TECHNISCHE UNIVERSITÄT CHEMNITZ

Existence of Solutions of a Thermoviscoplastic Model and Associated Optimal Control Problems

Roland Herzog* Christian Meyer† Ailyn Stötzner‡

October 25, 2016

A quasistatic, thermoviscoplastic model at small strains with linear kinematic hardening, von Mises yield condition and mixed boundary conditions is considered. The existence of a unique weak solution is proved by means of a fixed-point argument, and by employing maximal parabolic regularity theory. The weak continuity of the solution operator is also shown. As an application, the existence of a global minimizer of a class of optimal control problems is proved.

Keywords: thermoviscoplasticity; variational inequality of second kind; mixed boundary conditions; Banach fixed-point theorem; maximal parabolic regularity; optimal control

*Technische Universität Chemnitz, Faculty of Mathematics, Professorship Numerical Mathematics (Partial Differential Equations), 09107 Chemnitz, Germany, roland.herzog@mathematik.tu-chemnitz.de, http://www.tu-chemnitz.de/herzog

†Technische Universität Dortmund, Fakultät für Mathematik, Lehrstuhl LSX, 44227 Dortmund, Germany, cmeyer@math.tu-dortmund.de, http://www.mathematik.uni-dortmund.de/de/personen/person/Christian+Meyer.html

‡Technische Universität Chemnitz, Faculty of Mathematics, Professorship Numerical Mathematics (Partial Differential Equations), 09107 Chemnitz, Germany, ailyn.stoetzner@mathematik.tu-chemnitz.de, http://www.tu-chemnitz.de/mathematik/part_dgl/people/stoetzner

1. Introduction

We consider the following quasistatic, thermovisco(elasto)plastic model at small strains with linear kinematic hardening and von Mises yield condition:

$$\text{stress-strain relation:} \qquad \boldsymbol{\sigma} = \mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)\big), \tag{1.1}$$

$$\text{conjugate forces:} \qquad \boldsymbol{\chi} = -\mathbb{H}\,\boldsymbol{p}, \tag{1.2}$$

$$\text{viscoplastic flow rule:} \qquad \epsilon\,\dot{\boldsymbol{p}} + \partial_{\dot{\boldsymbol{p}}} D(\dot{\boldsymbol{p}},\theta) \ni [\boldsymbol{\sigma} + \boldsymbol{\chi}]\,, \tag{1.3}$$

$$\text{balance of momentum:} \qquad -\operatorname{div}\big(\boldsymbol{\sigma} + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\big) = \boldsymbol{\ell}, \tag{1.4}$$

$$\text{heat equation:} \qquad \varrho\, c_p\, \dot{\theta} - \operatorname{div}(\kappa\,\nabla\theta) = r + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} - \theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}}). \tag{1.5}$$

The unknowns are the stress $\boldsymbol{\sigma}$, back-stress $\boldsymbol{\chi}$, plastic strain $\boldsymbol{p}$, displacement $\boldsymbol{u}$ and temperature θ. Further, $\mathbb{C}$ and $\mathbb{H}$ denote the elastic and hardening moduli, respectively. $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the symmetrized gradient or linearized strain associated with $\boldsymbol{u}$. The temperature dependent term $\boldsymbol{t}(\theta)$ expresses thermally induced strains. D denotes the dissipation function. The right hand sides $\boldsymbol{\ell}$ and r represent mechanical and thermal volume and boundary loads, respectively. ϱ, c_p and κ describe the density, specific heat capacity and thermal conductivity of the material. The positive parameters ϵ and γ represent viscous effects in the evolution of the plastic strain and in the balance of momentum. For the derivation of the system (1.1)–(1.5) and more on its physical background, we refer the reader to [Ottosen and Ristinmaa, 2005, Chapter 22 and 23].

The analysis of thermoplastic models poses numerous mathematical challenges, mainly due to the low integrability of the nonlinear terms on the right hand side of the heat equation. Several approaches have been considered in the literature to deduce the existence and uniqueness of a solution, and we mention the following.

