Pointwise Constraints in Vector-Valued Sobolev Spaces
with Applications in Optimal Control

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We consider a set $C$ with pointwise constraints in a vector-valued Sobolev space. We characterize its tangent and normal cone. Under the additional assumption that the pointwise constraints are affine and satisfy the linear independence constraint qualification, we show that the set $C$ is polyhedric. The results are applied to the optimal control of a string in a polyhedral tube.

**Keywords:** tangent cone, normal cone, polyhedricity, vector-valued function, vector-valued measure

**MSC:** 46N10, 49K21

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open set and let $C \subset \mathbb{R}^m$ be a closed, convex set with $0 \in \text{int}(C)$. We consider the set

$$ C := \{ v \in H_0^1(\Omega)^m : v(\omega) \in C \text{ for almost all } \omega \in \Omega \}. \tag{1.1} $$

We mention that such sets $C$ appear, e.g., in the study of vector-valued evolution variational inequalities, see [Krejčí, 1999, Section 7], and in vector-valued obstacle problems, see, e.g., [Hildebrandt, Widman, 1979; Dal Maso, Musina, 1989; Mancini, Musina, 1989]. Note that the latter case also includes phase-field models in which the concentrations of the phases belong to the simplex $C = \{ x \in \mathbb{R}^m : x \geq 0, \sum_{i=1}^m x_i \leq 1 \}$.

In this work, we are going to characterize the tangent and normal cone to the set $C$. Finally, we prove the polyhedricity of $C$ in case that $C$ is described by affine constraints which satisfy the linear independence constraint qualification (LICQ). We recall that $C$ is called polyhedric if

$$ T_C(u) \cap \xi^\perp = \text{cl}(R_C(u) \cap \xi^\perp) \tag{1.2} $$

holds for all $u \in C$ and $\xi \in T_C(u)^\circ$, see [Haraux, 1977]. We refer to Section 2.1 for the corresponding notation. These results are applied to an optimal control problem in which the state equation is given by a variational inequality over the set $C$.

To our knowledge, the convex analysis of the set $C$ was not studied previously in the vector-valued case $m > 1$. In the scalar case $m = 1$, the set $C$ was studied in [Mignot, 1976]. Therein, the set $C$
was allowed to depend also on the spatial variable \( \omega \in \Omega \), and Mignot characterized the tangent and normal cone. Moreover, he proved the polyhedricity of \( C \). However, the arguments in [Mignot, 1976] heavily exploit the fact that the non-negative functions induce a lattice structure in \( H^1_0(\Omega) \). Similarly, [Haraux, 1977, Corollary 2] and [Bonnans, Shapiro, 2000, Theorem 3.58] prove the polyhedricity of a cone which induces a lattice structure. These results are extended to more general function spaces in [Frémiot et al., 2009, Section 4], see also [Rao, Sokołowski, 1993] for the case \( H^2_0(\Omega) \).

This approach for proving polyhedricity even fails for some polyhedral cones in finite dimensions. To give an example, the polyhedral cone

\[
K = \{ x \in \mathbb{R}^3 : (-1,1,0)x \leq 0, (-1,-1,0)x \leq 0, (-1,0,1)x \leq 0, (-1,0,-1)x \leq 0 \}
\]

does not induces a lattice structure on \( \mathbb{R}^3 \). Hence, the polyhedricity of \( K \) cannot be deduced from the above mentioned theorems. On the other hand, since \( K \) is a polyhedral set, its polyhedricity is evident.

Finally, we mention the results [Bonnans, 1998, Lemma 4.2, Proposition 4.3], [Bonnans, Shapiro, 2000, Lemma 6.34, Proposition 6.35]. These results characterize the tangent and normal cone and show the polyhedricity of a set with pointwise affine constraints in the Lebesgue space \( L^s(\Omega)^m \), \( s \in [1,\infty) \). The arguments utilize the Lipschitz continuity of a pointwise projection in \( L^s(\Omega)^m \). Since a pointwise projection is not Lipschitz continuous in \( H^1_0(\Omega) \), the arguments cannot be generalized to the situation at hand.

We mention that the polyhedricity of a set has some important applications. First of all, one can show that the projection onto a polyhedral set in a Hilbert space is directionally differentiable, see [Mignot, 1976; Haraux, 1977]. This can also be extended to a (directional) shape sensitivity for the solutions of variational inequalities, see [Sokołowski, Zolésio, 1987] and [Sokołowski, Zolésio, 1992, Section 4]. Second, in the presence of polyhedricity, one can provide no-gap second-order optimality conditions, see [Bonnans, Shapiro, 2000, Section 3.3.3]. Finally, polyhedricity is utilized to obtain optimality conditions of strongly stationary type for infinite-dimensional optimization problems with complementarity constraints, see, e.g., [Mignot, 1976, Proposition 4.1] and [Wachsmuth, 2015, Section 5].

We briefly describe the content of this work. Some notation from convex analysis and capacity theory is introduced in Section 2. We characterize the tangent cone \( T_C(u) \) and the normal cone \( T_C(u)^\circ \) of \( C \) in Section 3 and Section 4, respectively. Under an additional assumption, we deduce the polyhedricity of \( C \) in Section 5 and this result is applied to an optimal control problem in Section 6. In the appendices, we give various auxiliary results, which are of independent interest. A chain rule for the Nemytskii operator associated with a vector-valued truncation is given in Appendix A. In Appendix B, we show that the positive part of a measure in \( H^{-1}(\Omega) \) may not belong to \( H^{-1}(\Omega) \). Appendix C contains auxiliary results on polyhedral sets in \( \mathbb{R}^m \), which are utilized to infer the polyhedricity of \( C \) in Section 5.

### 2. Notation and Preliminaries

The Euclidean norm in \( \mathbb{R}^m \) is denoted by \(|\cdot|_{\mathbb{R}^m}\), and we use \((\cdot,\cdot)_{\mathbb{R}^m}\) for the associated scalar product. For \( x \in \mathbb{R}^m \) and \( \gamma > 0 \) we define the closed ball

\[
B_\gamma(x) := \{ y \in \mathbb{R}^m : |x - y|_{\mathbb{R}^m} \leq \gamma \}
\]

and we set

\[
B_\gamma := B_\gamma(0).
\]

#### 2.1. Notation from convex analysis

For a Banach space \( X \) and an arbitrary subset \( A \subset X \), we denote by \( \text{cl}(A) \), \( \text{conv}(A) \) and \( \text{cone}(A) \) the closure, convex hull (smallest convex superset of \( A \)) and the conical hull (smallest convex cone
Lemma 2.1. Assume that $A \subset \Omega$ is quasi-open and has measure zero. Then, $A$ has zero capacity.
Proof. Assume that the set $A$ is quasi-open and has measure zero. Then, for the function $f = 0$ we have $f \geq 1$ a.e. on $A$, since $A$ has measure zero. The quasi-openness of $A$ implies $f \geq 1$ q.e. on $A$, see [Wachsmuth, 2014, Lemma 2.3]. Hence, $f$ is an admissible function for the computation of the capacity of $A$, cf. [Heinonen, Kilpeläinen, Martio, 1993, Lemma 4.7], and this implies $\text{cap}(A) = 0$.

By using this lemma, we can give a characterization of the set $\mathcal{C}$ in terms of quasi-everywhere.

**Lemma 2.2.** For $\mathcal{C}$ defined in (1.1) we have

$$\mathcal{C} = \{v \in H^1_0(\Omega)^m : v(\omega) \in C \text{ for q.a. } \omega \in \Omega\}.$$

**Proof.** The inclusion “$\supset$” is clear since sets of capacity zero have measure zero.

To prove the converse inclusion, let $v \in \mathcal{C}$ be given. Then, $v(\omega) \in C$ for almost all $\omega \in \Omega$. We set $N = \{\omega \in \Omega : v(\omega) \notin C\}$. Then, $N$ has measure zero and is quasi-open. Now, the assertion follows from Lemma 2.1.

### 3. Characterization of the Tangent Cone

In this section, we are going to characterize the tangent cone of $\mathcal{C}$. We will see that the tangent cone of $\mathcal{C}$ consists of functions whose point values lie in the tangent cone of $C$. As in the scalar case $m = 1$, we have to use the notion of quasi-everywhere. In particular, we will show that

$$\mathcal{T}_C(u) = \{v \in H^1_0(\Omega)^m : v(\omega) \in \mathcal{T}_C(u(\omega)) \text{ for q.a. } \omega \in \Omega\}. \quad (3.1)$$

Throughout this section, we fix $u \in \mathcal{C}$. One inclusion in (3.1) is easy to prove.

**Lemma 3.1.** We have

$$\mathcal{T}_C(u) \subset A_1 := \{v \in H^1_0(\Omega)^m : v(\omega) \in \mathcal{T}_C(u(\omega)) \text{ for q.a. } \omega \in \Omega\}.$$

**Proof.** Let $v \in \mathcal{T}_C(u)$ be given. By definition of the tangent cone, there exist sequences $\{t_n\} \subset (0, \infty)$, $\{u_n\} \subset \mathcal{C}$ such that $t_n \searrow 0$ and $(u_n - u)/t_n \to v$ in $H^1_0(\Omega)^m$ as $n \to \infty$. Since $u_n(\omega), u(\omega) \in C$ for q.a. $\omega \in \Omega$, we have $[u_n(\omega) - u(\omega)]/t_n \in \mathcal{T}_C(u(\omega))$ for q.a. $\omega \in \Omega$. Since $(u_n - u)/t_n \to v$ in $H^1_0(\Omega)^m$, we have (up to a subsequence) $[u_n(\omega) - u(\omega)]/t_n \to v(\omega)$ as $n \to \infty$ for q.a. $\omega \in \Omega$. Since $\mathcal{T}_C(u(\omega))$ is closed, this shows $v(\omega) \in \mathcal{T}_C(u(\omega))$ for q.a. $\omega \in \Omega$.

In what follows, we show the converse embedding $A_1 \subset \mathcal{T}_C(u)$, which is harder to obtain. To this end, we will define sets $A_2, A_3, A_4 \subset H^1_0(\Omega)^m$ with the properties

$$A_1 \subset \text{cl}(A_2), \quad A_2 \subset \text{cl}(A_3), \quad A_3 \subset \text{cl}(A_4), \quad A_4 \subset \mathcal{R}_C(u).$$

Taking the closure, we obtain

$$\mathcal{T}_C(u) \subset A_1 \subset \text{cl}(A_1) \subset \text{cl}(A_2) \subset \text{cl}(A_3) \subset \text{cl}(A_4) \subset \text{cl}(\mathcal{R}_C(u)) = \mathcal{T}_C(u),$$

and the characterization of the tangent cone follows, see Theorem 3.12. Note that the same result is achieved in Section 4 by owing to the bipolar theorem. However, we present the direct approach via approximation, since it can be adopted to infer the polyhedricity of $\mathcal{C}$ under some additional assumptions on $\mathcal{C}$, see Section 5.

First, we use a simple truncation argument in order to work in the space $H^1_0(\Omega)^m \cap L^\infty(\Omega)^m$. 

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Lemma 3.2. Each function $v \in A_1$ can be approximated by functions from $A_2 := \{ v \in H^1_0(\Omega)^m : v \in L^\infty(\Omega)^m, v(\omega) \in T_C(u(\omega)) \text{ for q.a. } \omega \in \Omega \}$. That is, $A_1 \subset \text{cl}(A_2)$.

Proof. The truncation $T_M v : \Omega \to \mathbb{R}^m$ of $v \in A_1$ which is defined via 
$$(T_M v)(\omega) := \min \left( 1, \frac{M}{|v(\omega)|_{\mathbb{R}^m}} \right) v(\omega)$$
belongs to $(L^\infty(\Omega) \cap H^1_0(\Omega))^m$, and converges towards $v$ in $H^1_0(\Omega)^m$, see Theorem A.2. Since $T_C(u(\omega))$ is a cone for all $\omega \in \Omega$, $T_M v \in A_2$ follows.

To proceed, we need an assumption on the set $C$. Roughly speaking, we assume that for each point $x \in C$ there is a ball of radius $c_2$ and the distance between $x$ and the center of the ball is at most $c_1$.

Assumption 3.3. There exist $c_1, c_2 > 0$, such that for every $x \in C$, there exists a point $w(x) \in C$ such that $|x - w(x)|_{\mathbb{R}^m} \leq c_1$ and $B_{c_2}(w(x)) \subset C$. Moreover, the mapping $w$ is Lipschitz continuous with modulus $L$ and $w(0) = 0$.

Note that Assumption 3.3 implies $0 \in \text{int}(C)$. Moreover, in the case that $C$ is bounded and $0 \in \text{int}(C)$, we can choose $w \equiv 0$. However, the unbounded, closed, convex set 
$$\{(x, y, z) : x^2 - y \leq 1, x^2/(y + 1) + |z| \leq 1\}$$
has $(0, 0, 0)$ in its interior, but it does not satisfy Assumption 3.3. Hence, Assumption 3.3 is stronger than $0 \in \text{int}(C)$.

Lemma 3.4. Let Assumption 3.3 be satisfied. For all $x \in C$ and $n \in T_C(x)^\circ$, we have 
$$(n, x - w(x))_{\mathbb{R}^m} \geq c_2 |n|_{\mathbb{R}^m}.$$ 

Proof. Since $B_{c_2}(w(x)) \subset C$, we have 
$$(n, x - (w(x) + p))_{\mathbb{R}^m} \geq 0$$
for all $|p|_{\mathbb{R}^m} \leq c_2$. Hence, 
$$(n, x - w(x))_{\mathbb{R}^m} \geq \sup_{|p|_{\mathbb{R}^m} \leq c_2} (n, p)_{\mathbb{R}^m} = c_2 |n|_{\mathbb{R}^m}.$$ 

Let us consider a function $v \in A_2$. Then, the angle between $v(\omega)$ and $n \in T_C(u(\omega))$ can be a right angle. The next lemma shows, that we can approximate $v$ with functions $\tilde{v}$, for which the angle between $\tilde{v}(\omega)$ and $n \in T_C(u(\omega))$ is obtuse, uniformly in $\omega \in \Omega$.

Lemma 3.5. Let Assumption 3.3 be satisfied. Each function $v \in A_2$ can be approximated by functions from $A_3 := \{ v \in H^1_0(\Omega)^m \cap L^\infty(\Omega)^m : \exists \varepsilon > 0 : (v(\omega), n)_{\mathbb{R}^m} \leq -\varepsilon |n|_{\mathbb{R}^m} \text{ for all } n \in T_C(u(\omega))^\circ \text{ for q.a. } \omega \in \Omega \}$. That is, $A_2 \subset \text{cl}(A_3)$. 

