

# Numerik partieller Differentialgleichungen

## Quadraturformeln

weak formulation	local contributions of the cell $K$
$\int_{\Omega} \nabla \varphi_j^\top A \nabla \varphi_i \, dx$	$ \det B_K  \sum_{\ell=1}^q \omega_{\widehat{K},\ell} [B_K^{-\top} \widehat{\nabla} \widehat{p}_n(\xi_{\widehat{K},\ell})]^\top A(T_K(\xi_{\widehat{K},\ell})) [B_K^{-\top} \widehat{\nabla} \widehat{p}_m(\xi_{\widehat{K},\ell})]$
$\int_{\Omega} \nabla \varphi_j \cdot \beta \varphi_i \, dx$	$ \det B_K  \sum_{\ell=1}^q \omega_{\widehat{K},\ell} [B_K^{-\top} \widehat{\nabla} \widehat{p}_n(\xi_{\widehat{K},\ell})] \cdot \beta(T_K(\xi_{\widehat{K},\ell})) \widehat{p}_m(\xi_{\widehat{K},\ell})$
$\int_{\Omega} \varphi_j c_0 \varphi_i \, dx$	$ \det B_K  \sum_{\ell=1}^q \omega_{\widehat{K},\ell} \widehat{p}_n(\xi_{\widehat{K},\ell}) c_0(T_K(\xi_{\widehat{K},\ell})) \widehat{p}_m(\xi_{\widehat{K},\ell})$
$\int_{\Omega} f \varphi_i \, dx$	$ \det B_K  \sum_{\ell=1}^q \omega_{\widehat{K},\ell} f(T_K(\xi_{\widehat{K},\ell})) \widehat{p}_m(\xi_{\widehat{K},\ell})$

Table 1: Transformation of volume integrals in the weak formulation to the reference cell and approximation via a quadrature rule

weak formulation	local contributions of the cell $K$
$\int_{\Gamma} \varphi_j \alpha(s) \varphi_i \, ds$	$\frac{ F_j }{ \widehat{F}_j } \sum_{\ell=1}^{q'} \omega_{\widehat{F}_j,\ell} \widehat{p}_n(\xi_{\widehat{F}_j,\ell}) \alpha(T_K(s)) \widehat{p}_m(\xi_{\widehat{F}_j,\ell})$
$\int_{\Gamma} g(s) \varphi_i \, ds$	$\frac{ F_j }{ \widehat{F}_j } \sum_{\ell=1}^{q'} \omega_{\widehat{F}_j,\ell} g(T_K(s)) \widehat{p}_m(\xi_{\widehat{F}_j,\ell})$

Table 2: Transformation of boundary integrals in the weak formulation to a reference facet and approximation via a quadrature rule

The **order** of a quadrature rule

$$\int_{\widehat{K}} \widehat{g}(x) \, dx \approx \sum_{\ell=1}^q \omega_{\widehat{K},\ell} \widehat{g}(\xi_{\widehat{K},\ell})$$

is the largest  $r \in \mathbb{N}$ , such that equality holds for all polynomials  $\widehat{g} \in P_r(\widehat{K})$  (analogously: boundary integrals). The numbers  $\omega_{\widehat{K},\ell}$  are called **weights** and the points  $\xi_{\widehat{K},\ell}$  are called **nodes** of the quadrature rule.

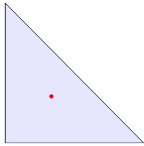
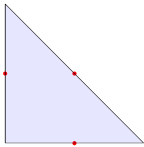
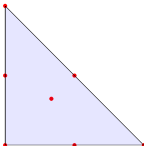
$r$	$q$		coordinates	number	weights
1	1		$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1	$ \hat{K} $
2	3		$(\frac{1}{2}, \frac{1}{2}, 0)$	3	$\frac{1}{3} \hat{K} $
3	7		$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ $(\frac{1}{2}, \frac{1}{2}, 0)$ $(1, 0, 0)$	1 3 3	$\frac{9}{20} \hat{K} $ $\frac{2}{15} \hat{K} $ $\frac{1}{20} \hat{K} $

Table 3: Quadrature rules on triangles given in barycentric coordinates. If “number” is larger than one, the coordinates of further nodes are given by cyclic permutations.

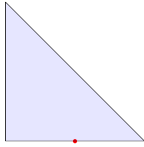
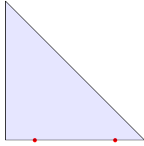
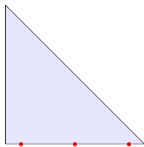
$r$	$q$		coordinates	number	weights
1	1		$(\frac{1}{2}, \frac{1}{2}, 0)$	1	$ \hat{F}_j $
3	2		$(\frac{1}{2} + \frac{1}{6}\sqrt{3}, \frac{1}{2} - \frac{1}{6}\sqrt{3}, 0)$ $(\frac{1}{2} - \frac{1}{6}\sqrt{3}, \frac{1}{2} + \frac{1}{6}\sqrt{3}, 0)$	1 1	$\frac{1}{2} \hat{F}_j $ $\frac{1}{2} \hat{F}_j $
5	3		$(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}, \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}, 0)$ $(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}, 0)$ $(\frac{1}{2}, \frac{1}{2}, 0)$	1 1 1	$\frac{5}{18} \hat{F}_j $ $\frac{5}{18} \hat{F}_j $ $\frac{8}{18} \hat{F}_j $

Table 4: Quadrature rules on facets of a triangle given in barycentric coordinates. The quadrature rules on the other edges are obtained by permutations of the coordinates.