\varphi_i\,\mathrm{d}x$ | $\displaystyle\lvert\det B_K\rvert\sum_{\ell=1}^{q}\omega_{\widehat{K},\ell}\,\big[B_K^{-\top}\widehat{\nabla}\widehat{p}_n(\xi_{\widehat{K},\ell})\big]^\top A(T_K(\xi_{\widehat{K},\ell}))\,\big[B_K^{-\top}\widehat{\nabla}\widehat{p}_m(\xi_{\widehat{K},\ell})\big]$ |
| $\displaystyle\int_\Omega \nabla\varphi_j\cdot\beta\,\varphi_i\,\mathrm{d}x$ | $\displaystyle\lvert\det B_K\rvert\sum_{\ell=1}^{q}\omega_{\widehat{K},\ell}\,\big[B_K^{-\top}\widehat{\nabla}\widehat{p}_n(\xi_{\widehat{K},\ell})\big]\cdot\beta(T_K(\xi_{\widehat{K},\ell}))\,\widehat{p}_m(\xi_{\widehat{K},\ell})$ |
| $\displaystyle\int_\Omega \varphi_j\,c_0\,\varphi_i\,\mathrm{d}x$ | $\displaystyle\lvert\det B_K\rvert\sum_{\ell=1}^{q}\omega_{\widehat{K},\ell}\,\widehat{p}_n(\xi_{\widehat{K},\ell})\,c_0(T_K(\xi_{\widehat{K},\ell}))\,\widehat{p}_m(\xi_{\widehat{K},\ell})$ |
| $\displaystyle\int_\Omega f\,\varphi_i\,\mathrm{d}x$ | $\displaystyle\lvert\det B_K\rvert\sum_{\ell=1}^{q}\omega_{\widehat{K},\ell}\,f(T_K(\xi_{\widehat{K},\ell}))\,\widehat{p}_m(\xi_{\widehat{K},\ell})$ |

Table 1: Transformation of volume integrals in the weak formulation to the reference cell and approximation via a quadrature rule

| weak formulation | local contributions of the cell $K$ |
|---|---|
| $\displaystyle\int_\Gamma \varphi_j\,\alpha(s)\,\varphi_i\,\mathrm{d}s$ | $\displaystyle\frac{\lvert F_j\rvert}{\lvert\widehat{F}_j\rvert}\sum_{\ell=1}^{q'}\omega_{\widehat{F}_j,\ell}\,\widehat{p}_n(\xi_{\widehat{F}_j,\ell})\,\alpha(T_K(s))\,\widehat{p}_m(\xi_{\widehat{F}_j,\ell})$ |
| $\displaystyle\int_\Gamma g(s)\,\varphi_i\,\mathrm{d}s$ | $\displaystyle\frac{\lvert F_j\rvert}{\lvert\widehat{F}_j\rvert}\sum_{\ell=1}^{q'}\omega_{\widehat{F}_j,\ell}\,g(T_K(s))\,\widehat{p}_m(\xi_{\widehat{F}_j,\ell})$ |

Table 2: Transformation of boundary integrals in the weak formulation to a reference facet and approximation via a quadrature rule

The **order** of a quadrature rule

$$\int_{\widehat{K}}\widehat{g}(x)\,\mathrm{d}x\approx\sum_{\ell=1}^{q}\omega_{\widehat{K},\ell}\,\widehat{g}(\xi_{\widehat{K},\ell})$$

is the largest $r\in\mathbb{N}$, such that equality holds for all polynomials $\widehat{g}\in P_r(\widehat{K})$ (analogously: boundary integrals). The numbers $\omega_{\widehat{K},\ell}$ are called **weights** and the points $\xi_{\widehat{K},\ell}$ are called **nodes** of the quadrature rule.

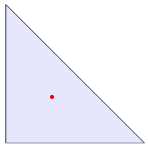
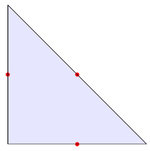
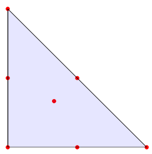
| $r$ | $q$ | | coordinates | number | weights |
|---|---|---|---|---|---|
| 1 | 1 |  | $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ | 1 | $\lvert\widehat{K}\rvert$ |
| 2 | 3 |  | $(\frac{1}{2},\frac{1}{2},0)$ | 3 | $\frac{1}{3}\lvert\widehat{K}\rvert$ |
| 3 | 7 |  | $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ | 1 | $\frac{9}{20}\lvert\widehat{K}\rvert$ |
| | | | $(\frac{1}{2},\frac{1}{2},0)$ | 3 | $\frac{2}{15}\lvert\widehat{K}\rvert$ |
| | | | $(1,0,0)$ | 3 | $\frac{1}{20}\lvert\widehat{K}\rvert$ |

Table 3: Quadrature rules on triangles given in barycentric coordinates. If "number" is larger than one, the coordinates of further nodes are given by cyclic permutations.

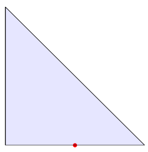
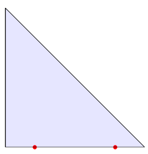
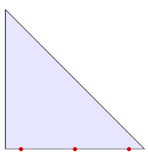
| $r$ | $q$ | | coordinates | number | weights |
|---|---|---|---|---|---|
| 1 | 1 |  | $(\frac{1}{2},\frac{1}{2},0)$ | 1 | $\lvert\widehat{F}_j\rvert$ |
| 3 | 2 |  | $(\frac{1}{2}+\frac{1}{6}\sqrt{3},\ \frac{1}{2}-\frac{1}{6}\sqrt{3},0)$ | 1 | $\frac{1}{2}\lvert\widehat{F}_j\rvert$ |
| | | | $(\frac{1}{2}-\frac{1}{6}\sqrt{3},\ \frac{1}{2}+\frac{1}{6}\sqrt{3},0)$ | 1 | $\frac{1}{2}\lvert\widehat{F}_j\rvert$ |
| 5 | 3 |  | $(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3}{5}},\ \frac{1}{2}-\frac{1}{2}\sqrt{\frac{3}{5}},0)$ | 1 | $\frac{5}{18}\lvert\widehat{F}_j\rvert$ |
| | | | $(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{3}{5}},\ \frac{1}{2}+\frac{1}{2}\sqrt{\frac{3}{5}},0)$ | 1 | $\frac{5}{18}\lvert\widehat{F}_j\rvert$ |
| | | | $(\frac{1}{2},\frac{1}{2},0)$ | 1 | $\frac{8}{18}\lvert\widehat{F}_j\rvert$ |

Table 4: Quadrature rules on facets of a triangle given in barycentric coordinates. The quadrature rules on the other edges are obtained by permutations of the coordinates.