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Proof. Let $v \in A_2$ be given. By definition, we have $(v(\omega), n)_{\mathbb{R}^m} \leq 0$ for all $n \in T_C(u(\omega))^o$ for q.a. $\omega \in \Omega$, since $v(\omega) \in T_C(u(\omega))$. Let $w : C \to C$ be the Lipschitz continuous mapping from Assumption 3.3. Then, $w(u) \in H^1_0(\Omega)^m$, see Lemma A.1, and $w(u) \in L^\infty(\Omega)^m$. For $\varepsilon > 0$, we set

$$v_\varepsilon := v - \varepsilon (u - w(u)).$$

Now, let $\omega \in \Omega$ and $n \in T_C(u(\omega))^o$ be given. By owing to Lemma 3.4, we have

$$(v_\varepsilon(\omega), n)_{\mathbb{R}^m} = (v(\omega) - \varepsilon (u - w(u)), n)_{\mathbb{R}^m} \leq -\varepsilon c_2 |n|_{\mathbb{R}^m}.$$ 

The assertion follows since $v_\varepsilon \to v$ in $H^1_0(\Omega)^m$ as $\varepsilon \to 0$ and $v_\varepsilon \in L^\infty(\Omega)^m$.

The following lemma is a replacement for a partition of unity in the setting of capacities. Under certain conditions, it allows to approximate a function $f$ with $f \leq 0$ on some set $M_0$ with functions $f_\delta$ satisfying $f_\delta \leq 0$ on some prescribed, larger sets $M_\delta$.

**Lemma 3.6.** Let $f \in H^1_0(\Omega)$ be given. We assume that the quasi-closed sets $M_\delta$, $\delta \geq 0$, are non-decreasing in $\delta$, in the sense that $M_\delta \subset M_{\delta'}$ for $0 < \delta < \delta'$. Moreover, we suppose that $f \leq 0$ q.e. on $M_0$ and $M_0 = \bigcap_{\delta > 0} M_\delta$.

(a) Then, for all $\delta > 0$ there exists $f_\delta \in H^1_0(\Omega)$, such that $f_\delta \leq 0$ q.e. on $M_\delta$, $f_\delta = 0$ q.e. on $M_\delta \cap \{\omega \in \Omega : f(\omega) = 0\}$ and $f_\delta \to f$ in $H^1_0(\Omega)$ as $\delta \to 0$. 

(b) In case $\|f\|_{L^\infty(\Omega)} < \infty$, the same can be achieved with $\|f_\delta\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$.

(c) If, additionally, $f \leq -\varepsilon$ on $M_0$ for some $\varepsilon > 0$ and if $\text{cap}(M_\delta) < \infty$ for some $\delta > 0$, then, we find $f_\delta \in H^1_0(\Omega)$ with $f_\delta \leq -\varepsilon$ q.e. on $M_\delta$ for $\delta$ small enough, such that $f_\delta \to f$ in $H^1_0(\Omega)$. In case $f \in L^\infty(\Omega)$, we can achieve $\|f_\delta\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} + 2 \varepsilon$.

Proof.

(a) The sets $\Omega \setminus M_\delta$ are quasi-open and $\{\Omega \setminus M_\delta\}_{\delta > 0}$ is a covering of $\Omega \setminus M_0$. Moreover, $\max(f, 0) = 0$ q.e. on $M_0$ implies $\max(f, 0) \in H^1_0(\Omega \setminus M_0)$, see [Kilpeläinen, Malý, 1992, Theorem 2.10]. Now, we can invoke [Kilpeläinen, Malý, 1992, Lemma 2.4 and Theorem 2.10], and find $g_\delta \in H^1_0(\Omega \setminus M_\delta)$, $g_\delta \geq 0$ and $g_\delta \to \max(f, 0)$ in $H^1_0(\Omega)$.

We define $f_\delta := f - (\max(f, 0) - g_\delta)$ and obtain $f_\delta \to f$ in $H^1_0(\Omega)$, $f_\delta = f + \min(-f, 0) + g_\delta = \min(0, f) + g_\delta \leq 0$ on $M_\delta$. Since $g_\delta = 0$ q.e. on $M_\delta$, this shows $f_\delta = 0$ q.e. on $M_\delta \cap \{\omega \in \Omega : f(\omega) = 0\}$.

(b) We use the same technique as in (a), and obtain additionally $\|g_\delta\|_{L^\infty(\Omega)} \leq \|\max(f, 0)\|_{L^\infty(\Omega)}$, see [Kilpeläinen, Malý, 1992, Lemma 2.4]. Hence, $\|f_\delta\|_{L^\infty(\Omega)} = \|\min(0, f) + g_\delta\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$.

(c) Let $\hat{\delta} > 0$ be given such that $\text{cap}(M_{\hat{\delta}}) < \infty$. By definition of the capacity, there exists a $h \in H^1_0(\Omega)$, $0 \leq h \leq 1$ q.e. on $\Omega$ and $h = 1$ q.e. on $M_{\hat{\delta}}$.

We have $f + \varepsilon h \leq 0$ on $M_0$. By owing to (a), we find $g_\delta \in H^1_0(\Omega)$ with $g_\delta \leq 0$ q.e. on $M_\delta$ and $g_\delta \to f + \varepsilon h$ in $H^1_0(\Omega)$. We set $f_\delta := g_\delta - \varepsilon h$ and obtain the first part of the assertion, since $h = 1$ on $M_{\hat{\delta}}$ for $\delta \leq \hat{\delta}$. By owing to (b), we have $\|g_\delta\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} + \varepsilon \|h\|_{L^\infty(\Omega)}$ and, hence, $\|f_\delta\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} + 2 \varepsilon \|h\|_{L^\infty(\Omega)}$.

Let us recall the well-known result that the normal cone mapping of the convex set $C$ is upper semicontinuous, since this will be important in the sequel.

**Lemma 3.7.** Let $\{x_i\}_{i \in \mathbb{N}} \subset C$ and $\{n_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^m$ be sequences, such that $x_i \to x$, $n_i \to n$ and $n_i \in T_C(x_i)^o$ for all $i \in \mathbb{N}$. Then, $n \in T_C(x)^o$. 


The next lemma is a preparation for the proof of Lemma 3.9. We recall that \( B_\gamma = \{ x \in \mathbb{R}^m : |x|_{\mathbb{R}^n} \leq \gamma \} \) for \( \gamma \geq 0 \).

**Lemma 3.8.** Suppose that Assumption 3.3 is satisfied. Let \( \gamma \geq 0 \) and \( n \in \mathbb{R}^m \) be given. For \( \lambda \geq 0 \) we define the set
\[
M_\lambda := \{ \omega \in \Omega : \exists \tilde{u} \in B_\lambda (u(\omega)) : n \in B_\gamma + \mathcal{T}_C(\tilde{u})^o \}.
\]
Then, the sets \( M_\lambda \) possess the following properties.

(a) The set \( M_\lambda \) is quasi-closed for all \( \lambda \geq 0 \).

(b) The sets \( M_\lambda \) are non-decreasing in \( \lambda \) and \( M_0 = \bigcap_{\lambda > 0} M_\lambda \).

(c) In case \( \gamma < |n|_{\mathbb{R}^n} \) and \( \lambda \leq c_2/2 \) we have \( \text{cap}(M_\lambda) \leq \infty \).

(d) The inequality \((n, u(\omega) - w(u(\omega)))_{\mathbb{R}^m} \geq c_2 |n|_{\mathbb{R}^n}/3\) is satisfied for all \( \omega \in M_\lambda \), \( \lambda \leq c_2/(2 L) \) and \( \gamma \leq c_2 |n|_{\mathbb{R}^n}/(6 (c_1 + 2 c_2)) \).

**Proof.**

(a) We have the chain of equivalences
\[
\omega \in M_\lambda \iff n \in B_\gamma + \mathcal{T}_C(u(\omega) + B_\lambda)^o \\
\iff u(\omega) \in [\mathcal{T}_C(\cdot)^o]^{-1} (n + B_\gamma + B_\lambda).
\]

Here, \( \mathcal{T}_C(u(\omega) + B_\lambda)^o \) denotes the image of the set \( u(\omega) + B_\lambda \) under the set-valued mapping \( \mathcal{T}_C(\cdot)^o \). Since \( u \) is quasi-continuous, it remains to show that the set \( [\mathcal{T}_C(\cdot)^o]^{-1} (n + B_\gamma + B_\lambda) \) is closed. Since \( B_\lambda \) is compact, we have to show that \( [\mathcal{T}_C(\cdot)^o]^{-1} (n + B_\gamma) \) is closed. To this end, let \( \{ \tilde{u}_i \} \subset C \) be a convergent sequence, such that \( n + p_i \in \mathcal{T}_C(\tilde{u}_i)^o \) for some \( p_i \in B_\gamma \). Since \( B_\gamma \) is compact, we have (up to a subsequence) \( n + p_i \to n + p \) for some \( p \in B_\gamma \). Now, Lemma 3.7 implies \( n + p \in \mathcal{T}_C(\lim_{i \to \infty} \tilde{u}_i)^o \), hence \( \lim_{i \to \infty} \tilde{u}_i \in [\mathcal{T}_C(\cdot)^o]^{-1} (n + B_\gamma) \). This shows the desired closedness.

(b) The first assertion is clear. Let \( \omega \in \bigcap_{\lambda > 0} M_\lambda \) be given. For \( \lambda > 0 \), there exist \( \tilde{u}_\lambda \in B_\lambda (u(\omega)) \) and \( p_\lambda \in B_\gamma \), such that \( n \in p_\lambda + \mathcal{T}_C(u(\lambda)) \). By compactness of \( B_\gamma \), there exists a sequence \( \{ \lambda_i \} \subset \mathbb{N} \) with \( \lambda_i \to 0 \) as \( i \to \infty \) and \( p_{\lambda_i} \to p \) for some \( p \in B_\gamma \). Hence, \( u_{\lambda_i} \to u(\omega) \). Using \( n - p_{\lambda_i} \in \mathcal{T}_C(u(\lambda_i))^o \) and Lemma 3.7, we infer \( n - p \in \mathcal{T}_C(u(\omega))^o \) and this yields \( \omega \in M_0 \).

(c) For \( \gamma < |n|_{\mathbb{R}^n} \) no element in \( n + B_\gamma \) is equal to zero. Moreover, Assumption 3.3 implies \( B_{c_2}(0) \subset C \). Hence, \( n \in B_\gamma + \mathcal{T}_C(\tilde{u})^o \). Together with \( \tilde{u} \in B_\lambda (u(\omega)) \), we find \( |u(\omega)|_{\mathbb{R}^n} \geq c_2 - \lambda \) for q.a. \( \omega \in M_\lambda \). Hence, \( |u(\omega)|_{\mathbb{R}^n} \geq c_2/2 \) for \( \lambda \leq c_2/2 \) for q.a. \( \omega \in M_\lambda \). Thus, \( \text{cap}(M_\lambda) < \infty \).

(d) Let \( \omega \in M_\lambda \) be given. Hence, there exist \( \tilde{u} \in B_\lambda (u(\omega)) \) such that \( n + p \in \mathcal{T}_C(\tilde{u})^o \) for some \( |p|_{\mathbb{R}^n} \leq \gamma \). This yields
\[
(n, u(\omega) - w(u(\omega)))_{\mathbb{R}^m} \geq (n + p, u(\omega) - w(u(\omega)))_{\mathbb{R}^m} - \gamma c_1 \\
\geq (n + p, \tilde{u} - w(\tilde{u}))_{\mathbb{R}^m} - (|n|_{\mathbb{R}^n} + \gamma) (L + 1) \lambda - \gamma c_1 \\
\geq c_2 |n + p|_{\mathbb{R}^m} - (|n|_{\mathbb{R}^n} + \gamma) (L + 1) \lambda - \gamma c_1 \\
\geq (c_2 - (L + 1) \lambda) |n|_{\mathbb{R}^n} - \gamma (c_1 + c_2 + (L + 1) \lambda),
\]
where we used Lemma 3.4. This yields the claim.

In the next lemma, we use the approximation result Lemma 3.6, in order to approximate \( v \in A_3 \) with a function \( \tilde{v} \), such that \( \tilde{v}(\omega) \) does belong to the tangent cone of \( C \) not only at \( u(\omega) \), but also at all points in the neighborhood of \( u(\omega) \).
Lemma 3.9. Let Assumption 3.3 be satisfied. Each function \( v \in A_3 \) can be approximated by functions from

\[
A_4 := \left\{ v \in H_0^1(\Omega)^m \cap L^\infty(\Omega)^m : \exists \lambda > 0 : (v(\omega), n)_{\mathbb{R}^m} \leq 0 \text{ for all } n \in T_C(\tilde{u})^o, \tilde{u} \in B_\lambda(u(\omega)) \cap C \text{ for q.a. } \omega \in \Omega \right\}
\]

That is, \( A_3 \subset \text{cl}(A_4) \).

Proof. Let \( v \in A_3 \) be given. W.l.o.g. assume \( \|v\|_{L^\infty(\Omega)} \leq 1 \). There exists \( \varepsilon > 0 \), such that \( (v(\omega), n)_{\mathbb{R}^m} \leq -\varepsilon \) for all \( n \in T_C(u(\omega))^o, |n|_{\mathbb{R}^m} = 1 \) for q.a. \( \omega \in \Omega \). Let \( \gamma > 0 \) be chosen such that \( \gamma \leq \frac{\varepsilon}{2(1 + \gamma)} \leq \varepsilon/2 \) and \( \gamma \leq \frac{c_2}{b c_1 + 12 c_2} \).

By compactness, we find a \( \gamma \)-net \( \{n_i\}_{i=1}^N \) of the unit sphere in \( \mathbb{R}^m \). That is, \( |n_i|_{\mathbb{R}^m} = 1 \) and for every \( n \in \mathbb{R}^m \) with \( |n|_{\mathbb{R}^m} = 1 \), there is \( i \in \{1, \ldots, N\} \) with \( |n - n_i|_{\mathbb{R}^m} \leq \gamma \).

We define the sets

\[
M_0^i := \{ \omega \in \Omega : \exists u \in B_\lambda(u(\omega)) : n_i \in B_{\gamma} + T_C(\tilde{u})^o \}.
\]

Let \( \omega \in M_0^i \) be given. Then, by definition of \( M_0^i \), there exists \( n \in T_C(u(\omega))^o \), such that \( |n - n_i|_{\mathbb{R}^m} \leq \gamma \).

