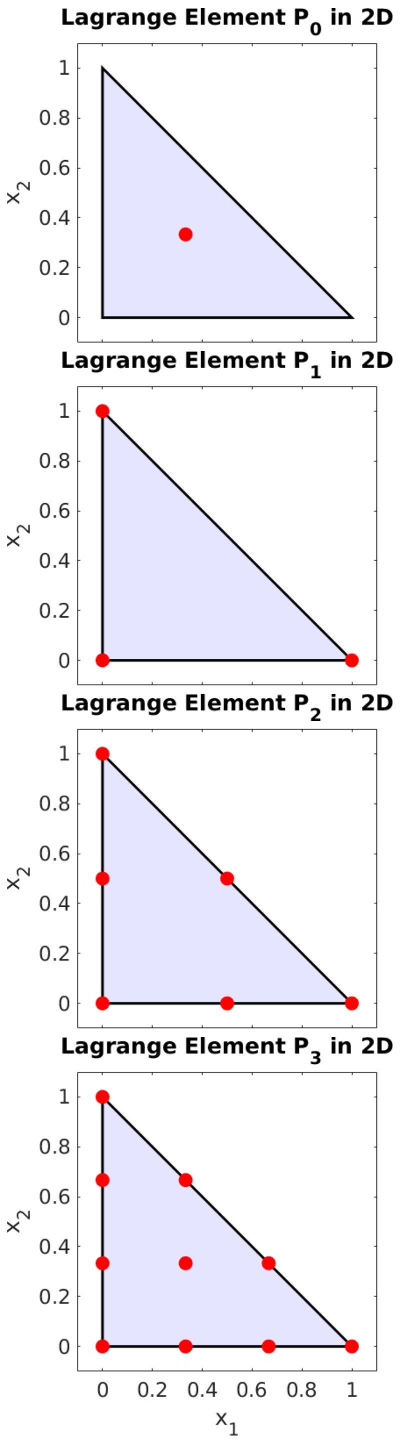
Numerical Methods for Partial Differential Equations

Lagrange elements $\mathbb{P}_k$ in 2D



Constant triangular element $\mathbb{P}_0$

$P = P_0, \quad \dim(P) = 1$

$p_1 = 1$

Linear triangular element $\mathbb{P}_1$

$P = P_1, \quad \dim(P) = 3$

$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2$

Quadratic triangular element $\mathbb{P}_2$

$P = P_2, \quad \dim(P) = 6$

$$p_1 = \lambda_0\,(2\,\lambda_0 - 1), \quad p_4 = 4\,\lambda_0\,\lambda_1$$
$$p_2 = \lambda_1\,(2\,\lambda_1 - 1), \quad p_5 = 4\,\lambda_0\,\lambda_2$$
$$p_3 = \lambda_2\,(2\,\lambda_2 - 1), \quad p_6 = 4\,\lambda_1\,\lambda_2$$

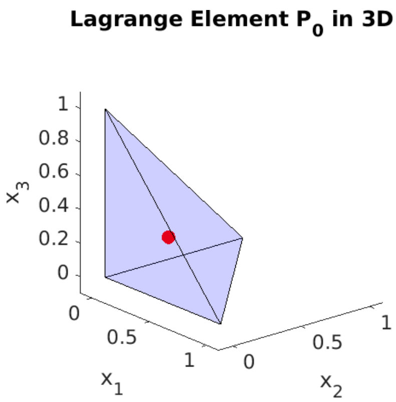
Cubic triangular element $\mathbb{P}_3$

$P = P_3, \quad \dim(P) = 10$

$$p_{i+1} = \tfrac{1}{2}\,\lambda_i\,(3\,\lambda_i - 1)\,(3\,\lambda_i - 2), \quad i = 0, 1, 2$$
$$p_4 = \tfrac{9}{2}\,\lambda_0\,(3\,\lambda_0 - 1)\,\lambda_1, \quad p_5 = \tfrac{9}{2}\,\lambda_0\,(3\,\lambda_0 - 1)\,\lambda_2$$
$$p_6 = \tfrac{9}{2}\,\lambda_1\,(3\,\lambda_1 - 1)\,\lambda_0, \quad p_7 = \tfrac{9}{2}\,\lambda_1\,(3\,\lambda_1 - 1)\,\lambda_2$$
$$p_8 = \tfrac{9}{2}\,\lambda_2\,(3\,\lambda_2 - 1)\,\lambda_0, \quad p_9 = \tfrac{9}{2}\,\lambda_2\,(3\,\lambda_2 - 1)\,\lambda_1$$
$$p_{10} = 27\,\lambda_0\,\lambda_1\,\lambda_2$$

