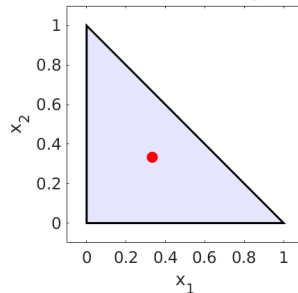


Numerik partieller Differentialgleichungen

Lagrange elements \mathbb{P}_k in 2D

Lagrange Element \mathbb{P}_0 in 2D

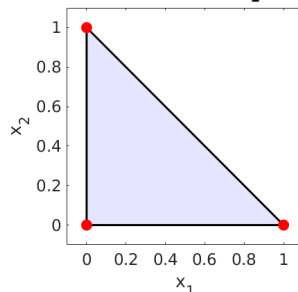


Constant triangular element \mathbb{P}_0

$$P = P_0, \quad \dim(P) = 1$$

$$p_1 = 1$$

Lagrange Element \mathbb{P}_1 in 2D

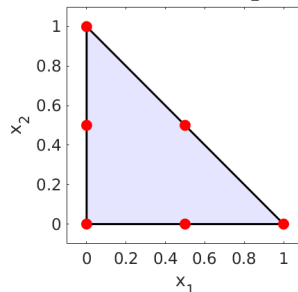


Linear triangular element \mathbb{P}_1

$$P = P_1, \quad \dim(P) = 3$$

$$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2$$

Lagrange Element \mathbb{P}_2 in 2D



Quadratic triangular element \mathbb{P}_2

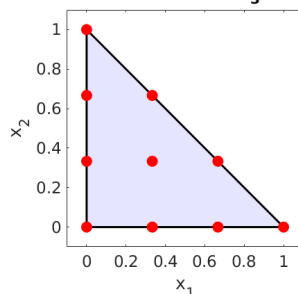
$$P = P_2, \quad \dim(P) = 6$$

$$p_1 = \lambda_0 (2 \lambda_0 - 1), \quad p_4 = 4 \lambda_0 \lambda_1$$

$$p_2 = \lambda_1 (2 \lambda_1 - 1), \quad p_5 = 4 \lambda_0 \lambda_2$$

$$p_3 = \lambda_2 (2 \lambda_2 - 1), \quad p_6 = 4 \lambda_1 \lambda_2$$

Lagrange Element \mathbb{P}_3 in 2D



Cubic triangular element \mathbb{P}_3

$$P = P_3, \quad \dim(P) = 10$$

$$p_{i+1} = \frac{1}{2} \lambda_i (3 \lambda_i - 1) (3 \lambda_i - 2), \quad i = 0, 1, 2$$

$$p_4 = \frac{9}{2} \lambda_0 (3 \lambda_0 - 1) \lambda_1, \quad p_5 = \frac{9}{2} \lambda_0 (3 \lambda_0 - 1) \lambda_2$$

$$p_6 = \frac{9}{2} \lambda_1 (3 \lambda_1 - 1) \lambda_0, \quad p_7 = \frac{9}{2} \lambda_1 (3 \lambda_1 - 1) \lambda_2$$

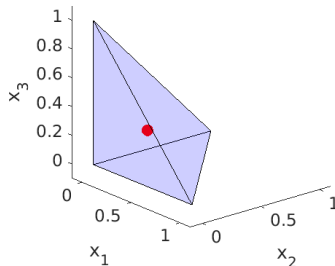
$$p_8 = \frac{9}{2} \lambda_2 (3 \lambda_2 - 1) \lambda_0, \quad p_9 = \frac{9}{2} \lambda_2 (3 \lambda_2 - 1) \lambda_1$$

$$p_{10} = 27 \lambda_0 \lambda_1 \lambda_2$$

Figure 1: Lagrange elements \mathbb{P}_k on the unit triangle and shape functions in barycentric coordinates λ_0 , λ_1 and λ_2