Hence,

\[
(n_i, v(\omega))_{\mathbb{R}^m} \leq (n, v(\omega))_{\mathbb{R}^m} + \gamma \leq -\varepsilon + \gamma \leq -\frac{\varepsilon}{2} < 0
\]

for q.a. \( \omega \in M_0^i \). We set \( f_i(\omega) := (n_i, v(\omega))_{\mathbb{R}^m} \).

Now, we choose \( k \in \mathbb{N} \).

By Lemma 3.6, Lemma 3.8, there exist \( \lambda^k_i > 0 \) and \( \tilde{f}_i^k \in H_0^1(\Omega) \), such that \( \|f_i - \tilde{f}_i^k\|_{H_0^1(\Omega)} \leq 1/k \), \( \|\tilde{f}_i^k\|_{L^\infty(\Omega)} \leq 1 + 2(\varepsilon - \gamma) \) and \( \tilde{f}_i^k \leq -\varepsilon + \gamma \) on \( M_{\lambda^k}^i \).

We set \( \lambda^k := \min\{\lambda^1_k, \ldots, \lambda^N_k, c_2/(2L)\} \) and

\[
\tilde{v}_k^k(\omega) := v(\omega) - \frac{3}{c_2} \max_{i=1, \ldots, N} \left[ (f_i(\omega) - \tilde{f}_i^k(\omega), 0) \right].
\]

Then, for \( \omega \in M_{\lambda^k}^i \) we have

\[
(n_i, \tilde{v}_k^k(\omega))_{\mathbb{R}^m} = f_i(\omega) - \frac{3}{c_2} \max_{i=1, \ldots, N} \left[ (f_i(\omega) - \tilde{f}_i^k(\omega), 0) \right] \leq f_i(\omega) - f_i(\omega) - (f_i(\omega) - \tilde{f}_i^k(\omega)) \leq -\varepsilon + \gamma
\]

for q.a. \( \omega \in M_{\lambda^k}^i \).

Let us show that \( \tilde{v}_k \in A_4 \). To this end, let \( \omega \in \Omega, \tilde{u} \in B_\lambda(u(\omega)) \cap C \) and \( n \in T_C(\tilde{u})^o \) be given. By definition of \( \{n_i\} \), there exists \( i \in \{1, \ldots, N\} \) with \( |n - n_i|_{\mathbb{R}^m} \leq \gamma \). This shows \( \omega \in M_{\lambda^k}^i \). Hence,

\[
(n, \tilde{v}_k^k(\omega))_{\mathbb{R}^m} \leq (n_i, \tilde{v}_k^k(\omega))_{\mathbb{R}^m} + \gamma \|\tilde{v}_k^k\|_{L^\infty(\Omega)} \leq -\varepsilon + \gamma + (1 + 3 \frac{c_1}{c_2} + 2 (1 + \varepsilon - \gamma)) \leq -\varepsilon + \gamma (2 + 3 \frac{c_1}{c_2} 2 (1 + \varepsilon - \gamma)) \leq 0.
\]

This shows \( \tilde{v}_k \in A_4 \).

Let us estimate \( \|v - \tilde{v}_k\|_{H_0^1(\Omega)} \). We have for some generic constant \( c > 0 \), which may change from line to line,

\[
\|v - \tilde{v}_k\|_{H_0^1(\Omega)} \leq c \left( \|u - w(u)\|_{L^\infty(\Omega)} \sum_{i=1}^N \|f_i - \tilde{f}_i^k\|_{H_0^1(\Omega)} + \|u - w(u)\|_{H_0^1(\Omega)} \sum_{i=1}^N \|f_i - \tilde{f}_i^k\|_{L^\infty(\Omega)} \right)
\]

\[
\leq c \left( \frac{c_1}{k} + \|u - w(u)\|_{H_0^1(\Omega)} 2 (1 + \varepsilon - \gamma) \right) \leq \text{const}
\]
and for some \( p \in (1, 2) \) we choose \( q \in (2, \infty) \) such that \( 1/p = 1/2 + 1/q \) and \( \theta = 2/q \in (0, 1) \) and obtain
\[
\|v - \tilde{v}^k\|_{W^{0, p}(\Omega)} \leq c \left( \|u - w(u)\|_{L^\infty(\Omega)} \right)^N \sum_{i=1}^N\|f_i - \tilde{f}_i^k\|_{H^1_0(\Omega)} + \|u - w(u)\|_{W^{1, q}(\Omega)} \sum_{i=1}^N\|f_i - \tilde{f}_i^k\|_{L^q(\Omega)}
\leq c \left( \frac{c_1}{k} + \|u - w(u)\|_{H^1_0(\Omega)} \right)^N \sum_{i=1}^N\|f_i - \tilde{f}_i^k\|_{L^q(\Omega)} \|f_i - \tilde{f}_i^k\|_{L^q(\Omega)}
\leq c \frac{1}{k^\theta} \to 0.
\]
This shows that the sequence \( \tilde{v}^k \) converges weakly towards \( v \) in \( H^1_0(\Omega) \). By convexity of \( A_4 \) and owing to Mazur’s lemma, there is a sequence in \( A_4 \) converging strongly in \( H^1_0(\Omega) \) towards \( v \).

The next lemma will be used in the proof of Lemma 3.11, but it is also interesting for itself. It shows that a direction \( h \) which belongs to the tangent cone at all points in a neighborhood of \( x \in C \) actually belongs to \( R_C(x) \).

**Lemma 3.10.** Let \( x \in C \) and \( h \in \mathbb{R}^m \), \( |h|_{\mathbb{R}^m} \leq 1 \) be given, such that \( h \in \bigcap_{\varepsilon \in B(x) \cap C} T_C(\varepsilon) \) for some \( \varepsilon > 0 \). Then, \( x + \varepsilon h \in C \).

**Proof.** We set \( y = \text{Proj}_C(x + \varepsilon h) \). Then, \( |y - x|_{\mathbb{R}^m} \leq \varepsilon \). Hence, \( h \in T_C(y) \). By the properties of the projection, we have \((x + \varepsilon h) - y \in T_C(y)^\circ \) Hence,
\[
0 \geq (h, x + \varepsilon h - y)_{\mathbb{R}^m} = (h, x - y)_{\mathbb{R}^m} + \varepsilon \geq -\varepsilon + \varepsilon = 0.
\]
Hence, we have \((h, x - y)_{\mathbb{R}^m} = -\varepsilon \geq -|h|_{\mathbb{R}^m} |x - y|_{\mathbb{R}^m} \). This yields \( h = -(x - y)/\varepsilon \). Hence, \( x + \varepsilon h = y \in C \).

Using this lemma, we can show that directions in \( A_4 \) belong to the radial cone \( R_C(u) \).

**Lemma 3.11.** Let \( v \in A_4 \) be given. Then, there exists \( \delta > 0 \) such that \( u + \delta v \in C \). This shows \( A_4 \subset R_C(u) \).

**Proof.** Let \( v \in A_4 \) be given. There is \( \lambda > 0 \), such that \((v(\omega), n)_{\mathbb{R}^m} \leq 0 \) for all \( n \in T_C(\tilde{u}) \), \( \tilde{u} \in B_\lambda(v(\omega)) \cap C \) for q.a. \( \omega \in \Omega \). This gives \( v(\omega) \in T_C(\tilde{u}) \) for all \( \tilde{u} \in B_\lambda(v(\omega)) \cap C \) for q.a. \( \omega \in \Omega \). By applying the previous lemma, we obtain \( u(\omega) + \lambda v(\omega)/\|v\|_{L^\infty(\Omega)} \in C \) for q.a. \( \omega \in \Omega \). Hence, \( u + \lambda v/\|v\|_{L^\infty(\Omega)} \in C \), see Lemma 2.2.

The following theorem collects the previous lemmas and is the main result of this section.

**Theorem 3.12.** Suppose Assumption 3.3 is satisfied. Then,
\[
T_C(u) = \{v \in H^1_0(\Omega)^m : v(\omega) \in T_C(u(\omega)) \text{ q.e. in } \Omega\}.
\]

**Proof.** From the previous lemmas, we have
\[
T_C(u) \subset A_1 \subset \overline{\text{cl}}(A_2), \quad A_2 \subset \overline{\text{cl}}(A_3), \quad A_3 \subset \overline{\text{cl}}(A_4), \quad A_4 \subset R_C(u).
\]
Taking the closures yields
\[
T_C(u) \subset A_1 \subset \overline{\text{cl}}(A_2) \subset \overline{\text{cl}}(A_3) \subset \overline{\text{cl}}(A_4) \subset \overline{\text{cl}}(R_C(u)) = T_C(u),
\]
and the assertion follows.
4. Characterization of the Normal Cone

In order to characterize the normal cone of $\mathcal{C}$, we have to work with vector-valued measures, which act on $H_0^1(\Omega)^m \cap C_0(\Omega)^m$. For an introduction to vector-valued measures, we refer to [Diestel, Uhl, 1977]. Since the values of our measures will belong to $\mathbb{R}^m$, we can identify a vector-valued measure with a tuple of $m$ scalar-valued measures.

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a tuple of regular, signed Borel measures on $\Omega$, i.e., $\mu_i : B \to \mathbb{R}$, $i = 1, \ldots, m$, where $B$ denotes the Borel $\sigma$-algebra of $\Omega$. The variation $|\mu|$ is defined for $E \in B$ by

$$|\mu|(E) := \sum_{A \in \pi} |\mu(A)|_{\mathbb{R}^m},$$

where the supremum ranges over all finite partitions $\pi$ of $E$ into pairwise disjoint Borel sets. We have $|\mu|(\Omega) < \infty$, see [Rudin, 1987, Section 6.6], and the variation $|\mu|$ is again a regular Borel measure, see [Diestel, Uhl, 1977, Proposition I.1.9] and [Rudin, 1987, Theorem 2.18].

We typically identify $\mu$ with its completion, i.e., with the complete measure on the smallest $\sigma$-algebra containing $B$ and the $|\mu|$-null sets, cf. [Rudin, 1987, Theorem 1.36].

Similar to the polar decomposition of a complex measure, cf. [Rudin, 1987, Theorem 6.12], we can obtain a decomposition of $\mu$ over $|\mu|$. In particular, the Radon-Nikodym derivative

$$\mu' := \frac{d\mu}{d|\mu|}$$

satisfies $\mu' \in L^\infty(|\mu|)^m$ (in fact, we have $|\mu'(\omega)|_{\mathbb{R}^m} = 1$ everywhere after changing $\mu'$ on a $|\mu|$-null set) and

$$\mu(E) = \int_E \mu' d|\mu|$$

holds for all $E \in B$.

By the Riesz representation theorem, see [Rudin, 1987, Theorem 6.19], the dual of $C_0(\Omega)^m$ can be identified with $\mathcal{M}(\Omega)^m$ which is the space of regular Borel measures $\mu : B \to \mathbb{R}^m$ of bounded variation, i.e., $|\mu|(\Omega) < +\infty$, and the duality pairing is

$$\langle \mu, f \rangle_{\mathcal{M}(\Omega)^m, C_0(\Omega)^m} := \int_{\Omega} f \ d\mu := \int_{\Omega} (f, \mu')_{\mathbb{R}^m} d|\mu|.$$

Suppose we have a functional $\mu \in H^{-1}(\Omega)^m = (H_0^1(\Omega)^m)^*$ such that $\mu$ is bounded w.r.t. the supremum norm on $H_0^1(\Omega)^m \cap C_0(\Omega)^m$. Since $H_0^1(\Omega)^m \cap C_0(\Omega)$ is a dense subspace of $C_0(\Omega)$, see [Fukushima, Oshima, Takeda, 1994, p. 100], $\mu$ can be extended uniquely to $C_0(\Omega)^m$. By the Riesz representation theorem, this functional can be represented as a regular Borel measure $\mu$ with bounded variation. With a slight abuse of notation, we shall write $\mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m$. In particular, we have

$$\langle \mu, f \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \langle \mu, f \rangle_{\mathcal{M}(\Omega)^m, C_0(\Omega)^m} = \int_{\Omega} (f, \mu')_{\mathbb{R}^m} d|\mu|$$

for all $f \in H_0^1(\Omega)^m \cap C_0(\Omega)^m$.

It would be tempting to assume that $\langle \mu, f \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \int_{\Omega} (f, \mu')_{\mathbb{R}^m} d|\mu|$ holds for all $f \in H_0^1(\Omega)^m$. This is, in general, not true, since $f \in L^1(|\mu|)^m$ may fail to hold, see Appendix B. We will prove, however, that this formula holds if $\int_{\Omega} (f, \mu')_{\mathbb{R}^m} d|\mu|$ is interpreted as a suitable limit, see (4.4) below.

The following lemma is well-known for scalar, non-negative measures $\mu$, see, e.g., [Bonnans, Shapiro, 2000, Lemma 6.56]. We state an extension to signed, vector-valued measures which is due to [Gruen-Rehonne, 1977, Proposition 1]. For convenience of the reader, we give its proof.
Lemma 4.1. Let $\mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m$ be given. Then, $|\mu|$ does not charge Borel sets of zero capacity.

Moreover, all sets of zero capacity belong to the completion of Borel $\sigma$-algebra $\mathcal{B}$ (over $\Omega$) w.r.t. $|\mu|$ and are not charged by $|\mu|$. Hence, a property which holds q.e. (i.e., up to a set of zero capacity) holds $|\mu|$-a.e.

Proof. We follow the ideas of the proof of [Grun-Rehomme, 1977, Proposition 1].

We first consider the (scalar) case $m = 1$. By the Hahn decomposition theorem, see, e.g. [Rudin, 1987, Theorem 6.14], we can decompose $\Omega$ into disjoint Borel sets $\Omega^+$ and $\Omega^-$, such that $\mu$ is non-negative on subsets of $\Omega^+$ and non-positive on subsets of $\Omega^-$. Let $A \subset \Omega$ be a Borel set with zero capacity. Since $|\mu|(A) = |\mu|(A \cap \Omega^+) + |\mu|(A \cap \Omega^-)$ and $\text{cap}(A \cap \Omega^+) = \text{cap}(A \cap \Omega^-) = 0$, it is sufficient to consider the case $A \subset \Omega^+$ (otherwise, apply the following arguments to $A \cap \Omega^+$ and $A \cap \Omega^-$).