- Chełmiński and Racke [2006]: In this model without viscosity terms the dissipation function is only allowed to depend linearly on the temperature and a simplified mechanical heat source is used which does not account for plastic dissipation and is cut off at large temperatures. The authors use a Yosida regularization to prove the existence of a solution.
- Bartels and Roubíček [2008]: The model does not account for hardening and thermal strains, it contains a hyperbolic viscous balance of momentum and a simplified right hand side of the heat equation. The authors prove the existence of a solution in a weak sense via a discretization strategy.
- Bartels and Roubíček [2011]: In contrast to Bartels and Roubíček [2008] the authors take into account thermal strains, linear kinematic and isotropic hardening and the same right hand side of the heat equation as in (1.5) but they consider a temperature independent flow rule. The authors require a growth condition for the heat capacity w.r.t. the temperature to obtain the existence of a solution in a weak sense, again via a discretization procedure.

- Paoli and Petrov [2012]: In contrast to our model the authors assume a C^2 regular boundary in addition to homogeneous boundary conditions for the displacement, which leads to better regularity. Moreover, the dissipation function is assumed to be independent of the temperature. The authors use a growth condition for the heat capacity w.r.t. the temperature to show the existence of a solution in a classical sense by means of Schauder's fixed point theorem.

Our approach is closest to the one in Paoli and Petrov [2012]. We emphasize that we admit more general domains and boundary conditions. The overall strategy to show the existence and uniqueness of a solution is an application of Banach's fixed point theorem, applied to a reduced problem formulated in the temperature variable alone. In order to apply the fixed-point argument, we make use of the theory of maximal parabolic regularity. The same strategy was used in Hömberg et al. [2009/10] for the analysis and optimal control of a thermistor problem. Furthermore, we focus our discussion on the case of constant heat capacities. We mention that this case is not included in Paoli and Petrov [2012] since a linear growth of the heat capacity is assumed there. In contrast to the linear dependence of the thermal strain on the temperature in Paoli and Petrov [2012], we allow more general thermal strains $\boldsymbol{t}$ and only assume them to be globally bounded w.r.t. the temperature. This can be achieved w.l.o.g. by a cut-off outside the relevant temperature regime.

Under the assumptions made precise in Section 2, our main result is as follows. (We refer the reader to Theorem 3.1 for a re-iteration of the theorem.)
Theorem (Main Theorem)**.** There exists $\bar{p} > 2$ such that for all $2 < p \leq \bar{p}$, there exists $\bar{q} > 2$ (depending on p) such that for all $\bar{q} \leq q < \infty$ and sufficiently smooth right hand sides $(\boldsymbol{\ell}, r)$ and initial conditions $(\boldsymbol{u}_0, \boldsymbol{p}_0, \theta_0)$, there exists a unique weak solution $(\boldsymbol{u}, \boldsymbol{p}, \theta)$ of (1.1)–(1.5) such that

$$\boldsymbol{u} \in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)), \qquad \boldsymbol{p} \in W^{1,q}(0,T;\boldsymbol{L}^{p}(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})) \text{ trace-free},$$
$$\theta \in W^{1,\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))$$

with $v(p)$ given in (2.1). The stress components $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ are obtained from (1.1)–(1.2).

Note that compared to Bartels and Roubíček [2008, 2011], we obtain solutions of higher regularity, working with a different notion of a solution. Although we hope that the main theorem is a result of independent interest, we consider in this paper also the existence of global minimizers of certain optimal control problems involving (1.1)–(1.5). To this end, we prove a result about the weak sequential continuity of the solution map w.r.t. the right hand side data, see Proposition 4.8. It will be proved using a technique developed in Bartels and Roubíček [2008].

The paper is organized as follows. After an introduction of the exact setting and the detailed assumptions in Section 2, the existence and uniqueness of a solution to the system (1.1)–(1.5) is shown in Section 3. A short roadmap is presented at the beginning of Section 3, describing the break up of the proof into smaller parts. In particular, we prove in Section 3.1 the existence and uniqueness of a solution to system (1.1)–(1.4) for a given temperature field, and in Section 3.2 the contractivity of the fixed-point map on small time intervals is shown, along with a continuation argument. The proof of the main theorem is given in Section 3.3. Finally, we prove Proposition 4.8 on the weak sequential continuity of the solution map in Section 4, and deduce the existence of a global minimizer of associated optimal control problems as an application.