Figure 1: Lagrange elements $\mathbb{P}_k$ on the unit triangle and shape functions in barycentric coordinates λ_0, λ_1 and λ_2

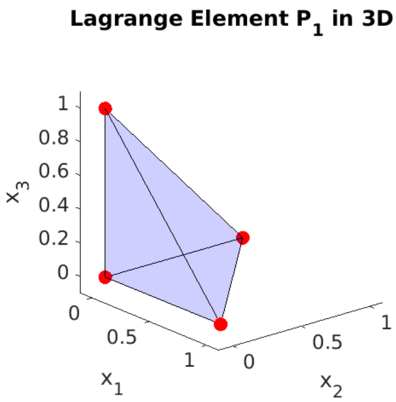
Lagrange elements $\mathbb{P}_k$ in 3D



Constant tetrahedral element $\mathbb{P}_0$

$P = P_0, \quad \dim(P) = 1$

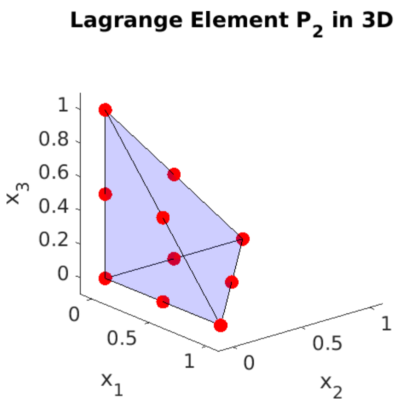
$p_1 = 1$



Linear tetrahedral element $\mathbb{P}_1$

$P = P_1, \quad \dim(P) = 4$

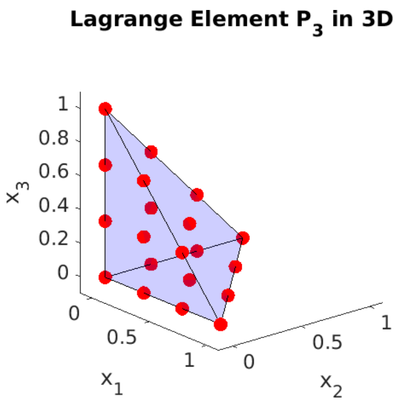
$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2, \quad p_4 = \lambda_3$



Quadratic tetrahedral element $\mathbb{P}_2$

$P = P_2, \quad \dim(P) = 10$

$$
\begin{aligned}
p_1 &= \lambda_0\,(2\,\lambda_0 - 1), & p_6 &= 4\,\lambda_0\,\lambda_2\\
p_2 &= \lambda_1\,(2\,\lambda_1 - 1), & p_7 &= 4\,\lambda_0\,\lambda_3\\
p_3 &= \lambda_2\,(2\,\lambda_2 - 1), & p_8 &= 4\,\lambda_1\,\lambda_2\\
p_4 &= \lambda_3\,(2\,\lambda_3 - 1), & p_9 &= 4\,\lambda_1\,\lambda_3\\
p_5 &= 4\,\lambda_0\,\lambda_1, & p_{10} &= 4\,\lambda_2\,\lambda_3
\end{aligned}
$$

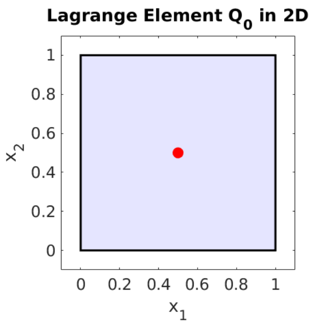


Cubic tetrahedral element $\mathbb{P}_3$

$P = P_3, \quad \dim(P) = 20$

Figure 2: Lagrange elements $\mathbb{P}_k$ on the unit tetrahedron and shape functions in barycentric coordinates λ_0, λ_1, λ_2 and λ_3

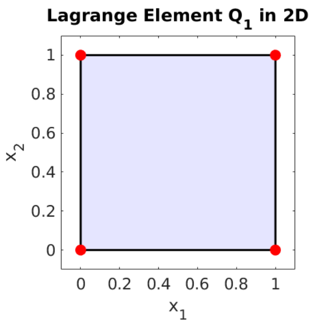
Lagrange elements $\mathbb{Q}_k$ in 2D



Constant rectangular element $\mathbb{Q}_0$

$P = Q_0, \quad \dim(P) = 1$

$p_1 = 1$

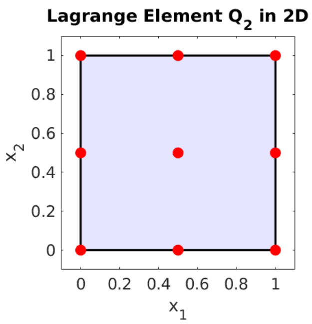


Bilinear rectangular element $\mathbb{Q}_1$

$P = Q_1, \quad \dim(P) = 4$

$p_1 = t_1\,t_2, \qquad p_2 = t_1\,(1-t_2)$

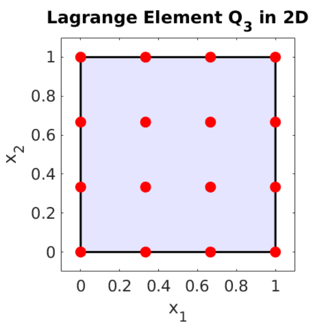
$p_3 = (1-t_1)\,t_2, \quad p_4 = (1-t_1)\,(1-t_2)$