Let $\varepsilon > 0$ be arbitrary. Since $|\mu|$ is outer regular, we find an open set $O \subset \Omega$ with $A \subset O$ and $|\mu|(O) \leq |\mu|(A) + \varepsilon$. Since $\text{cap}(A) = 0$, we find a sequence $\{u_n\} \subset L^\infty(O) \cap H^1_0(O) \subset L^\infty(\Omega) \cap H^1_0(\Omega)$ with $0 \leq u_n \leq 1$ in $\Omega$, $\|u_n\|_{H^1_0(\Omega)} \to 0$, and $u_n = 1$ on a neighborhood of $A$, see [Heinonen, Kilpeläinen, Martio, 1993, Lemma 2.9]. Using Lemma 4.2 below, this yields

$$
\|u_n\|_{H^1_0(\Omega)} \|\mu\|_{H^{-1}(\Omega)} \geq \langle \mu, u_n \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_{\Omega} u_n \, d\mu = \int_{\Omega} u_n \, d\mu = \int_A u_n \, d\mu + \int_{O \setminus A} u_n \, d\mu \\
= \mu(A) - |\mu|(O \setminus A) = \mu(A) - |\mu|(O) + |\mu|(A) \geq \mu(A) - (|\mu|(A) + \varepsilon) + |\mu|(A) \\
\geq \mu(A) - \varepsilon.
$$

Since $\|u_n\|_{H^1_0(\Omega)} \to 0$, this yields $\varepsilon \geq \mu(A)$ for all $\varepsilon > 0$. Together with $A \subset \Omega^+$, we get $\mu(A) = 0$.

Now, let us consider the (vector-valued) case $m > 1$. Let $A \subset \Omega$ be a Borel set with zero capacity. By definition of $|\mu|$, we have

$$
|\mu|(A) = \sup_{\pi} \sum_{B \in \pi} |\mu(B)|_{\mathbb{R}^m},
$$

where the supremum ranges over all finite partitions $\pi$ of $A$ into pairwise disjoint Borel sets. By the definition of the Euclidean norm and the first part of the proof, we thus have

$$
|\mu|(A) = \sup_{\pi} \sum_{B \in \pi} |\mu(B)|_{\mathbb{R}^m} = \sup_{\pi} \left( \sum_{i=1}^m \mu_i(B) \right)^{1/2} = 0,
$$

since $\mu_i(B) = 0$ for all $B \subset A$ and $i \in \{1, \ldots, m\}$.

Finally, let $A \subset \Omega$ have capacity zero, but be otherwise arbitrary. By the outer regularity of the capacity, see [Attouch, Buttazzo, Michaille, 2006, Definition 5.8.1(b) and Proposition 5.8.3(a)], we find

$$
\text{cap}(A) = \inf_{O \text{ open}, A \subset O} \text{cap}(O).
$$

Hence, there exist open sets $O_n \subset \Omega$, $A \subset O_n$ with $\text{cap}(O_n) \leq 1/n$, $n \in \mathbb{N}$. Hence, $A \subset B := \bigcap_{n \in \mathbb{N}} O_n$. This readily yields $B \in \mathcal{B}$ and $\text{cap}(B) = 0$. Hence, $|\mu|(B) = 0$ by the first part of the proof. Thus, $A$ belongs to the completion of $\mathcal{B}$ w.r.t. $|\mu|$ and $|\mu|(A) = 0$.

We recall that (the quasi-continuous representative of) $f \in H^1_0(\Omega)$ is uniquely determined quasi-everywhere, hence, $|\mu|$-almost-everywhere, for every $\mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m$.

Now, we prove an integral formula for $\langle \mu, f \rangle_{H^{-1}(\Omega)^m, H^1_0(\Omega)^m}$ with $\mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m$ and $f \in H^1_0(\Omega)^m$. This yields $|\mu| = 0$ on a set of zero capacity.
The next lemma is a direct consequence of [Brézis, Browder, 1979, Theorem 1].

\[ H_0^1(\Omega)^m. \]

Therefore, we use the truncation \( T_M: \mathbb{R}^m \to \mathbb{R}^m \), defined for \( M > 0 \) via

\[
T_M(x) := \max\left(1, \frac{M}{|x|_{\mathbb{R}^m}}\right) x = \begin{cases} x & \text{if } |x|_{\mathbb{R}^m} \leq M, \\ \frac{x}{|x|_{\mathbb{R}^m}} & \text{if } |x|_{\mathbb{R}^m} > M. \end{cases}
\]

(4.2)

Note that \( T_M: \mathbb{R}^m \to \mathbb{R}^m \) is Lipschitz continuous with modulus 1. By Theorem A.2, we have \( T_M f \in H_0^1(\Omega)^m \) and \( T_M f \to f \) in \( H_0^1(\Omega)^m \) for all \( f \in H_0^1(\Omega) \).

The next lemma is a direct consequence of [Brézis, Browder, 1979, Theorem 1].

**Lemma 4.2.** Let \( \mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m \) be given. Then, any \( f \in H_0^1(\Omega)^m \cap L^\infty(\Omega)^m \) belongs to \( L^\infty(|\mu|)^m \) and

\[
\langle \mu, f \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \int_\Omega (f, \mu')_{\mathbb{R}^m} \, d|\mu|.
\]

(4.3)

Moreover, we have

\[
\langle \mu, f \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \lim_{M \to \infty} \int_\Omega (T_M f, \mu')_{\mathbb{R}^m} \, d|\mu|.
\]

(4.4)

for all \( f \in H_0^1(\Omega)^m \).

**Proof.** Formula (4.3) follows directly from [Brézis, Browder, 1979, Theorem 1] using \( f_i = d\mu_i/d|\mu| \) therein. It can also be proven directly by approximating \( f \in H_0^1(\Omega)^m \) with a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \) such that \( \|f_n\|_{L^\infty(\Omega)^m} \leq \|f\|_{L^\infty(\Omega)^m} \).

Now, let \( f \in H_0^1(\Omega)^m \) be given. The formula (4.4) follows from

\[
\langle \mu, f \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \lim_{M \to \infty} \langle \mu, T_M f \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \lim_{M \to \infty} \int_\Omega (T_M f, \mu')_{\mathbb{R}^m} \, d|\mu|,
\]

since \( T_M f \to f \) in \( H_0^1(\Omega)^m \) and \( T_M f \in H_0^1(\Omega)^m \cap L^\infty(\Omega)^m \) we can use (4.2) for \( T_M f \).

We emphasize that, in general, we cannot pass to the limit under the integral in (4.4). Although we have \( T_M f \to f \) pointwise \(|\mu|-\text{a.e.}, \) we cannot find an integrable function dominating \( T_M f \), since \( f \) may not belong to \( L^1(|\mu|) \), see Appendix B. We also refer to [Brézis, Browder, 1979, Example 2], which shows that (4.3) does not hold for all \( f \in H_0^1(\Omega)^m \).

With this preparation, we are able to compute the normal cone. A similar characterization can be found in [Grun-Rehomme, 1977, Théorème 3], but there the Lebesgue decomposition of the measure \( \mu \in T_C(u)^\circ \) is used.

**Theorem 4.3.** Suppose \( 0 \in \text{int}(C) \). For \( u \in C \) we have

\[
T_C(u)^\circ = \{ \mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m : \mu'(\omega) \in T_C(u(\omega))^\circ \text{ for } |\mu|-\text{a.a. } \omega \in \Omega \}.
\]

**Proof.** Since \( (C - u)^\circ = T_C(u)^\circ \), it is sufficient to show

\[
(C - u)^\circ = \{ \mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m : \mu'(\omega) \in T_C(u(\omega))^\circ \text{ for } |\mu|-\text{a.a. } \omega \in \Omega \}.
\]

Let \( \mu \in H^{-1}(\Omega)^m \cap \mathcal{M}(\Omega)^m \) be given such that \( \mu'(\omega) \in T_C(u(\omega))^\circ \) for \(|\mu|-\text{a.a. } \omega \in \Omega \). For \( v \in C \) we have by using (4.4)

\[
\langle \mu, v - u \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \lim_{M \to \infty} \int_\Omega (T_M(v - u), \mu')_{\mathbb{R}^m} \, d|\mu| \leq 0.
\]


Here, we employed \( \{ T_M(v(\omega) - u(\omega)), \mu'(\omega) \}_{\mathbb{R}^m} \leq 0 \) for \(|\mu|\)-a.a. \( \omega \in \Omega \) since \( v(\omega) - u(\omega) \in \mathcal{T}_C(u(\omega)) \), \( T_M(v(\omega) - u(\omega)) \in \mathcal{T}_C(u(\omega)) \) and \( \mu'(\omega) \in \mathcal{T}_C(u(\omega))^{\circ} \) for \(|\mu|\)-a.a. \( \omega \in \Omega \). This shows \( \mu \in (C - u)^\circ \).

“\( \subset \)”: Let \( \mu \in (C - u)^\circ \) be given. Now, let \( h \in H^1_0(\Omega) \) with \( \|h\|_{L^\infty(\Omega)} \leq 1 \) be given. Since \( B_r(0) \subset C \) for some \( r > 0 \), we have \( \pm r \in C \). Now,

\[
\langle \mu, \pm r h - u \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega) \leq 0
\]

implies

\[
|\langle \mu, h \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega)| \leq \frac{1}{r} \langle \mu, u \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega).
\]

This shows

\[
|\langle \mu, h \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega)| \leq \frac{\|h\|_{L^\infty(\Omega)}}{r} \langle \mu, u \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega)
\]

for all \( h \in H^1_0(\Omega) \) by the bipolar theorem, we also obtain a characterization of the tangent cone.

Now, let \( h \in C^\infty_0(\Omega) \), \( 0 \leq h \leq 1 \), \( M \in \mathbb{N} \) and \( v \in C \) be arbitrary. Then, \( h T_M(v - u) + u \in C \) and by Lemma 4.2 this yields

\[
0 \geq \langle \mu, h T_M(v - u) + u - u \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega) = \langle \mu, h T_M(v - u) \rangle_{H^{-1}(\Omega)'} - H^1_0(\Omega) = \int_\Omega h (T_M(v - u), \mu')_{\mathbb{R}^m} d|\mu| \quad \forall h \in C^\infty_0(\Omega), 0 \leq h \leq 1.
\]

Using a mollification argument and \( (T_M(v - u), \mu')_{\mathbb{R}^m} \in L^\infty(|\mu|) \), we get

\[
0 \geq \int_\Omega h (T_M(v - u), \mu')_{\mathbb{R}^m} d|\mu| \quad \forall h \in C_c(\Omega), 0 \leq h \leq 1.
\]

Here, \( C_c(\Omega) \) is the space of continuous functions whose support is compact in \( \Omega \). By scaling, we can drop the upper bound \( h \leq 1 \). Owing to [Rudin, 1987, Theorem 3.14], \( C_c(\Omega) \) is dense in \( L^2(|\mu|) \), thus

\[
0 \geq \int_\Omega h (T_M(v - u), \mu')_{\mathbb{R}^m} d|\mu| \quad \forall h \in L^2(|\mu|), 0 \leq h.
\]

This yields \( (T_M(v - u(\omega)), \mu'(\omega))_{\mathbb{R}^m} \leq 0 \) for \(|\mu|\)-a.a. \( \omega \in \Omega \). Since \( M \in \mathbb{N} \) was arbitrary, this yields \( (v - u(\omega), \mu'(\omega))_{\mathbb{R}^m} \leq 0 \) for \(|\mu|\)-a.a. \( \omega \in \Omega \). Since \( v \in C \) was arbitrary, this yields \( \mu'(\omega) \in \mathcal{T}_C(u(\omega))^{\circ} \) for \(|\mu|\)-a.a. \( \omega \in \Omega \).

Note that the above theorem does not use any result from Section 3.

By the bipolar theorem, we also obtain a characterization of the tangent cone.

**Theorem 4.4.** Suppose that \( 0 \in \text{int}(C) \) and let \( u \in C \) be given. Then,

\[
\mathcal{T}_C(u) = \{ v \in H^1_0(\Omega) : v(\omega) \in \mathcal{T}_C(u(\omega)) \text{ q.e. in } \Omega \}.
\]

**Proof.** From Lemma 3.1 we obtain

\[
\mathcal{T}_C(u) \subset \{ v \in H^1_0(\Omega) : v(\omega) \in \mathcal{T}_C(u(\omega)) \text{ q.e. in } \Omega \}.
\]

From Theorem 4.3 and the bipolar theorem, we infer

\[
\mathcal{T}_C(u) = \{ \mu \in H^{-1}(\Omega) \cap \mathcal{M}(\Omega) : \mu'(\omega) \in \mathcal{T}_C(u(\omega))^{\circ} \text{ for } |\mu|\text{-a.a. } \omega \in \Omega \}^{\circ}.
\]
Now, a direct calculation yields
\[
\{ v \in H_0^1(\Omega)^m : v(\omega) \in T_C(u(\omega)) \text{ q.e. in } \Omega \} 
\subset \{ \mu \in H^{-1}(\Omega)^m \cap M(\Omega)^m : \mu'(\omega) \in T_C(u(\omega))^\circ \text{ for } |\mu|\text{-a.a. } \omega \in \Omega \}^\circ
\]
and this shows the claim.

The following remark compares the techniques and results from Section 3 and Section 4.

**Remark 4.5.** In Section 3 as well as in Section 4, we obtained a characterization of the tangent cone of \( C \), and both sections used rather different techniques. Whereas in Section 3 we directly approximated \( v \in T_C(u) \) by feasible directions (which was rather subtle), we used the bipolar theorem in Section 4, which required to work with vector valued measures in \( H^{-1}(\Omega)^m \). To this end, the novel representation result (4.4) was crucial.

In the next section, we use similar arguments as in Section 3 to show the polyhedricity of \( C \) (under additional assumptions). Since elements in \( (T_C(u) \cap \mu^\perp)^\circ \) might not be measures, the technique of Section 4 cannot be used.

Finally, we highlight that the technique in Section 3 requires Assumption 3.3 to be satisfied, whereas in Section 4 the weaker assumption \( 0 \in \text{int}(C) \) is sufficient. It is not clear whether the arguments of Section 3 can be adapted if Assumption 3.3 fails, but we still have \( 0 \in \text{int}(C) \).

5. Polyhedricity under LICQ

In this section, we consider the polyhedricity of \( C \). Recall, that \( C \) is called polyhedric w.r.t. \((u, \mu)\), where \( u \in C, \mu \in T_C(u)^\circ \), if
\[
R_C(u) \cap \mu^\perp \text{ is dense in } T_C(u) \cap \mu^\perp.
\]

To this end, we put an additional assumption on \( C \subset \mathbb{R}^m \).