Lagrange elements \mathbb{P}_k in 3D

Lagrange Element \mathbb{P}_0 in 3D

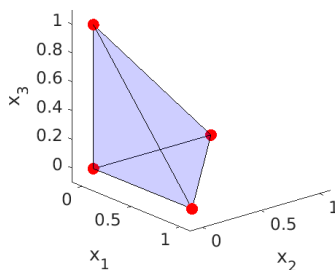


Constant tetrahedral element \mathbb{P}_0

$$P = P_0, \quad \dim(P) = 1$$

$$p_1 = 1$$

Lagrange Element \mathbb{P}_1 in 3D

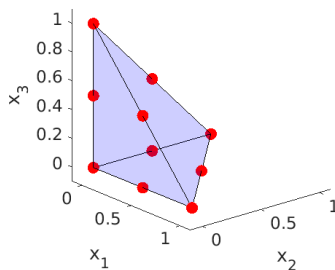


Linear tetrahedral element \mathbb{P}_1

$$P = P_1, \quad \dim(P) = 4$$

$$p_1 = \lambda_0, \quad p_2 = \lambda_1, \quad p_3 = \lambda_2, \quad p_4 = \lambda_3$$

Lagrange Element \mathbb{P}_2 in 3D



Quadratic tetrahedral element \mathbb{P}_2

$$P = P_2, \quad \dim(P) = 10$$

$$p_1 = \lambda_0 (2 \lambda_0 - 1), \quad p_6 = 4 \lambda_0 \lambda_2$$

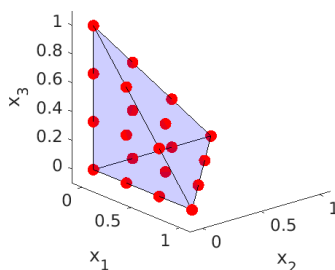
$$p_2 = \lambda_1 (2 \lambda_1 - 1), \quad p_7 = 4 \lambda_0 \lambda_3$$

$$p_3 = \lambda_2 (2 \lambda_2 - 1), \quad p_8 = 4 \lambda_1 \lambda_2$$

$$p_4 = \lambda_3 (2 \lambda_3 - 1), \quad p_9 = 4 \lambda_1 \lambda_3$$

$$p_5 = 4 \lambda_0 \lambda_1, \quad p_{10} = 4 \lambda_2 \lambda_3$$

Lagrange Element \mathbb{P}_3 in 3D



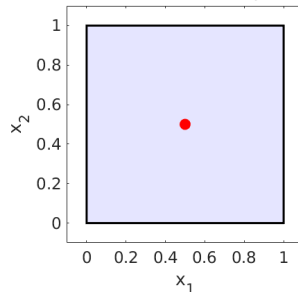
Cubic tetrahedral element \mathbb{P}_3

$$P = P_3, \quad \dim(P) = 20$$

Figure 2: Lagrange elements \mathbb{P}_k on the unit tetrahedron and shape functions in barycentric coordinates $\lambda_0, \lambda_1, \lambda_2$ and λ_3

Lagrange elements \mathbb{Q}_k in 2D

Lagrange Element \mathbb{Q}_0 in 2D

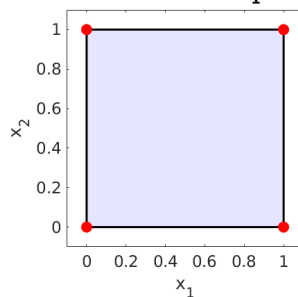


Constant rectangular element \mathbb{Q}_0

$$P = Q_0, \quad \dim(P) = 1$$

$$p_1 = 1$$

Lagrange Element \mathbb{Q}_1 in 2D



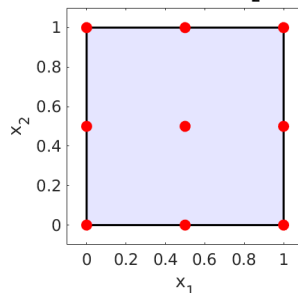
Bilinear rectangular element \mathbb{Q}_1