2. Notation, Assumptions, and Weak Formulation

In what follows, Ω denotes a bounded domain in $\mathbb{R}^3$ and $T > 0$ is a fixed end time point. For brevity, we write for the space-time cylinder $Q = \Omega \times (0, T)$. The spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$ denote Lebesgue and Sobolev spaces, respectively. For a Banach space X and its dual space X', we denote the duality product as $\langle \cdot, \cdot \rangle_X$ or simply $\langle \cdot, \cdot \rangle$ if no ambiguity arises. The norm of X is always denoted as $\|\cdot\|_X$. In the case $X = W^{1,p}(\Omega)$ we denote the dual by $W_\diamond^{-1,p'}(\Omega)$ where $1/p + 1/p' = 1$.

The space $\mathrm{Lin}(X)$ denotes the space of bounded linear functions from X into itself. Furthermore the space $L^p(0,T;X)$ denotes a Bochner space and the space $W^{1,p}(0,T;X)$ is the subset of $L^p(0,T;X)$ such that distributional time derivative of the elements are again in $L^p(0,T;X)$, see, e.g., [Showalter, 1997, Chapter III]. The space $W_0^{1,p}(0,T;X)$ denotes the subspace of functions which vanish in $t = 0$.

Vector-valued functions, and spaces containing such functions are written in bold-face notation. The spaces $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 3}_{\text{sym}}$ represent the (symmetric) 3×3 matrices. Furthermore, $\mathbb{R}^{3\times 3}_{\text{dev}}$ denotes the symmetric and trace-free (deviatoric) 3×3 matrices. For $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^{3\times 3}$, the inner product and the associated Frobenius norm are denoted by $\boldsymbol{p} : \boldsymbol{q}$ $(:= \mathrm{trace}(\boldsymbol{p}^\top \boldsymbol{q}))$ and $|\boldsymbol{p}|$, respectively. The symmetrized gradient $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is defined as $\frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$. The distributional time derivative of a function f defined on $\Omega \times (0, T)$ is denoted by $\dot{f}$. Further, we denote by g' the Fréchet derivative of a function g defined on $\mathbb{R}$. The symbol $\partial_{\boldsymbol{q}} D$ stands for the partial convex subdifferential of the dissipation function $D(\boldsymbol{q}, \theta)$. Finally, C denotes a generic nonnegative constant and it is written as $C(\cdot)$ to indicate dependencies.

Now we are able to state our assumptions on the quantities in the thermoviscoplastic model (1.1)–(1.5). We begin with the physical constants and functions. We then proceed to make precise the assumptions on the initial conditions and mechanical and thermal loads, and give the weak formulation of the model. We conclude the section with the assumptions on the domain Ω.

Assumption 2.1.

1. The moduli $\mathbb{C}, \mathbb{H} : \Omega \to \mathrm{Lin}(\mathbb{R}^{3\times 3}_{\text{sym}})$ are
   a) elements of $L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^{3\times 3}_{\text{sym}}))$,
   b) symmetric in the sense that

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij} \qquad \text{and} \qquad \mathbb{H}_{ijkl} = \mathbb{H}_{jikl} = \mathbb{H}_{klij},$$

c) coercive on $\mathbb{R}^{3\times 3}_{\text{sym}}$ with coercivity constants $\underline{c}, \underline{h} > 0$, i.e.

$$\boldsymbol{\varepsilon} : \mathbb{C}(\boldsymbol{x})\, \boldsymbol{\varepsilon} \geq \underline{c}\, |\boldsymbol{\varepsilon}|^2 \text{ and } \boldsymbol{p} : \mathbb{H}(\boldsymbol{x})\, \boldsymbol{p} \geq \underline{h}\, |\boldsymbol{p}|^2$$

for all $\boldsymbol{\varepsilon}, \boldsymbol{p} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ and almost all $\boldsymbol{x} \in \Omega$.

