Scalability of a FETI-DP Method for Optimal Control Problems

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A parallel FETI-DP (Finite Element Tearing and Interconnecting) domain decomposition method is applied to optimal control problems with control constraints. We show parallel scalability for up to 1024 cores of a Cray XT6.

1 Introduction

We consider problems of the form

\[
\min_{y,u} \frac{1}{2} \int_{\Omega} (y(x) - y_d(x))^2 \, dx + \frac{\alpha}{2} \int_{\Omega} (u(x))^2 \, dx,
\]

where \( y \) denotes the unknown state and \( u \in U \) the unknown control, subject to a PDE constraint

\[
a(y, v) = (f, v)_0 + (u, v)_0, \quad \text{for all } v \in V.
\]

The function \( y_d \) denotes a given desired state and \( \alpha > 0 \) a cost parameter. Here, \( (\cdot, \cdot)_0 \) is the \( L_2 \) inner product and \( a(\cdot, \cdot) \) is the bilinear form associated with two dimensional linear elasticity. The displacement field \( y \) is sought in \( V = H^1(\Omega, \partial \Omega_D)^2 = \{ y \in H^1(\Omega)^2 : y = 0 \text{ on } \partial \Omega_D \} \), where \( \Omega \subset \mathbb{R}^2 \) and \( \partial \Omega_D \) is part of its boundary. For simplicity, we restrict ourselves to the case of volume control, i.e., \( U = L_2(\Omega)^2 \).

Dual-primal FETI methods were first introduced by Farhat, Lesoinne, Le Tallec, Pierson, and Rixen \cite{Farhat1998} and have scaled to 10\textsuperscript{5} processor cores already in 2009 \cite{Farhat2009}. In \cite{Heinkenschloss2006} a first convergence bound for scalar problems in 2D was provided. A condition number bound for FETI-DP methods applied to linear elasticity was first proven in \cite{Farhat2003}.

Balancing Neumann-Neumann domain decomposition methods are closely related to FETI methods and have been considered for the optimal control of scalar problems in Heinkenschloss and Nguyen \cite{Heinkenschloss2003,Heinkenschloss2005}. There, a Balancing Neumann-Neumann preconditioner is constructed for the indefinite Schur complement. Also see the recent works of Ketelaer \cite{Ketelaer2013} and Heinkenschloss, Heuveline, and Ketelaer \cite{Heinkenschloss2008} on Neumann-Neumann preconditioners for the optimal control of flow problems. For some first results on the present FETI-DP approach to optimal control problems, see \cite{Rheinbach2014}.

2 Discretization

We discretize \( y, p \) by \( P1 \) finite elements, \( u \) by \( P0 \) finite elements and obtain the discrete problem

\[
\min_{y,u} \frac{1}{2} y^T M y + \frac{\alpha}{2} u^T Q u - c^T y \quad \text{s.t.} \quad Ay = f + Nu,
\]

where \( c = M y_d \). We obtain the discrete optimality system

\[
\begin{bmatrix}
M & 0 & A^T \\
0 & \alpha Q & -N^T \\
A & -N & 0
\end{bmatrix}
\begin{bmatrix}
y \\ u \\ p
\end{bmatrix}
= \begin{bmatrix}
c \\ 0 \\ 0
\end{bmatrix},
\]

where \( A \in \mathbb{R}^{n \times m}, \; Q \in \mathbb{R}^{m \times m}, \; M \in \mathbb{R}^{m \times m} \). Here, \( A = A^T = (a(\varphi_i, \varphi_j))_{i,j} \) is a stiffness matrix, whereas \( Q = (\langle \psi_i, \psi_j \rangle)_{i,j}, \; M = (\langle \phi_i, \psi_j \rangle)_{i,j} \) and \( N = (\langle \psi_i, \psi_j \rangle)_{i,j} \) are mass matrices. Our elastic body \([0,1]^2\) is clamped on the left side. We also consider inequality constraints on the control \( u \), i.e., \( u_0 \leq u \leq u_b \). We consider local optimal control problems on nonoverlapping subdomains with left hand sides \( K_i, i = 1, \ldots, N_\Omega \). Following the all-at-once dual-primal FETI domain decomposition approach from \cite{Rheinbach2014}, we construct the FETI-DP master system,

\[
\begin{bmatrix}
\tilde{K} & \tilde{B}^T \\
\tilde{B} & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x} \\ \lambda
\end{bmatrix}
= \begin{bmatrix}
b \\ 0
\end{bmatrix}, \quad u \in \mathbb{R}^n, \; \lambda \in \mathbb{R}^m,
\]

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α active. The number of GMRES iterations is scalable wrt. to the number of subdomains; Table 3:

Table 2: Weak Parallel Scalability on a Cray XT6 of Universität Duisburg-Essen from 4 to 1024 processor cores. The problem size is $2 \times 2 \times \text{Points} + 2 \times \text{Elements}$. $H$ is the size of a subdomain, $h$ is the size of a finite element; $\alpha = 0.01$, $E = 1$, $\nu = 0.3$, rtol=1e$-6$

<table>
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<tr>
<th>$\Omega$</th>
<th>#Cores</th>
<th>$H/h$</th>
<th>#Points</th>
<th>#Elem</th>
<th>Problem Size</th>
<th>GMRES(30)</th>
<th>Solution Time</th>
</tr>
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<td>8+9+9</td>
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<td>13</td>
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<td>81945604</td>
<td>56</td>
<td>37.9s</td>
<td></td>
</tr>
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</table>

Table 2: Dual-primal active set iteration for the constraint $u_a \leq u \leq u_b$. In these problems between 17% and 18% of all constraints are active. The number of GMRES iterations is scalable wrt. to the number of subdomains; $\alpha = 0.01$, $E = 1$, $\nu = 0.3$, rtol=1e$-6$/||\text{rhs}||$.

$\nu$ from the matrices $K^{(i)}$, $i = 1, \ldots, N_\Omega$; for details, see [9]. Here, the jump operator $\hat{B}$ only acts on the dual variables $y_\Delta$ and $p_\Delta$. We use vertices as our primal variables $y_{\Omega}$, $p_{\Omega}$. After the elimination of $\tilde{e}$ in (5) it remains to solve a system $F\lambda = d$ where $F$ is symmetric indefinite, i.e., with positive and negative eigenvalues, by a suitable Krylov subspace method.

3 Numerical Results

We apply GMRES to the symmetric indefinite FETI-DP system (2), using the Dirichlet preconditioner $M_D$, which is symmetric indefinite in our case. We use the sparse direct solver UMFPACK to solve the local optimal control problems on the subdomains. We denote the number of subdomains by $N_\Omega$. In Tab. 1 parallel scalability for up to 1024 Cray XT cores (12-Core 1.9 GHz) is shown, using restarted GMRES(30). In Tab. 2 the numerical scalalility using an active set strategy for the constraint $u_a \leq u \leq u_b$ is shown. In Tab. 3 recycling of $\lambda$-values is used to reduce the number of GMRES iterations.

References