**Assumption 5.1.** There exist \( N \in \mathbb{N} \) and \( n_i \in \mathbb{R}^m, b_i \in (0, \infty) \) for \( i = 1, \ldots, N \), such that
\[
C = \{ x \in \mathbb{R}^m : (x, n_i)_{\mathbb{R}^m} \leq b_i \forall i = 1, \ldots, N \}.
\]

Further, we assume that the linear independence constraint qualification (LICQ) is satisfied, that is, the family \( \{ n_i : (x, n_i)_{\mathbb{R}^m} = b_i \} \) is linearly independent for all \( x \in C \).

Note that Assumption 5.1 implies \( 0 \in \text{int}(C) \), since we required \( b_i > 0 \).

Throughout this section, we fix \( u \in C \) and \( \mu \in T_C(u)^\circ \).

First, we provide a characterization of \( T_C(u) \cap \mu^\perp \) which does not require Assumption 5.1 to be satisfied.

**Lemma 5.2.** We have \( v \in T_C(u) \cap \mu^\perp \) if and only if
\[
v(\omega) \in T_C(u(\omega)) \text{ for q.a. } \omega \in \Omega \quad \text{and} \quad (v(\omega), \mu'(\omega))_{\mathbb{R}^m} = 0 \text{ for } |\mu|\text{-a.a. } \omega \in \Omega.
\]

Here, \(|\mu|\) is the total variation of \( \mu \) and \( \mu' \) the Radon-Nikodým derivative of \( \mu \) w.r.t. \(|\mu|\), see Section 4.
Finally, for q.a. see Lemma 5.2. the claim follows from Theorem A.2. By definition, we have

\[ g(\omega) = 0 \text{ for } |\mu|\text{-a.a. } \omega \in \Omega. \]

Proof. From Theorem 4.4, we obtain

\[ v \in T_C(u) \iff v(\omega) \in T_C(u(\omega)) \text{ for q.a. } \omega \in \Omega. \]

It remains to show the equivalence

\[ v \in \mu^\perp \iff (v(\omega), \mu'(\omega))_{R^m} = 0 \text{ for } |\mu|\text{-a.a. } \omega \in \Omega. \]

for all \( v \in T_C(u) \).

To this end, let \( v \in T_C(u) \) be given. Due to

\[ v(\omega) \in T_C(u(\omega)) \text{ and } \mu'(\omega) \in T_C(u(\omega))^{\circ} \text{ for } |\mu|\text{-a.a. } \omega \in \Omega, \]

see Lemma 4.1 and Theorem 4.3, we find

\[ (v(\omega), \mu'(\omega))_{R^m} \leq 0 \text{ for } |\mu|\text{-a.a. } \omega \in \Omega. \]

For \( M > 0 \), the truncation \( T_M v \), see (4.2) for the definition of \( T_M \), is given by

\[ T_M v(\omega) = \min\left(1, \frac{M}{|v(\omega)|_{R^m}}\right) v(\omega). \]

The factor \( \min(1, \frac{M}{|v|_{R^m}}) \) is monotonically increasing in \( M \) and converges to 1 pointwise as \( M \to \infty \).

Hence, the function \( g : (0, \infty) \to \mathbb{R} \), defined via

\[ g(M) := \langle \mu, T_M v \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = \int_{\Omega} \min\left(1, \frac{M}{|v|_{R^m}}\right) (v, \mu')_{R^m} d|\mu|, \]

is monotonically decreasing, takes non-positive values and \( g(M) \) converges towards \( \langle \mu, v \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} \) as \( M \to \infty \) by Lemma 4.2.

Hence, we get the chain of equivalences

\[ v \in \mu^\perp \iff \lim_{M \to \infty} g(M) = 0 \iff \forall M > 0 : g(M) = 0 \]

\[ \iff (v(\omega), \mu'(\omega))_{R^m} = 0 \text{ for } |\mu|\text{-a.a. } \omega \in \Omega. \]

Note that the last “\( \iff \)” follows from \( \min(1, \frac{M}{|v|_{R^m}}) > 0 \) for q.a. \( \omega \in \Omega \) and hence for \( |\mu|\text{-a.a. } \omega \in \Omega. \)

As in Section 3, we start with a truncation argument.

Lemma 5.3. Each function \( v \in T_C(u) \cap \mu^\perp \) can be approximated in \( H_0^1(\Omega)^m \) from functions in

\[ T_C(u) \cap \mu^\perp \cap L^\infty(\Omega)^m. \]

Proof. Let \( v \in T_C(u) \cap \mu^\perp \) be given. For \( M > 0 \), we show that \( T_M v \in T_C(u) \cap \mu^\perp \cap L^\infty(\Omega)^m \). Then, the claim follows from Theorem A.2. By definition, we have \( T_M v \in L^\infty(\Omega)^m \).

Since \( v \in T_C(u) \), we have \( v(\omega) \in T_C(u(\omega)) \) for q.a. \( \omega \in \Omega \). This yields \( (T_M v)(\omega) = T_M(v(\omega)) \in T_C(u(\omega)) \) for q.a. \( \omega \in \Omega \) since \( T_C(u(\omega)) \) is a cone. By Theorem 4.4 we find \( T_M v \in T_C(u) \).

Finally, \( \langle \mu, T_M v \rangle_{H^{-1}(\Omega)^m, H_0^1(\Omega)^m} = 0 \) follows from

\[ (v(\omega), \mu'(\omega))_{R^m} = 0 \text{ for } |\mu|\text{-a.a. } \omega \in \Omega, \]

see Lemma 5.2.
Now, by using Assumption 5.1, we can show that
\[ \mathcal{R}_C(u) \cap \mu \} \) is dense in \( \mathcal{T}_C(u) \cap \mu \} \) \( L_\infty(\Omega)^m. \)

**Lemma 5.4.** Assume that Assumption 5.1 is satisfied. Let \( v \in \mathcal{T}_C(u) \cap \mu \} \) \( L_\infty(\Omega)^m \) be given. Then, \( v \) can be approximated in \( H_0^1(\Omega)^m \) by functions in \( \mathcal{R}_C(u) \cap \mu \} \).

**Proof.** W.l.o.g. we assume \( |n_i|_{\mathbb{R}^m} = 1 \) for all \( i \in \{1, \ldots, N\} \). By \( \tilde{h}_i : C \rightarrow \mathbb{R}^m \) we denote the Lipschitz continuous functions from Lemma C.1.

In order to apply Lemma 3.6, we define for \( i \in \{1, \ldots, N\} \) and \( \delta \geq 0 \) the sets
\[ M^i_\delta := \{ \omega \in \Omega : (u(\omega), n_i)_{\mathbb{R}^m} = b_i - \delta \} \]
and \( f^i := (v, n_i)_{\mathbb{R}^m} \in H_0^1(\Omega) \cap L_\infty(\Omega) \). Then, the assumptions of Lemma 3.6 are satisfied and we find \( f^i_\delta \in H_0^1(\Omega) \cap L_\infty(\Omega) \) with \( f^i_\delta \rightarrow f^i \) in \( H_0^1(\Omega) \) as \( \delta \searrow 0 \),
\[ f^i_\delta \leq 0 \text{ q.e. on } M^i_\delta \quad \text{and} \quad f^i_\delta = 0 \text{ q.e. on } M^i_\delta \cap \{ \omega \in \Omega : f^i(\omega) = 0 \} \].

Moreover, the \( L_\infty(\Omega) \)-norm of \( f^i_\delta \) is bounded by the \( L_\infty(\Omega)^m \)-norm of \( v \).

Now, we define
\[ v_\delta := v + \sum_{i=1}^N (f^i_\delta - f^i) \tilde{h}_i(u) = v + \sum_{i=1}^N (f^i_\delta - (v, n_i)_{\mathbb{R}^m}) h_i(u) \]
Since \( \tilde{h}_i \) is globally bounded and Lipschitz continuous, we get \( v_\delta \in H_0^1(\Omega)^m \cap L_\infty(\Omega)^m \), see also Lemma A.1.

We proceed by showing the weak convergence of \( v_\delta \) towards \( v \) in \( H_0^1(\Omega)^m \). Since we have \( f^i_\delta \rightarrow f^i \) in \( H_0^1(\Omega) \) as \( \delta \searrow 0 \), we find \( v_\delta \rightarrow v \) in \( W_0^{1,1}(\Omega)^m \). Further, we can show that \( v_\delta \) is bounded in \( H_0^1(\Omega)^m \), and this yields \( v_\delta \rightarrow v \) in \( H_0^1(\Omega)^m \).

As a next step, we show \( v_\delta \in \mathcal{R}_C(u) \cap \mu \} \).

We denote by \( M, \delta \) the constants from Lemma C.3 and by \( L \) the largest Lipschitz constant of the functions \( \tilde{h}_i \). From now on, we assume \( \delta \leq \delta \).

Then, we set \( \lambda := \min \{ (2NLM \| u \|_{L_\infty(\Omega)^m})^{-1}, \delta \| v \|_{L_\infty(\Omega)^m}^{-1} \} \) and show \( u + \lambda v_\delta \in \mathcal{C} \). By owing to Lemma C.3, we find that for \( i \in \{1, \ldots, N\} \) and for q.a. \( \omega \in M^i_\delta \), there exists \( \tilde{x} \in C \), such that
\[ (\tilde{x}, n_i)_{\mathbb{R}^m} = b_i \quad \text{and} \quad |u(\omega) - \tilde{x}|_{\mathbb{R}^m} \leq M(b_i - (u(\omega), n_i)_{\mathbb{R}^m}). \]

In particular, we have \( (h_j(\tilde{x}, n_i)_{\mathbb{R}^m} = \delta_{ij} \). Hence, we have for \( i \in \{1, \ldots, N\} \) and for q.a. \( \omega \in M^i_\delta \) the estimate
\[ (u(\omega) + \lambda v_\delta(\omega), n_i)_{\mathbb{R}^m} = (u(\omega) + \lambda v(\omega), n_i)_{\mathbb{R}^m} + \lambda \sum_{j=1}^N (f^j_\delta(\omega) - f^j(\omega)) (h_j(u(\omega)), n_i)_{\mathbb{R}^m} \]
\[ \leq (u(\omega) + \lambda v(\omega), n_i)_{\mathbb{R}^m} + \lambda \sum_{j=1}^N (f^j_\delta(\omega) - f^j(\omega)) (h_j(\tilde{x}), n_i)_{\mathbb{R}^m} \]
\[ + \lambda N 2 \| v \|_{L_\infty(\Omega)^m} L M (b_i - (u(\omega), n_i)_{\mathbb{R}^m}) \]
\[ \leq (u(\omega) + \lambda v(\omega), n_i)_{\mathbb{R}^m} + \lambda (f^i_\delta(\omega) - f^i(\omega)) + (b_i - (u(\omega), n_i)_{\mathbb{R}^m}) \]
\[ = \lambda (v(\omega), n_i)_{\mathbb{R}^m} + \lambda (f^i_\delta(\omega) - (v(\omega), n_i)_{\mathbb{R}^m}) + b_i \leq b_i. \]

On the other hand, for \( i \in \{1, \ldots, N\} \) and for q.a. \( \omega \notin M^i_\delta \), we have
\[ (u(\omega) + \lambda v_\delta(\omega), n_i)_{\mathbb{R}^m} = (u(\omega), n_i)_{\mathbb{R}^m} + \lambda \| v_\delta \|_{L_\infty(\Omega)^m} < b_i - \delta + \delta = b_i. \]
This shows $u + \lambda v_\delta \in \mathcal{C}$. 

Now, we show $\langle \mu, v_\delta \rangle_{H^{-1}(\Omega)\cap H_0^1(\Omega)^m} = 0$. By Theorem 4.3, we have $\mu'(\omega) \in T_C(u(\omega))$ for $|\mu|$-a.a. $\omega \in \Omega$. By $v \in \mathcal{T}_C(u)$, Theorem 4.4 and Lemma 4.2, we have $\langle v(\omega), \mu'(\omega) \rangle_{\mathbb{R}^m} = 0$ for $|\mu|$-a.a. $\omega \in \Omega$. Moreover, there exist functions $\alpha_i : \Omega \to [0, \infty)$, such that $\alpha_i(\omega) = 0$ if $u(\omega), n_i \in \mathbb{R}^m < b_i$ and $\mu'(\omega) = \sum_{i=1}^N \alpha_i(\omega) n_i$ holds for $|\mu|$-a.a. $\omega \in \Omega$. We do not claim that the functions $\alpha_i$ are measurable. Hence, we have

$$0 = \langle v(\omega), \mu'(\omega) \rangle_{\mathbb{R}^m} = \sum_{i=1}^N \alpha_i(\omega) \langle v(\omega), n_i \rangle_{\mathbb{R}^m} = \sum_{i=1}^N \alpha_i(\omega) f^i(\omega). \quad (5.1)$$

Note that $\alpha_i = 0$ on $\Omega \setminus M_i^j$ and $f^i \geq 0$ on $M_i^j$. Thus, all summands in (5.1) are non-negative and we have for $|\mu|$-a.a. $\omega \in \Omega$

$$\alpha_i(\omega) \neq 0 \quad \Rightarrow \quad \omega \in M_i^j \text{ and } f^i(\omega) = 0 \quad \Rightarrow \quad f^j_\delta(\omega) = 0.$$ 

Moreover, $\omega \in M_i^j$ if and only if $(u(\omega), n_i)_{\mathbb{R}^m} = b_i$, which implies $(h_j(u(\omega)), n_i)_{\mathbb{R}^m} = \delta_{ij}$ for all $j = 1, \ldots, N$. This shows

$$\langle v_\delta(\omega), \mu'(\omega) \rangle_{\mathbb{R}^m} = \sum_{i=1}^N \alpha_i(\omega) \langle v_\delta(\omega), n_i \rangle_{\mathbb{R}^m} = \sum_{i=1}^N \alpha_i(\omega) \sum_{j=1}^N (f^j_\delta - f^i) (h_j(u(\omega)), n_i)_{\mathbb{R}^m}$$

$$= \sum_{i=1}^N \alpha_i(\omega) (f^j_\delta - f^i) = 0$$

for $|\mu|$-a.a. $\omega \in \Omega$. By using Lemma 4.2, we get

$$\langle \mu, v_\delta \rangle_{H^{-1}(\Omega)^m, H^m_0(\Omega)^m} = \int_\Omega \langle v_\delta, \mu' \rangle_{\mathbb{R}^m} \, d|\mu| = 0.$$ 

Hence, we have $v_\delta \in \mathcal{R}_C(u) \cap \mu^\perp$. 