2. The temperature dependent uni-axial yield stress $\sigma_0 : \mathbb{R} \to \mathbb{R}$ is positive and belongs to $W^{1,\infty}(\mathbb{R})$.
3. The temperature dependent dissipation function $D : \mathbb{R}^{3\times 3}_{\text{sym}} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$D(\boldsymbol{q}, \theta) := \sqrt{\frac{2}{3}}\, \sigma_0(\theta)\, |\boldsymbol{q}|.$$

4. The temperature dependent thermal strain function $\boldsymbol{t} : \mathbb{R} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ is
   a) of class $C^2_b(\mathbb{R}, \mathbb{R}^{3\times 3}_{\text{sym}})$ (the space of bounded C^2 functions with bounded derivatives),
   b) such that $\mathbb{R} \ni \theta \mapsto \theta\, \boldsymbol{t}'(\theta) \in \mathbb{R}^{3\times 3}_{\text{sym}}$ is Lipschitz continuous and bounded.
5. The density ϱ, specific heat capacity c_p, thermal conductivity κ and heat transfer coefficient β are positive constants independent of the temperature. W.l.o.g. we set $\varrho\, c_p = 1$ in the analysis.
6. The viscosity parameters ϵ and γ are positive.

Remark 2.2. If the thermal strain $\boldsymbol{t}$ fulfills Assumption 2.1 item 4a and satisfies

$$\boldsymbol{t}(\theta) = \boldsymbol{t}_{-\infty} \text{ if } \theta \leq -M \qquad \text{and} \qquad \boldsymbol{t}(\theta) = \boldsymbol{t}_{\infty} \text{ if } \theta \geq M$$

for some $M > 0$ and matrices $\boldsymbol{t}_{-\infty}$ and $\boldsymbol{t}_{\infty}$ in $\mathbb{R}^{3\times 3}_{\text{sym}}$, then the product $\theta\, \boldsymbol{t}'(\theta)$ is Lipschitz continuous and bounded.

Next we introduce suitable function spaces for the weak formulation of (1.3)–(1.5).

Definition 2.3.

1. We define for $p \geq 2$ the (vector-valued) Sobolev space

$$\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega) := \left\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega; \mathbb{R}^3) : \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_{\mathfrak{D}} \right\}.$$

2. We denote the dual space of $\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$ by $\boldsymbol{W}^{-1,p'}_{\mathfrak{D}}(\Omega)$, where $1/p + 1/p' = 1$.
3. We define for $p \geq 2$ the (matrix-valued) Lebesgue space

$$\boldsymbol{Q}^p(\Omega) := \{ \boldsymbol{q} \in \boldsymbol{L}^p(\Omega; \mathbb{R}^{3\times 3}_{\text{dev}}) \}.$$

The following regularities for the initial conditions and the mechanical and thermal loads are assumed.

Assumption 2.4. Let $p, q \geq 2$ be fixed and define

$$v(p) \begin{cases} = 3p/(6-p) & \text{if } p < 6 \\ \in (\frac{3p}{3+p}, \infty) \text{ arbitrary} & \text{if } p \geq 6. \end{cases} \tag{2.1}$$

1. The initial conditions $\boldsymbol{u}_0$, $\boldsymbol{p}_0$ and θ_0 have regularity

$$\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega), \quad \boldsymbol{p}_0 \in \boldsymbol{Q}^p(\Omega) \quad \text{and} \quad \theta_0 \in W^{1,v(p)}(\Omega).$$

2. The volume and boundary loads $\boldsymbol{\ell}$ and r belong to the spaces

$$\boldsymbol{\ell} \in L^q(0,T; \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)) \quad \text{and} \quad r \in L^{\frac{q}{2}}(0,T; W^{-1,v(p)}_{\diamond}(\Omega)).$$

Remark 2.5. The distinction of cases in the definition of $v(p)$ is due to the Sobolev embedding $L^{\frac{p}{2}}(\Omega) \hookrightarrow W^{-1,v(p)}_{\diamond}(\Omega)$ which becomes saturated for $p \geq 6$.