Biquadratic rectangular element $\mathbb{Q}_2$

$P = Q_2, \quad \dim(P) = 9$

$p_{3j+i+1} = L_i^{(2)}(t_1)\,L_j^{(2)}(t_2), \quad 0 \le i,j \le 2$



Bicubic rectangular element $\mathbb{Q}_3$

$P = Q_3, \quad \dim(P) = 16$

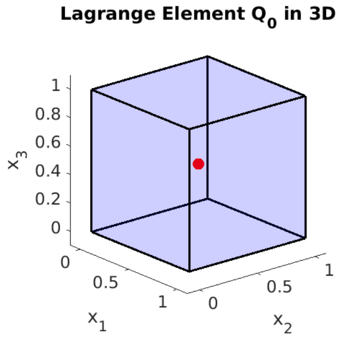
$p_{4j+i+1} = L_i^{(3)}(t_1)\,L_j^{(3)}(t_2), \quad 0 \le i,j \le 3$

Here, $L_i^{(k)}$ denotes the Lagrange polynomial i corresponding to the equidistant points $s_j = j/k$, $j = 0,1,\ldots,k$, i.e.,

$$L_i^{(k)}(t) = \prod_{j=0,\, j\neq i}^{k} \frac{t-s_j}{s_i-s_j}, \quad i = 0,1,\ldots,k.$$

Figure 3: Lagrange elements $\mathbb{Q}_k$ on the unit square and shape functions in coordinates t_1 and t_2.

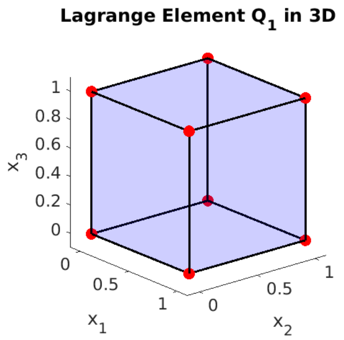
Lagrange elements $\mathbb{Q}_k$ in 3D



Constant cubiform element $\mathbb{Q}_0$

$P = Q_0, \quad \dim(P) = 1$

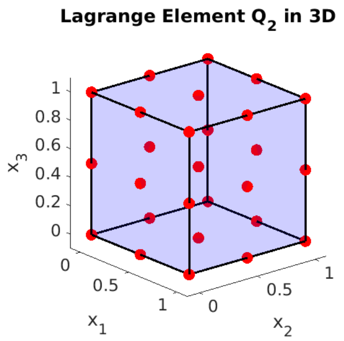
$p_1 = 1$



Trilinear cubiform element $\mathbb{Q}_1$

$P = Q_1, \quad \dim(P) = 8$

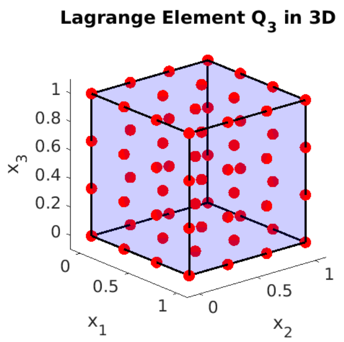
$$p_{4k+2j+i+1} = L_i^{(1)}(t_1)\, L_j^{(1)}(t_2)\, L_k^{(1)}(t_3), \quad 0 \le i,j,k \le 1$$



Triquadratic cubiform element $\mathbb{Q}_2$

$P = Q_2, \quad \dim(P) = 27$

$$p_{9k+3j+i+1} = L_i^{(2)}(t_1)\, L_j^{(2)}(t_2)\, L_k^{(2)}(t_3), \quad 0 \le i,j,k \le 2$$



Tricubic cubiform element $\mathbb{Q}_3$

$P = Q_3, \quad \dim(P) = 64$

$$p_{16k+4j+i+1} = L_i^{(3)}(t_1)\, L_j^{(3)}(t_2)\, L_k^{(3)}(t_3), \quad 0 \le i,j,k \le 3$$

Figure 4: Lagrange elements $\mathbb{Q}_k$ on the unit cube and shape functions t_1, t_2 and t_3.