It remains to show that $v$ can be approximated strongly in $H^1_0(\Omega)^m$ by functions in $\mathcal{R}_C(u) \cap \mu^\perp$. The set $\mathcal{R}_C(u) \cap \mu^\perp$ is convex and $v_\delta \rightharpoonup v$ in $H^1_0(\Omega)^m$. Owing to Mazur’s lemma, there is a sequence in $\mathcal{R}_C(u) \cap \mu^\perp$ which converges strongly in $H^1_0(\Omega)$ towards $v$. 

By combining Lemma 5.3 and Lemma 5.4, we obtain the main result of this section.

**Theorem 5.5.** Assumption 5.1 implies that $C$ is polyhedric w.r.t. all $u \in C$ and $\mu \in T_C(u)^\circ$.

It is expected, that $C$ is polyhedric, if $C \subset \mathbb{R}^m$ is a polyhedral set (i.e., a finite intersection of closed half-spaces), but we were not able to prove the result in this generality. In particular, the violation of LICQ impedes the existence of the functions $h_i$ and these functions are crucial in the proof of Lemma 5.4.

### 6. Optimal Control of a String in a Polyhedral Tube

In this section, we consider the optimal control of a string whose deflection is constrained by a polyhedral tube. We apply the results of the previous sections and show that a local minimizer satisfies a system of strong stationarity. For simplicity, we consider the two-dimensional situation $m = 2$.

For a given length $L > 0$, we set $\Omega = (0, L)$. Due to $\Omega \subset \mathbb{R}^1$, we have $\text{cap}(A) > 0$ for all $A \subset \Omega$, $A \neq \emptyset$. Hence, the notion of “quasi-everywhere” is equivalent to “everywhere”.

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**Pointwise Constraints in Sobolev Spaces (November 4, 2016)**

G. Wachsmuth
The deflection of the string is modelled by \( y \in H^1_0(\Omega)^2 \). If we apply a force \( u \in L^2(\Omega)^2 \), the unconstrained deflection \( y \) would satisfy the differential equation
\[
-\Delta y = u \quad \text{in} \; \Omega,
\]
\[
y = 0 \quad \text{in} \; \{0, L\}.
\]
Now, we choose a polygon \( C \subset \mathbb{R}^2 \) satisfying \( 0 \in \text{int}(C) \) and consider a string which is constrained to the tube \( \Omega \times C \). Then, the deflection of the constrained string is given by the solution of the energy minimization problem
\[
\text{Minimize} \; \int_{\Omega} \frac{1}{2} \|\nabla y\|_{\mathbb{R}^2}^2 - (y, u)_{\mathbb{R}^2} \, d\omega
\]
such that \( y(\omega) \in C \) for a.a. \( \omega \in \Omega \).

The unique solution \( y = S(u) \) of this problem is characterized by the (necessary and sufficient) optimality conditions
\[
y \in C, \quad \xi \in T_C(y)^o, \quad Ay - u + \xi = 0,
\]
where \( A : H^1_0(\Omega)^2 \to H^{-1}(\Omega)^2 \) is given by
\[
(Ay, v)_{H^{-1}(\Omega)^2, H^1_0(\Omega)^2} := \int_{\Omega} (\nabla y, \nabla v)_{\mathbb{R}^2} \, d\omega.
\]

For a given objective \( f : H^1_0(\Omega)^2 \times L^2(\Omega)^2 \to \mathbb{R} \), which is assumed to be Fréchet differentiable, we consider the optimal control problem
\[
\text{Minimize} \; f(y, u) \\
\text{with respect to} \; y \in H^1_0(\Omega)^2, u \in L^2(\Omega)^2, \xi \in H^{-1}(\Omega)^2
\]
such that
\[
y \in C \quad \xi \in T_C(y)^o \quad Ay - u + \xi = 0.
\]

Using additional assumptions on the objective \( f \), we may deduce the existence of global solutions by standard arguments, but this will not be discussed here.

The main result of this section is the following optimality system.

**Theorem 6.1.** Let \( (\bar{y}, \bar{u}, \bar{\xi}) \) be a locally optimal solution of (6.1). Then, there exist \( p \in H^1_0(\Omega)^2 \) and \( \mu \in H^{-1}(\Omega)^2 \), such that the system
\[
f_y(\bar{y}, \bar{u}) + \mu + A^*p = 0, \quad -p \in T_C(\bar{y}) \cap \bar{\xi},
\]
\[
f_u(\bar{y}, \bar{u}) - p = 0, \quad \mu \in (T_C(\bar{y}) \cap \bar{\xi})^o
\]
is satisfied. Here, \( f_y, f_u \) denote the partial derivatives of \( f \), w.r.t. \( y \) and \( u \), respectively.

**Proof.** First, we remark that every polygon in \( \mathbb{R}^2 \) can be described by affine constraints which satisfy LICQ. Hence, Assumption 5.1 is satisfied and we can apply Theorem 5.5 to obtain the polyhedricity of \( C \).

Now, we can argue as in the proof of [Mignot, 1976, Proposition 4.1] and get the existence of \( p \in H^1_0(\Omega)^2 \) and \( \mu \in H^{-1}(\Omega)^2 \), such that the system (6.2) is satisfied. We also refer to [Hintermüller, Surowiec, 2011, Theorem 4.6] and [Wachsmuth, 2015, Section 6.1] for different approaches to obtain this optimality system.
Using the nomenclature from finite dimensions, the system (6.2) is of strongly stationary type. For the extension of the finite-dimensional nomenclature to infinite dimensions, we refer to [Wachsmuth, 2015, Section 5.4].

For a characterization of the critical cone $\mathcal{T}_C(\bar{y}) \cap \xi^L$, which appears in (6.2), we refer to Lemma 5.2. Recall that we may replace “quasi-everywhere” in Lemma 5.2 by “everywhere” due to $\Omega \subset \mathbb{R}^1$. However, the polyhedricity of $\mathcal{C}$ which is established via Theorem 5.5 is a delicate issue even in dimension one.

Finally, we give a higher-dimensional application by means of a simplified phase-field model. The domain which is occupied by the $m+1$ phases is $\Omega \subset \mathbb{R}^d$. The concentration of phase $i$ is denoted by $y_i$ and, thus, we have $y_i \geq 0$ and $\sum_{i=1}^{m+1} y_i = 1$. Further, we assume that we have the Dirichlet boundary condition $y_i = 1/(m+1)$ on $\partial \Omega$ for all $i$. Now, we eliminate the phase $m+1$ via $y_{m+1} = 1 - \sum_{i=1}^{m} y_i$, and replace each phase $y_i$ by $y_i - 1/(m+1)$. Then, the phase-field vector $(y_1, \ldots, y_m)$ belongs to the simplex

$$C := \left\{ x \in \mathbb{R}^m : x \geq -\frac{1}{m+1} \text{ and } \sum_{i=1}^{m} x_i \geq \frac{1}{m+1} \right\}$$

and satisfies the homogeneous Dirichlet boundary condition $y = 0$ on $\partial \Omega$. It is easy to check that this set $C$ satisfies Assumption 5.1 and thus we can proceed as above to obtain a higher-dimensional application of Theorem 5.5.

A. Nemytskii Operators on Sobolev Spaces

First, we provide a result that Nemytskii operators associated with globally Lipschitz continuous functions $f : \mathbb{R}^m \to \mathbb{R}^n$, map $H^1_0(\Omega)^m$ to $H^1_0(\Omega)^n$. A more general result is provided in [Marcus, Mizel, 1973, Theorem 2.1] in case $\Omega$ satisfies the cone condition. Since we are only interested in the case of functions with zero trace, we can drop the assumption on $\Omega$.

**Lemma A.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We assume that the function $f : \mathbb{R}^m \to \mathbb{R}^n$ is globally Lipschitz continuous and $f(0) = 0$. Then, the associated Nemytskii operator $T$, which is defined for functions $u : \Omega \to \mathbb{R}^m$ via

$$(Tu)(\omega) = f(u(\omega)),$$

maps $H^1_0(\Omega)^m$ to $H^1_0(\Omega)^n$ and there is a constant $c > 0$, such that $\|Tu\|_{H^1_0(\Omega)^n} \leq c \|u\|_{H^1_0(\Omega)^m}$ holds for all $u \in H^1_0(\Omega)^m$.

**Proof.** Since $\Omega$ is bounded, there is $R > 0$, such that $U_R(0) := \{ \omega \in \mathbb{R}^d : |\omega|_{\mathbb{R}^d} < R \}$ contains $\bar{\Omega}$. We identify $u$ with its extension by zero and get $u \in H^1_0(U_R(0))^m$. Since $B_R(0)$ satisfies the cone condition, we can invoke [Marcus, Mizel, 1973, Theorem 2.1], and obtain $Tu \in H^1(B_R(0))^n$. Following the proof, we also find the bound $\|Tu\|_{H^1(B_R(0))^n} \leq c \|u\|_{H^1(B_R(0))^m}$.

It remains to show $(Tu)_{|\partial} \in H^1_0(\Omega)^n$. Since $u \in H^1_0(\Omega)^m$, its extension by zero belongs to $H^1(\mathbb{R}^d)^m$ and $u = 0$ q.e. on $\mathbb{R}^d \setminus \Omega$, see [Heinonen, Kilpeläinen, Martio, 1993, Theorem 4.5]. By using $f(0) = 0$, this shows $Tu = 0$ q.e. on $\mathbb{R}^d \setminus \Omega$ and thus, by using [Heinonen, Kilpeläinen, Martio, 1993, Theorem 4.5] again, we have $Tu \in H^1_0(\Omega)^m$.

Finally, we provide a chain rule for the truncation of vector-valued Sobolev functions. Similar to the classical argument in the scalar-valued case, we use a smooth approximation of the truncation and pass to the limit by using the dominated convergence theorem.
Theorem A.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. For $M > 0$, we define the truncation $T_M : \mathbb{R}^m \to \mathbb{R}^m$ by

$$T_M(x) := \min \left(1, \frac{M}{|x|_{\mathbb{R}^m}} \right) x = \begin{cases} x, & \text{if } |x|_{\mathbb{R}^m} \leq M, \\ \frac{M}{|x|_{\mathbb{R}^m}} x, & \text{if } |x|_{\mathbb{R}^m} > M. \end{cases}$$

Then, the associated Nemytskii operator, which is denoted by the same symbol, maps $H^1_0(\Omega)^m$ to itself and for all $u \in H^1_0(\Omega)^m$ we have

$$\frac{\partial}{\partial \omega_k}(T_M u)_i(\omega) = \begin{cases} \frac{\partial}{\partial \omega_k} u_i(\omega), & \text{if } |u(\omega)|_{\mathbb{R}^m} \leq M, \\ \frac{M}{|u(\omega)|_{\mathbb{R}^m}} \frac{\partial}{\partial \omega_k} u_i(\omega) + \frac{M}{|u(\omega)|_{\mathbb{R}^m}} u_i(\omega) \sum_{j=1}^m u_j(\omega) \frac{\partial}{\partial \omega_k} u_j(\omega), & \text{if } |u(\omega)|_{\mathbb{R}^m} > M \end{cases}$$

for almost all $\omega \in \Omega$. Moreover, $T_M u \to u$ in $H^1_0(\Omega)^m$ as $M \to \infty$.

Proof. Let $u \in H^1_0(\Omega)^m$ be given. By Lemma A.1, we get $T_M u \in H^1_0(\Omega)^m$. Next, we prove the expression for $\nabla (T_M u)$. Since $T_M$ is not differentiable on the set $\{ x \in \mathbb{R}^m : |x|_{\mathbb{R}^m} = M \}$, the chain rule of [Marcus, Mizel, 1972, Theorem 2.1] is not applicable.

Thus, we are going to provide a differentiable approximation $T^\sigma_M$ of $T_M$. For $\sigma \in (0,1)$, we define $m^\sigma : \mathbb{R} \to \mathbb{R}$ via

$$m^\sigma(x) = \begin{cases} \frac{\sigma}{2} + x, & \text{if } x < 1 - \sigma, \\ 1 - \frac{1}{2\sigma} (1 - x)^2, & \text{if } 1 - \sigma \leq x < 1, \\ 1, & \text{if } x \geq 1, \end{cases}$$

which is a differentiable approximation of $x \mapsto \min(1,x)$. We find

$$(m^\sigma)'(x) = \begin{cases} \frac{1}{\sigma} (1 - x), & \text{if } 1 - \sigma \leq x < 1, \\ 0, & \text{if } 1 \leq x. \end{cases}$$

Now, a differentiable approximation $T^\sigma_M : \mathbb{R}^m \to \mathbb{R}^m$ of $T_M$ is given by

$$T^\sigma_M(x) := m^\sigma\left(\frac{M}{|x|_{\mathbb{R}^m}}\right) x.$$

The partial derivatives of the components of $T^\sigma_M$ are

$$\frac{\partial}{\partial x_j}(T^\sigma_M)_i(x) = m^\sigma\left(\frac{M}{|x|_{\mathbb{R}^m}}\right) \delta_{ij} - (m^\sigma)'\left(\frac{M}{|x|_{\mathbb{R}^m}}\right) \frac{M}{|x|_{\mathbb{R}^m}^3} x_i x_j,$$

where $\delta_{ij}$ is the Kronecker delta. Here and in what follows, we use the conventions

$$m^\sigma\left(\frac{M}{|x|_{\mathbb{R}^m}}\right) = 0 \quad \text{and} \quad (m^\sigma)'\left(\frac{M}{|x|_{\mathbb{R}^m}}\right) \frac{M}{|x|_{\mathbb{R}^m}^3} x_i x_j = 0$$

in case $|x|_{\mathbb{R}^m} = 0$. By Lemma A.1, we find $T^\sigma_M u \in H^1_0(\Omega)^m$. Since $T^\sigma_M$ is differentiable, we can apply the chain rule of [Marcus, Mizel, 1972, Theorem 2.1] and obtain

$$\frac{\partial}{\partial \omega_k}(T^\sigma_M u)_i(\omega) = \sum_{j=1}^m \frac{\partial}{\partial \omega_j}(T^\sigma_M)_i(u(\omega)) \frac{\partial}{\partial \omega_k} u_j(\omega)$$

$$= m^\sigma\left(\frac{M}{|u(\omega)|_{\mathbb{R}^m}}\right) \frac{\partial}{\partial \omega_k} u_i(\omega) - (m^\sigma)'\left(\frac{M}{|u(\omega)|_{\mathbb{R}^m}}\right) \frac{M}{|u(\omega)|_{\mathbb{R}^m}^3} u_i(\omega) \sum_{j=1}^m u_j(\omega) \frac{\partial}{\partial \omega_k} u_j(\omega).$$
As an candidate for the derivative \( \frac{\partial}{\partial \omega_k} (T_M u) \), we define
\[
v_{ki}(\omega) = m^0 \left( \frac{M}{|u(\omega)|} \right) \frac{\partial}{\partial \omega_k} u_i(\omega) - (m^0)' \left( \frac{M}{|u(\omega)|} \right) \frac{M}{|u(\omega)|} u_i(\omega) \sum_{j=1}^m u_j(\omega) \frac{\partial}{\partial \omega_k} u_j(\omega),
\]
where the scalar functions \( m^0 \) and \( (m^0)' \) are defined by
\[
m^0(x) = \min\{1, x\}, \quad (m^0)'(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}
\]
Since \( m^\sigma(x) \to m^0(x) \) for all \( x \in \mathbb{R} \), we get \( T_M^\sigma u - T_M u \to 0 \) a.e. in \( \Omega \). Moreover, this difference is dominated by
\[
|T_M^\sigma u(\omega) - T_M u(\omega)|_{\mathbb{R}^m} \leq |m^\sigma(\frac{M}{|u(\omega)|}) - m^0(\frac{M}{|u(\omega)|})| u(\omega)|_{\mathbb{R}^m} \leq |m^\sigma(\frac{M}{|u(\omega)|}) - m^0(\frac{M}{|u(\omega)|})| u(\omega)|_{\mathbb{R}^m} \leq \frac{\sigma}{2} |u(\omega)|_{\mathbb{R}^m},
\]
where we used \( |m^\sigma(x) - m^0(x)| \leq \sigma/2 \) for all \( \sigma \in (0, 1) \) and \( x \in \mathbb{R} \). By the dominated convergence theorem, we get \( |T_M^\sigma u(\omega) - T_M u(\omega)|_{\mathbb{L}^2(\Omega)} \to 0 \).