Now, as the involved function spaces are defined, we are in the position to give a precise notion of (weak) solutions to (1.1)–(1.5).

Definition 2.6 (Weak solution of the thermoviscoplastic system)**.** Let $p, q > 2$. Given initial data and inhomogeneities according to Assumption 2.4, we call a quintuple

$$\begin{gathered} \boldsymbol{u} \in W^{1,q}(0,T; \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)), \quad \boldsymbol{p} \in W^{1,q}(0,T; \boldsymbol{Q}^p(\Omega)), \\ \boldsymbol{\sigma} \in W^{1,q}(0,T; \boldsymbol{L}^p(\Omega)), \quad \boldsymbol{\chi} \in W^{1,q}(0,T; \boldsymbol{L}^p(\Omega)), \\ \theta \in W^{1,\frac{q}{2}}(0,T; W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega)). \end{gathered}$$

a *weak solution* of the thermoviscoplastic system (1.1)–(1.5), if it fulfills for almost all $t \in (0,T)$

stress-strain relation:

$$\boldsymbol{\sigma} = \mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)\big) \quad \text{a.e. in } \Omega, \tag{1.1}$$

conjugate forces:

$$\boldsymbol{\chi} = -\mathbb{H}\,\boldsymbol{p} \quad \text{a.e. in } \Omega, \tag{1.2}$$

viscoplastic flow rule:

$$\epsilon \int_\Omega \dot{\boldsymbol{p}} : (\boldsymbol{q} - \dot{\boldsymbol{p}})\, \mathrm{d}x - \int_\Omega (\boldsymbol{\sigma} + \boldsymbol{\chi}) : (\boldsymbol{q} - \dot{\boldsymbol{p}})\, \mathrm{d}x$$

$$+ \int_\Omega D(\boldsymbol{q},\theta)\,\mathrm{d}\boldsymbol{x} - \int_\Omega D(\dot{\boldsymbol{p}},\theta)\,\mathrm{d}\boldsymbol{x} \geq 0 \quad \text{for all } \boldsymbol{q} \in \boldsymbol{Q}^p(\Omega), \tag{1.3'}$$

balance of momentum:

$$\int_\Omega (\boldsymbol{\sigma} + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{\ell},\, \boldsymbol{v}\rangle \quad \text{for all } \boldsymbol{v} \in \boldsymbol{W}^{1,p'}_{\mathfrak{D}}(\Omega), \tag{1.4'}$$

heat equation:

$$\begin{aligned} &\langle \dot\theta,\, z\rangle + \int_\Omega \kappa\,\nabla\theta\cdot\nabla z\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\,\theta\,z\,\mathrm{d}s \\ &= \langle r,\, z\rangle + \int_\Omega (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}}\,z\,\mathrm{d}\boldsymbol{x} - \int_\Omega \theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}})\,z\,\mathrm{d}\boldsymbol{x} \\ &\quad + \gamma \int_\Omega \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\,z\,\mathrm{d}\boldsymbol{x} \quad \text{for all } z \in W^{1,v(p)'}(\Omega), \end{aligned} \tag{1.5'}$$

along with the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{p}(0) = \boldsymbol{p}_0$, and $\theta(0) = \theta_0$.

Note that the associated strain $\boldsymbol{\sigma}$ and back-stress $\boldsymbol{\chi}$ are determined through $\boldsymbol{u}$, $\boldsymbol{p}$, and θ and can directly be calculated from the pointwise equations in (1.1) and (1.2). Their regularity then follows immediately from Assumption 2.1.