$$P = Q_1, \quad \dim(P) = 4$$

$$p_1 = t_1 t_2, \quad p_2 = t_1 (1 - t_2)$$

$$p_3 = (1 - t_1) t_2, \quad p_4 = (1 - t_1) (1 - t_2)$$

Lagrange Element \mathbb{Q}_2 in 2D

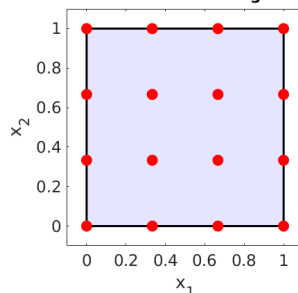


Biquadratic rectangular element \mathbb{Q}_2

$$P = Q_2, \quad \dim(P) = 9$$

$$p_{3j+i+1} = L_i^{(2)}(t_1) L_j^{(2)}(t_2), \quad 0 \leq i, j \leq 2$$

Lagrange Element \mathbb{Q}_3 in 2D



Bicubic rectangular element \mathbb{Q}_3

$$P = Q_3, \quad \dim(P) = 16$$

$$p_{4j+i+1} = L_i^{(3)}(t_1) L_j^{(3)}(t_2), \quad 0 \leq i, j \leq 3$$

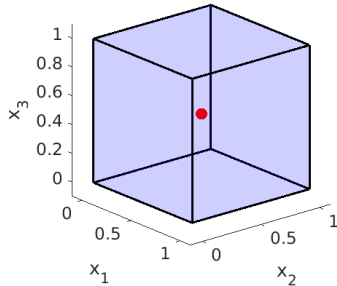
Here, $L_i^{(k)}$ denotes the Lagrange polynomial i corresponding to the equidistant points $s_j = j/k$, $j = 0, 1, \dots, k$, i.e.,

$$L_i^{(k)}(t) = \prod_{j=0, j \neq i}^k \frac{t - s_j}{s_i - s_j}, \quad i = 0, 1, \dots, k.$$

Figure 3: Lagrange elements \mathbb{Q}_k on the unit square and shape functions in coordinates t_1 and t_2 .

Lagrange elements \mathbb{Q}_k in 3D

Lagrange Element \mathbb{Q}_0 in 3D

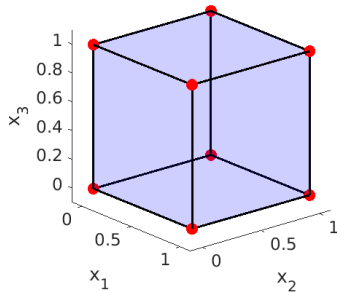


Constant cubiform element \mathbb{Q}_0

$$P = \mathbb{Q}_0, \quad \dim(P) = 1$$

$$p_1 = 1$$

Lagrange Element \mathbb{Q}_1 in 3D

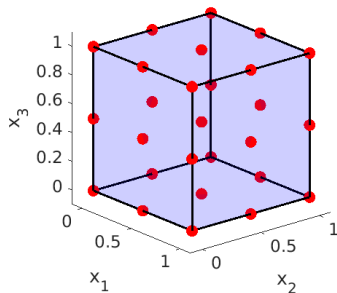


Trilinear cubiform element \mathbb{Q}_1

$$P = \mathbb{Q}_1, \quad \dim(P) = 8$$

$$p_{4k+2j+i+1} = L_i^{(1)}(t_1) L_j^{(1)}(t_2) L_k^{(1)}(t_3), \\ 0 \leq i, j, k \leq 1$$

Lagrange Element \mathbb{Q}_2 in 3D

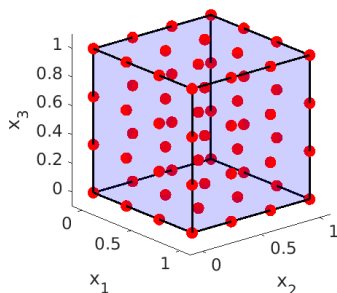


Triquadratic cubiform element \mathbb{Q}_2

$$P = \mathbb{Q}_2, \quad \dim(P) = 27$$

$$p_{9k+3j+i+1} = L_i^{(2)}(t_1) L_j^{(2)}(t_2) L_k^{(2)}(t_3), \\ 0 \leq i, j, k \leq 2$$

Lagrange Element \mathbb{Q}_3 in 3D



Tricubic cubiform element \mathbb{Q}_3

$$P = \mathbb{Q}_3, \quad \dim(P) = 64$$

$$p_{16k+4j+i+1} = L_i^{(3)}(t_1) L_j^{(3)}(t_2) L_k^{(3)}(t_3), \\ 0 \leq i, j, k \leq 3$$

Figure 4: Lagrange elements \mathbb{Q}_k on the unit cube and shape functions t_1 , t_2 and t_3 .