Since \( (m^\sigma)'(x) \to (m^0)'(x) \) for all \( x \in \mathbb{R} \), we similarly get \( \frac{\partial}{\partial \omega_k} (T_M^\sigma u) - v_{ki}(\omega) \to 0 \) for a.e. \( \omega \in \Omega \). Note that
\[
|(m^\sigma)'(x) - (m^0)'(x)| \leq 1
\]
for all \( \sigma \in (0, 1) \) and \( x \in \mathbb{R} \). Hence,
\[
\left| \frac{\partial}{\partial \omega_k} (T_M^\sigma u)_i(\omega) - v_{ki}(\omega) \right|
\]
\[
\leq |m^\sigma(\frac{M}{|u(\omega)|}) - m^0(\frac{M}{|u(\omega)|})| \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right|
\]
\[
+ \left| (m^\sigma)'(\frac{M}{|u(\omega)|}) - (m^0)'(\frac{M}{|u(\omega)|}) \right| \left| \frac{M}{|u(\omega)|} \right| u_i(\omega) \sum_{j=1}^m u_j(\omega) \left| \frac{\partial}{\partial \omega_k} u_j(\omega) \right|
\]
\[
\leq \frac{\sigma}{2} \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right| + \frac{1}{|u(\omega)|^2} \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right| \sum_{j=1}^m \left| \frac{\partial}{\partial \omega_k} u_j(\omega) \right|^2
\]
\[
\leq \frac{\sigma}{2} \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right| + \left( \sum_{j=1}^m \left| \frac{\partial}{\partial \omega_k} u_j(\omega) \right|^2 \right)^{1/2}.
\]
Thus, we can apply the dominated convergence theorem and obtain
\[
\left| \frac{\partial}{\partial \omega_k} (T_M^\sigma u)_i(\omega) - v_{ki}(\omega) \right|_{\mathbb{L}^2(\Omega)} \to 0.
\]
Finally, for \( \varphi \in C^\infty_0(\Omega) \) we find
\[
\int_\Omega \varphi \, v_{ki} \, d\omega \leftarrow \int_\Omega \varphi \, \frac{\partial}{\partial \omega_k} (T_M^\sigma u)_i \, d\omega = -\int_\Omega \varphi \, \frac{\partial}{\partial \omega_k} (T_M^\sigma u)_i \, d\omega \to -\int_\Omega \varphi \, \frac{\partial}{\partial \omega_k} (T_M u)_i \, d\omega.
\]
Hence,
\[
v_{ki} = \frac{\partial}{\partial \omega_k} (T_M u)_i,
\]
and this shows the first part of the claim.

It remains to show \( T_M u \to u \) in \( H^1_0(\Omega) \). The convergence \( T_M u \to u \) in \( L^2(\Omega) \) is clear. Using

\[
|m^0(x) - 1| \leq 1, \quad \text{and} \quad |(m^0)'(x) x| \leq 1,
\]

for all \( x \geq 0 \), the difference of the derivatives can be bounded by

\[
\left| \frac{\partial}{\partial \omega_k} (T_M u)_i(\omega) - \frac{\partial}{\partial \omega_k} u_i(\omega) \right| = \left| m^0 \left( \frac{M}{|u(\omega)|_{\mathbb{R}^m}} \right) - 1 \right| \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right|
\]

\[
+ \left| (m^0)' \left( \frac{M}{|u(\omega)|_{\mathbb{R}^m}} \right) \right| \frac{M}{|u(\omega)|_{\mathbb{R}^m}} \left| u_i(\omega) \sum_{j=1}^m u_j(\omega) \frac{\partial}{\partial \omega_k} u_j(\omega) \right|
\]

\[
\leq \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right| + \frac{1}{|u(\omega)|_{\mathbb{R}^m}^2} \left| u_i(\omega) \sum_{j=1}^m u_j(\omega) \frac{\partial}{\partial \omega_k} u_j(\omega) \right|
\]

\[
\leq \left| \frac{\partial}{\partial \omega_k} u_i(\omega) \right| + \left( \sum_{j=1}^m \left| \frac{\partial}{\partial \omega_k} u_j(\omega) \right|^2 \right)^{1/2}
\]

Hence, we can apply the dominated convergence theorem and obtain \( T_M u \to u \) in \( H^1_0(\Omega)^m \).

**Remark A.3.** In the proof of Theorem A.2, it is also possible to use different smooth approximations of \( T_M \), e.g.,

\[
\tilde{m}^\sigma(x) = \begin{cases} 
  x, & \text{if } x < 1, \\
  1 + \frac{x}{\sigma} - \frac{1}{\sigma^2} (1 + \sigma - x)^2, & \text{if } 1 \leq x < 1 + \sigma, \\
  1 + \frac{x}{\sigma}, & \text{if } 1 + \sigma \leq x.
\end{cases}
\]

Similarly, we obtain

\[
|\tilde{m}^\sigma(x) - m^0(x)| \leq \sigma/2 \quad \text{and} \quad |(\tilde{m}^\sigma)'(x) - (m^0)'(x)| |x| \leq 1
\]

for all \( \sigma \in (0, 1) \) and \( x \in \mathbb{R} \) and the arguments of the proof carry over. However, there is the crucial difference

\[
(m^0)'(1) = 0 \neq 1 = (\tilde{m}^\sigma)'(1).
\]

Thus, by using the approximation \( \tilde{m}^\sigma \), the arguments of the proof of Theorem A.2 lead to

\[
\frac{\partial}{\partial \omega_k} (T_M u)_i(\omega) = \begin{cases} 
 \frac{\partial}{\partial \omega_k} u_i(\omega), & \text{if } |u(\omega)|_{\mathbb{R}^m} < M, \\
 \frac{M}{|u(\omega)|_{\mathbb{R}^m}} \frac{\partial}{\partial \omega_k} u_i(\omega) + \frac{M}{|u(\omega)|_{\mathbb{R}^m}} u_i(\omega) \sum_{j=1}^m u_j(\omega) \frac{\partial}{\partial \omega_k} u_j(\omega), & \text{if } |u(\omega)|_{\mathbb{R}^m} \geq M
\end{cases}
\]

for a.a. \( \omega \in \Omega \) for all \( u \in H^1_0(\Omega)^m \). Together with the result of Theorem A.2, this shows

\[
\frac{1}{2} \nabla(|u|_{\mathbb{R}^m}^2) = \sum_{j=1}^m u_j \nabla u_j = 0 \quad \text{a.e. on the set } \{ \omega \in \Omega : |u(\omega)|_{\mathbb{R}^m} = M \}.
\]

Note that, in case \( m = 1 \), this reduces to the well-known formula

\[
\nabla u = 0 \quad \text{a.e. on the set } \{ \omega \in \Omega : |u(\omega)| = M \}
\]

for \( u \in H^1_0(\Omega) \).
B. Decomposition of Measures in $H^{-1}(\Omega)$

In this section, we give a counterexample which shows that the positive part of a measure in $H^{-1}(\Omega)$, i.e. of an element of $H^{-1}(\Omega) \cap \mathcal{M}(\Omega)$, may not belong to $H^{-1}(\Omega)$.

Let $\Omega = U_1(0) \subset \mathbb{R}^2$ be the (open) unit ball. We denote by $\delta_\hat{r}$ the uniform line measure which is located at the radius $\hat{r} \in (0, 1)$ and with total mass $2 \pi$ (i.e., line density $1/\hat{r}$). Note that $\delta_\hat{r} \in H^{-1}(\Omega) \cap \mathcal{M}(\Omega)$ for $\hat{r} > 0$.

By $(-\Delta_0)^{-1}$ we denote the solution mapping associated with the Laplace equation with homogeneous Dirichlet boundary condition on $\Omega$. It is easy to check that

$$v_\hat{r}(x, y) := (-\Delta_0)^{-1}(\delta_\hat{r})(x, y) = \begin{cases} \log(1/\hat{r}), & \text{if } r \leq \hat{r}, \\ \log(1/r), & \text{if } r > \hat{r}. \end{cases}$$

Here and in the sequel, we use $r = \sqrt{x^2 + y^2}$. We find

$$\frac{\partial}{\partial r} v_\hat{r}(x, y) = \begin{cases} 0, & \text{if } r \leq \hat{r}, \\ -1/r, & \text{if } r > \hat{r}, \end{cases}$$

and, thus,

$$\|v_\hat{r}\|_{H^1_0(\Omega)} = \int_{\Omega} |\nabla v_\hat{r}|^2 \, d(x, y) = \int_{\Omega} \left( \frac{\partial}{\partial r} v_\hat{r} \right)^2 \, d(x, y) = 2 \pi \int_{\hat{r}}^{1} 1/r \, dr = 2 \pi \log(1/\hat{r}).$$

Now, let $q \in (0, 1)$ and a sequence $\{c_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$ be given. We set $r_i = q^i$. We define a sequence $\{\mu_k\} \subset H^{-1}(\Omega) \cap \mathcal{M}(\Omega)$ by

$$\mu_k := \sum_{i=1}^{k} c_i \left( \delta_{r_{2i}} - \delta_{r_{2i-1}} \right),$$

where $\{c_i\}$ is a sequence of positive numbers. Since all line measures have mass $2 \pi$, the sequence $\{\mu_k\}$ is a Cauchy sequence in $\mathcal{M}(\Omega)$ if and only if $\{c_i\}$ is summable.

In order to compute the $H^{-1}(\Omega)$-norm of $\mu_k$, we set

$$v_k := (-\Delta_0)^{-1} \mu_k$$

and have

$$\|v_k\|_{H^1_0(\Omega)} = \|\mu_k\|_{H^{-1}(\Omega)}.$$

Since

$$\frac{\partial}{\partial r} v_k(x, y) = \begin{cases} -c_i/r, & \text{if } r_{2i} \leq r \leq r_{2i-1} \text{ with } i \in \{1, \ldots, k\}, \\ 0, & \text{else}, \end{cases}$$

we find for $n \leq k$

$$\|\mu_n - \mu_k\|_{H^{-1}(\Omega)}^2 = \|v_n - v_k\|_{H^1_0(\Omega)}^2 = \int_{\Omega} |\nabla (v_n - v_k)|^2 \, d(x, y) = \int_{\Omega} \left( \frac{\partial}{\partial r} (v_n - v_k) \right)^2 \, d(x, y)$$

$$= 2 \pi \sum_{i=n+1}^{k} c_i^2 \int_{r_{2i}}^{r_{2i-1}} \frac{1}{r} \, dr = 2 \pi \sum_{i=n+1}^{k} c_i^2 \log \left( \frac{r_{2i-1}}{r_{2i}} \right) = 2 \pi \log(1/q) \sum_{i=n+1}^{k} c_i^2.$$

Hence, the sequence $\{\mu_k\}$ is a Cauchy sequence in $H^{-1}(\Omega)$ if and only if $\{c_i\}$ is square summable.

In case $\{c_i\}$ is summable, the limits of $\mu_k$ in $H^{-1}(\Omega) = H^1_0(\Omega)^*$ and $\mathcal{M}(\Omega)$ coincide, since $C_0^\infty(\Omega)$ is a dense subspace of $H^1_0(\Omega)$ and of $C_0(\Omega)$.
Now, we choose \( c_i = i^p \) for some \(-3/2 < p < -1\). Then \( c_i \) is summable and, thus, square summable. Hence, \( \{\mu_k\} \) is a Cauchy sequence in \( H^{-1}(\Omega) \) and \( M(\Omega) \) and we set

\[
\mu := \lim_{k \to \infty} \mu_k = \sum_{i=1}^{\infty} c_i (\delta_{r_{2i}} - \delta_{r_{2i-1}}).
\]

Since the mapping \( \nu \mapsto \nu^+ \) is continuous on \( M(\Omega) \), the positive part of \( \mu \) is given by

\[
\mu^+ = \sum_{i=1}^{\infty} c_i \delta_{r_{2i}}.
\]

Now, we show that \( \mu^+ \) is not bounded w.r.t. the \( H^1_0(\Omega) \) norm on \( C_0(\Omega) \cap H^1_0(\Omega) \). Let \( \varphi \in C^\infty(\Omega) \) with \( 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) on \( B_q(0) \) be given. For \( 0 < s < 1/2 \), the function

\[
v(x, y) = \log\left(1/r^s\right) \varphi(x, y)
\]

belongs to \( H^1_0(\Omega) \).