We remark that $W^{1,v(p)'}(\Omega)$ is the dual space to $W^{-1,v(p)}_\diamond(\Omega)'$. Note that the balance of momentum (1.4) is equipped with mixed boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathfrak{D}} \qquad \text{and} \qquad (\gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \boldsymbol{\sigma})\,\boldsymbol{n} = \boldsymbol{s} \quad \text{on } \Gamma_{\mathfrak{N}},$$

where $\boldsymbol{n}$ is the outwards unit normal of Ω. The surface traction forces $\boldsymbol{s}$, together with volume loads, are summarized in $\boldsymbol{\ell}$. Moreover, the heat equation (1.5) is endowed with Robin boundary conditions whose right hand side enters r.

For simplicity, we will refer to (1.3') in the sequel as (1.3) and similarly for (1.4) and (1.5) but always have in mind the weak form of the respective equation.

Finally, we present the assumptions on the domain.

Assumption 2.7.

1. $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ, see, e.g., [Grisvard, 1985, Definition 1.2.1.1]. The boundary Γ is divided into disjoint measurable parts $\Gamma_{\mathfrak{N}}$ and $\Gamma_{\mathfrak{D}}$ such that $\Gamma = \Gamma_{\mathfrak{N}} \,\dot\cup\, \Gamma_{\mathfrak{D}}$. Furthermore, $\Gamma_{\mathfrak{N}}$ is an open and $\Gamma_{\mathfrak{D}}$ is a closed subset of Γ with positive measure.
2. The set $\Omega \cup \Gamma_{\mathfrak{N}}$ is regular in the sense of Gröger [1989], which will be necessary to obtain $\boldsymbol{W}^{1,p}$ regularity (for some $p > 2$) of a solution of (1.4), as well as for the following assumption on maximal parabolic regularity.
3. In addition, the domain Ω is assumed to be smooth enough such that the operator related to (3.2) satisfies maximal parabolic regularity in $W^{-1,v(p)}_\diamond(\Omega)$; for a precise definition see Definition A.2.

Remark 2.8.

1. In 3D, there is no simple characterization for Assumption 2.7 item 2; cf. [Haller-Dintelmann et al., 2009, Theorem 5.4]. For example $\Omega \cup \Gamma_{\mathfrak{N}}$ is regular in the sense of Gröger if $\Omega \subset \mathbb{R}^3$ is a Lipschitzian polyhedron and $\overline{\Gamma}_{\mathfrak{N}} \cap \Gamma_{\mathfrak{D}}$ is a finite union of line segments; see [Haller-Dintelmann et al., 2009, Corollary 5.5].
2. Assumption 2.7 item 3 is not very restrictive because there exists $\hat{v} > 2$ such that the operator related to (3.2) satisfies maximal parabolic regularity in $W_{\diamond}^{-1,v(p)}(\Omega)$ for $\hat{v}' \le v(p) \le \hat{v}$ (where $\hat{v}'$ is the conjugate exponent of $\hat{v}$); cf. Lemma A.4 and Lemma A.5.

Remark 2.9 (Thermodynamical consistency of the model)**.** Postulating the Helmholtz free energy, cf. [Ottosen and Ristinmaa, 2005, eq.(23.5)]

$$\begin{aligned} \psi(\theta, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{p}) = \varrho\, c_p \left(\theta - \theta \ln \theta\right) + \frac{1}{2}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p}) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p}) \\ - \boldsymbol{t}(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p}) + \frac{1}{2}\boldsymbol{p} : \mathbb{H}\, \boldsymbol{p}, \end{aligned} \tag{2.2}$$

we recover the equations (1.1)–(1.4) following the calculations of [Ottosen and Ristinmaa, 2005, Chapter 23]. The heat equation derived by Ottosen and Ristinmaa however differs from ours and reads as follows:

$$\begin{aligned} &\left[\varrho\, c_p + \theta\, \boldsymbol{t}''(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p})\right]\, \dot\theta - \operatorname{div}(\kappa\, \nabla \theta) \\ &\quad = -\frac{\partial^2}{\partial \theta^2}\psi(\theta, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{p})\, \dot\theta - \operatorname{div}(\kappa\, \nabla\theta) \\ &\quad = r + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} - \theta\, \boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}}), \end{aligned} \tag{2.3}$$

cf. [Ottosen and Ristinmaa, 2005, Eq. (23.23)]. This equation differs from (1.5) by the term $\theta\, \boldsymbol{t}''(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p})\, \dot\theta$, which is often small in applications, see [Ottosen and Ristinmaa, 2005, Section 23.2.1]. This observation led us to neglect this term in our heat equation. To check the thermodynamical consistency of our model, define the entropy associated with our thermoviscoplastic system by

$$s = s(\theta, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{p}) = -\frac{\partial}{\partial\theta}\psi(\theta, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{p}) = \varrho\, c_p \ln \theta + \boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p}).$$

Then our simplified heat equation (1.5) can be formulated in terms of the entropy equation as

$$\begin{aligned} \theta \dot{s} - \operatorname{div}(\kappa\, \nabla\theta) &= \varrho\, c_p\, \dot\theta + \theta\, \boldsymbol{t}''(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p})\, \dot\theta - \operatorname{div}(\kappa\, \nabla\theta) \\ &= r + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}}. \end{aligned} \tag{2.4}$$

From (1.3′) and the positive homogeneity of D it follows that $D(\dot{\boldsymbol{p}}, \theta) = (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} - \epsilon\, \dot{\boldsymbol{p}} : \dot{\boldsymbol{p}}$, cf. also [Bartels and Roubíček, 2008, Eq. (2.4)]. Therefore, assuming that the body is thermally isolated, i.e. $\beta = 0$, and $\theta > 0$ (measured in Kelvin), formally testing (2.4) with $1/\theta$ and integrating by parts yield

$$\begin{aligned}\int_\Omega \dot{s}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \frac{\kappa \nabla\theta\cdot\nabla\theta}{\theta^2} + \frac{r}{\theta} + \gamma\,\frac{\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}):\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})}{\theta} + \frac{D(\dot{\boldsymbol{p}},\theta)}{\theta} + \epsilon\,\frac{\dot{\boldsymbol{p}}:\dot{\boldsymbol{p}}}{\theta}\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \frac{\boldsymbol{t}''(\theta):\mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{p})\,\dot{\theta}}{\theta}\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.5}$$

Consequently, if $r \geq 0$, then the Clausius-Duhem inequality $\int_\Omega \dot{s}\,\mathrm{d}\boldsymbol{x} \geq 0$ would follow, provided that the last term in (2.5) is non-negative. Note that this term does not appear in case of (2.3), which implies the thermodynamical consistency of the latter.

To summarize, our model is thermodynamically consistent, if this last term in (2.5) is non-negative. This is the case, in particular, if the thermal strain $\boldsymbol{t}$ depends only linearly on the temperature. In addition, an affine-linear thermal strain function closes the gap between (1.5) and (2.3). However, an affine-linear thermal strain does not fulfill the global boundedness condition in Assumption 2.1, unless it is constant, which substantially simplifies our model.

We wish to point out that this lack of physical rigor of our model can be compensated in an optimization framework. In many application problems, it makes sense to impose pointwise bounds on the temperature in order to avoid destruction of the material. Then one can choose the thermal strain $\boldsymbol{t}$ to be an affine-linear function within the temperature range associated with these bounds and impose a bound on $\boldsymbol{t}$ for temperatures outside this range in order to fulfill Assumption 2.1. In this way we obtain both a thermodynamically consistent model, and the bound for $\boldsymbol{t}$ needed in our analysis implicitly through the restrictions on the temperature. Our model is therefore especially well suited to optimization problems involving pointwise constraints on the temperature as in Problem 4.1.

3. Existence and Uniqueness of a Solution to the Model

In this section we prove the existence and uniqueness of a weak solution to the thermoviscoplastic system (1.1)–(1.5). Let us first re-iterate our main theorem followed by a detailed roadmap of its proof. The major part of this section is a rigorous proof of the main theorem.