For given \( M > \log(1/q)^s \), we consider the truncation \( v_M := \min\{v, M\} \) of \( v \) at \( M \) and have

\[
v_M(x, y) = \begin{cases} M & r \leq \exp(-M^{1/s}), \\ v(x, y) & r > \exp(-M^{1/s}). \end{cases}
\]

Moreover, \( v_M \to v \) in \( H^1_0(\Omega) \) as \( M \to \infty \).

But

\[
\langle \mu^+, v_M \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = 2\pi \sum_{i=1}^{\infty} c_i v_M(r_{2i}) \geq 2\pi \sum_{i=1}^{n(M)} c_i \log(1/q^2)^s = 2\pi \log(1/q^2)^s \sum_{i=1}^{n(M)} i^{p+s},
\]

where \( n(M) = \lfloor M^{1/s}/(2 \log(1/q)) \rfloor \). Note that \( n(M) \to \infty \) as \( M \to \infty \) and, hence,

\[
\langle \mu^+, v_M \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \geq 2\pi \log\left(1/q^2\right)^s \sum_{i=1}^{n(M)} i^{p+s} \to \infty
\]

as \( M \to \infty \) if \( p + s \geq -1 \). Note that for all \( p \in (-3/2, -1) \), we can choose \( s \in (0, 1/2) \) such that \( p + s \geq -1 \).

This shows that \( \mu^+ \in M(\Omega) \) is not bounded on \( H^1_0(\Omega) \cap C_0(\Omega) \) w.r.t. the \( H^1_0(\Omega) \)-norm.

A similar reasoning shows that \( v \notin L^1(\mu^+) \). Indeed, if \( v \) would belong to \( L^1(\mu^+) \), \( v \) would be integrable and dominates \( v_M \) and, thus,

\[
\infty > \int_\Omega v \, d\mu^+ = \lim_{M \to \infty} \int_\Omega v_M \, d\mu^+ = \infty.
\]

Similarly, we can show \( v \notin L^1(\mu) \).

\textbf{Remark B.1.} We have constructed a counterexample in dimension \( d = 2 \). The construction can be adopted to dimensions \( d > 2 \).

In dimension \( d = 1 \) however, we have \( H^1_0(\Omega) \hookrightarrow C_0(\Omega) \) and this embedding is continuous and dense. Hence, we obtain \( M(\Omega) = C_0(\Omega)^\ast \hookrightarrow H^1_0(\Omega)^\ast = H^{-1}(\Omega) \). Thus, the positive part of a measure belongs to \( M(\Omega) \) and, in turn, to \( H^{-1}(\Omega) \). Therefore, it is not possible to construct a similar counterexample in dimension \( d = 1 \).
C. Lemmas on Polyhedral Sets Satisfying LICQ

In this section, we provide some results for polyhedral sets. In the first lemma, we construct Lipschitz continuous functions satisfying (C.1). The existence of these functions is crucial in Section 5 to infer the polyhedricity of $C$.

**Lemma C.1.** We define the set

$$ C := \{ x \in \mathbb{R}^m : (x, n_i)_{\mathbb{R}^m} \leq b_i \ \forall \ i = 1, \ldots, N \}, $$

where $n_i \in \mathbb{R}^m$, $b_i \in \mathbb{R}$ are given for $i = 1, \ldots, N$. Further, we assume that LICQ is satisfied, that is, the family $\{ n_i : (x, n_i)_{\mathbb{R}^m} = b_i \}$ is linearly independent for all $x \in C$.

Then, there exist functions $h_i : C \rightarrow \mathbb{R}^m$, $i = 1, \ldots, N$, which are globally bounded and Lipschitz continuous, and

$$ (h_i(x), n_j)_{\mathbb{R}^m} = \delta_{ij} \quad \forall \ i, j \in \{1, \ldots, N\}, x \in F_j. $$

(C.1)

Here, $F_j$ is the facet corresponding to inequality $j$, i.e., $F_j := \{ x \in C : (x, n_j)_{\mathbb{R}^m} = b_j \}$, and $\delta_{ij}$ is the Kronecker delta.

**Proof.** **Step I:** Triangulation of $C$.

First, let $L$ denote the orthogonal complement of the lineality space of $C$. Then, $C = L^\perp + (C \cap L)$ and all faces of $C \cap L$ have at least one vertex (i.e., extreme point), cf. [Grünbaum, 1967, 2.5.6]. Then, following [Clarkson, 1987, p. 200], we find a triangulation $T = \{ S_k : k = 1, \ldots, K \}$ of $C \cap L$, see also [Clarkson, 1985]. That is, $C \cap L = \bigcup_{k=1}^{K} S_k$ and each $S_k$ is a generalized dim($L$)-simplex, i.e., $S_k = \text{conv}\{ v_i^k \}_{i=1}^{N_k(k)} + \text{cone}\{ r_i^k \}_{i=1}^{N_2(k)}$, such that $N_1(k) \geq 1$, $N_2(k) \geq 0$, $N_1(k) + N_2(k) = \text{dim}(L) + 1$, all $v_i^k$ are vertices of $C \cap L$ and all $r_i^k$ are extremal rays of $C \cap L$, see [Clarkson, 1985, Section 4] for details. Moreover, if $S_{k_1} \cap S_{k_2}$ is not empty, it is a common face of $S_{k_1}$ and $S_{k_2}$.

**Step II:** Definition of $h_i$.

Let $v$ be a vertex of $C \cap L$. By the linear independence assumption, we can find vectors $h_i(v) \in \mathbb{R}^m$, such that $(h_i(v), n_j)_{\mathbb{R}^m} = \delta_{ij}$ for all $j$ satisfying $v \in F_j$. For any generalized simplex $S_k$, we extend $h_i$ to $S_k$ by $h_i(v + r) := h_i(v)$ for $v \in \text{conv}\{ v_i^k \}_{i=1}^{N_1(k)}$, $r \in \text{cone}\{ r_i^k \}_{i=1}^{N_2(k)}$. Let us check that $h_i$ is well defined on $C \cap L$. If $x \in S_{k_1} \cap S_{k_2}$, this intersection is a common face of $S_{k_1}$ and $S_{k_2}$. Hence, $S_{k_1} \cap S_{k_2}$ is the convex hull of some common vertices and some common extremal rays. Since $h_i$ is well defined on the vertices of $C \cap L$, and extended linearly to $S_{k_1}$ and $S_{k_2}$, both definitions of $h_i(x)$ coincide.

Since $h_i$ is piecewise affine on $C \cap L$ and continuous, it is Lipschitz continuous. The boundedness of $h_i$ follows since the range of $h_i$ is contained in the convex hull of $\{ h_i(v) : v \text{ is a vertex of } C \cap L \}$.

Finally, we set $h_i(x) := h_i(\hat{x})$ for $x \in C$, where $\hat{x} \in C \cap L$ and $x - \hat{x} \in L^\perp$. The Lipschitz continuity and the boundedness of $h_i$ on $C$ follows.

**Step III:** Verification of (C.1). Let $x \in F_j$ be given. By definition we have $h_i(x) = h_i(\hat{x})$, where $\hat{x} \in C \cap L$ and $x - \hat{x} \in L^\perp$. It is easy to check that $n_j \in L$ and, thus, we have $(\hat{x}, n_j)_{\mathbb{R}^m} = (x, n_j)_{\mathbb{R}^m}$. This shows $\hat{x} \in F_j$. The point $\hat{x} \in C \cap L$ belongs to $S_k$ for some $k \in \{1, \ldots, K\}$. Hence, $h_i(\hat{x})$ is a convex combination of $\{ h_i(v) : v \text{ is vertex of } S_k \cap F_j \}$. This shows $(h_i(x), n_j)_{\mathbb{R}^m} = (h_i(\hat{x}), n_j)_{\mathbb{R}^m} = \delta_{ij}$, since $(h_i(v), n_j)_{\mathbb{R}^m} = \delta_{ij}$ for all vertices $v$ of $S_k \cap F_j$.

In the next lemma, we show that LICQ even holds for constraints which are almost active.

**Lemma C.2.** Suppose that the assumptions of Lemma C.1 are satisfied. Then, there exists $\hat{\delta} > 0$, such that the family $\{ n_i : (x, n_i)_{\mathbb{R}^m} \geq b_i - \hat{\delta} \}$ is linear independent for all $x \in C$. 


For each subset $I$, we denote by $L$ the orthogonal complement of the lineality space of $C$ and have
\[ C = L^\perp + (C \cap L) = L^\perp + \text{conv}\{v_j\}_{j=1,\ldots,N_1} + \text{cone}\{r_j\}_{j=1,\ldots,N_2}, \]
where $v_j$ and $r_j$ are the vertices (extreme points) and extreme rays of $C \cap L$, respectively, see [Schneider, 2014, Corollary 1.4.4].

In a first step, we use a compactness argument to show that the assertion holds for all $x \in V := \text{conv}\{v_j\}_{j=1,\ldots,N_1}$. Let $x \in V$ be arbitrary. By assumption, the family $\{n_i : (x, n_i)_{R^m} = b_i\}$ is linear independent. Since there are only finitely many inactive constraints, there is $\delta_x > 0$, such that for all $i = 1, \ldots, N$ we have
\[ (x, n_i)_{R^m} = b_i \iff (x, n_i)_{R^m} \geq b_i - 2\delta_x, \]
and, hence, the family $\{n_i : (x, n_i)_{R^m} \geq b_i - 2\delta_x\}$ is linear independent. By continuity of the scalar product, we find $\varepsilon_x > 0$, such that for all $i = 1, \ldots, N$ and all $x' \in U_{\varepsilon_x}(x) = \{y \in \mathbb{R}^m : |y - x|_{R^m} < \varepsilon_x\}$ we have
\[ (x, n_i)_{R^m} = b_i \iff (x', n_i)_{R^m} \geq b_i - \delta_x. \]
Hence, the family $\{n_i : (x, n_i)_{R^m} \geq b_i - \delta_x\}$ is linear independent for all $x' \in U_{\varepsilon_x}(x)$. Now, $\{U_{\varepsilon_x}(x)\}_{x \in V}$ is an open cover of the compact set $V$ and there exists a finite subcover. We denote by $\delta$ the minimal $\delta_x$ corresponding to this subcover and obtain $\delta > 0$. This shows that the family $\{n_i : (x, n_i)_{R^m} \geq b_i - \delta\}$ is independent for all $x \in V$.

Now, an arbitrary point $x \in C$ can be written as $x = \ell + v + r$ with $\ell \in L^\perp$, $v \in V$ and $r \in \text{cone}\{r_j\}_{j=1,\ldots,N_2}$. Since $L^\perp$ is a subspace and $\text{cone}\{r_j\}_{j=1,\ldots,N_2}$ is a cone, we get $(\ell, n_i)_{R^m} = 0$ and $(r, n_i)_{R^m} \leq 0$ for all $i = 1, \ldots, N$. Hence, $(x, n_i)_{R^m} \geq b_i - \delta$ implies
\[ b_i - \delta \leq (x, n_i)_{R^m} = (\ell + v + r, n_i)_{R^m} \leq (v, n_i)_{R^m} \leq b_i. \]
This shows $\{n_i : (x, n_i)_{R^m} \geq b_i - \delta\} \subset \{n_i : (v, n_i)_{R^m} \geq b_i - \delta\}$ and the latter family is linear independent by the first step of the proof.

Finally, we show that if the constraint $i$ is almost active for a point $x \in C$, there exists a point in the neighborhood of $x$ in which the constraint $i$ is active.

**Lemma C.3.** Suppose that the assumptions of Lemma C.1 are satisfied and let $\hat{\delta} > 0$ be given from the previous lemma. Then, there is a constant $M > 0$, such that $x \in C$ and $(x, n_i)_{R^m} \geq b_i - \hat{\delta}$ implies the existence of $\tilde{x} \in C$ such that $(x, n_i)_{R^m} = b_i$ and $|x - \tilde{x}|_{R^m} \leq M (b_i - (x, n_i)_{R^m})$.

**Proof.** For each subset $I \subset \{1, \ldots, N\}$, for which $\{n_i\}_{i \in I}$ is linear independent, we choose vectors $\{p^I_j\}_{j \in I}$, such that
\[ (p^I_j, n_i)_{R^m} = \delta_{ij} \quad \text{for all } i, j \in I. \]
We set $M = \max_{i,j \in I} |p^I_j|_{R^m}$.

For $x \in C$, we set $I(x) = \{i \in \{1, \ldots, N\} : (x, n_i)_{R^m} \geq b_i - \hat{\delta}\}$. We prove the claim by backward induction over the number of elements $\# I(x)$ in the set $I(x)$. The case $\# I(x) > m$ cannot appear since $\{n_i : i \in I(x)\}$ are linear independent vectors in $\mathbb{R}^m$.

Now, let $x \in C$ be given and suppose that the claim already holds for all $\tilde{x} \in C$ with $\# I(\tilde{x}) > \# I(x)$. Let $i \in \{1, \ldots, N\}$ be given, such that $(x, n_i)_{R^m} \geq b_i - \hat{\delta}$. Then, we have $i \in I(x)$ by definition. Moreover,
\[ (x + tp^I(x), n_j)_{R^m} = \begin{cases} (x, n_j)_{R^m} & \text{if } j \in I(x) \setminus \{i\}, \\ (x, n_i)_{R^m} + t & \text{if } j = i. \end{cases} \]
If $x + T p_i^{I(x)} \in C$ for $T = b_i - (x, n_i)_{\mathbb{R}^m}$, we can use $\tilde{x} = x + T p_i^{I(x)}$. Otherwise, there is a smallest $t > 0$, such that $I(x + t p_i^{I(x)})$ is strictly larger than $I(x)$. By the induction hypothesis, we find $\tilde{x} \in C$, such that $(\tilde{x}, n_i)_{\mathbb{R}^m} = b_i$ and

$$|x + t p_i^{I(x)} - \tilde{x}|_{\mathbb{R}^m} \leq M \left( b_i - (x + t p_i^{I(x)}, n_i)_{\mathbb{R}^m} \right).$$

This implies

$$|x - \tilde{x}|_{\mathbb{R}^m} \leq |x + t p_i^{I(x)} - \tilde{x}|_{\mathbb{R}^m} + t |p_i^{I(x)}|_{\mathbb{R}^m} \leq M \left( b_i - (x + t p_i^{I(x)}, n_i)_{\mathbb{R}^m} \right) + t M$$

$$\leq M \left( b_i - (x, n_i)_{\mathbb{R}^m} - t + t \right) = M \left( b_i - (x, n_i)_{\mathbb{R}^m} \right).$$

This shows that the claim holds for $x$ and this finishes the induction step.

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**References**