Theorem 3.1 (Main Theorem)**.** Suppose that Assumption 2.1 and Assumption 2.7 hold. There exists $\bar{p} > 2$ such that for all $2 < p \leq \bar{p}$, there exists $\bar{q} > 2$ (depending on p) such that for all $\bar{q} \leq q < \infty$ and right hand sides $(\boldsymbol{\ell}, r)$ and initial conditions $(\boldsymbol{u}_0, \boldsymbol{p}_0, \theta_0)$ as in Assumption 2.4, there exists a unique weak solution $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \chi)$ of (1.1)–(1.5) according to Definition 2.6.

Next we present the roadmap of the proof. The conditions required for the indices p and q will be collected in the course of the proof. Throughout, capital Greek letters refer to solution operators of certain equations.

1. We first consider (1.1)–(1.4) for a fixed temperature field $\theta \in L^1(0,T;L^1(\Omega))$ and prove the existence of a unique weak solution $(\boldsymbol{u},\boldsymbol{p})$. This gives rise to the definition of the solution map

$$L^1(0,T;L^1(\Omega)) \ni \theta \mapsto \Lambda(\theta) := (\boldsymbol{u},\boldsymbol{p},\boldsymbol{\sigma},\boldsymbol{\chi}) \in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))\times \\ \times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega)) \times \left[W^{1,q}(0,T;\boldsymbol{L}^p(\Omega))\right]^2. \tag{3.1}$$

   Individual components of this map will be referred to as $\Lambda^{\boldsymbol{u}}(\theta)$ etc. or simply $\boldsymbol{u}(\theta)$. The final result is given in Proposition 3.6.

2. The results of item 1 naturally lead to the definition of a reduced problem for the temperature alone. To show the existence of a unique solution, we apply Banach's fixed point theorem, which requires a number of preparatory steps.

   a) In order to apply maximal parabolic regularity results, we split the temperature field $\theta = \vartheta + \vartheta_{\text{init}}$ into its homogeneous and inhomogeneous parts w.r.t. the initial conditions. They are defined by

$$\langle \dot\vartheta_{\text{init}},\, z\rangle + \int_\Omega \kappa\, \nabla\vartheta_{\text{init}} \cdot \nabla z \,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\, \vartheta_{\text{init}}\, z \,\mathrm{d}s = 0, \tag{3.2a}$$

$$\vartheta_{\text{init}}(0) = \theta_0 \tag{3.2b}$$

   and

$$\begin{aligned} &\langle \dot\vartheta,\, z\rangle + \int_\Omega \kappa\, \nabla\vartheta \cdot \nabla z \,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\, \vartheta\, z \,\mathrm{d}s \\ &\quad = \langle r,\, z\rangle + \int_\Omega \big(\boldsymbol{\sigma}(\vartheta+\vartheta_{\text{init}}) + \boldsymbol{\chi}(\vartheta+\vartheta_{\text{init}})\big) : \dot{\boldsymbol{p}}(\vartheta+\vartheta_{\text{init}})\, z \,\mathrm{d}\boldsymbol{x} \\ &- \int_\Omega (\vartheta+\vartheta_{\text{init}})\, \boldsymbol{t}'(\vartheta+\vartheta_{\text{init}}) : \mathbb{C}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta+\vartheta_{\text{init}})) - \dot{\boldsymbol{p}}(\vartheta+\vartheta_{\text{init}})\big)\, z \,\mathrm{d}\boldsymbol{x} \\ &\qquad + \gamma \int_\Omega \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta+\vartheta_{\text{init}})) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta+\vartheta_{\text{init}}))\, z \,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.3a}$$

$$\vartheta(0) = 0 \tag{3.3b}$$

   for all $z \in W^{1,v(p)'}(\Omega)$ and almost all $t \in (0,T)$. By standard results (see Lemma A.1), ϑ_{init} satisfies

$$\vartheta_{\text{init}} \in W^{1,\infty}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^\infty(0,T;W^{1,v(p)}(\Omega)). \tag{3.4}$$

   b) The right hand side of (3.3a), without the term involving r, defines a map

$$\mathcal{R} : L^q(0,T;L^p(\Omega)) \to L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega)),$$

$$\begin{aligned}\mathcal{R}(\vartheta) := {} & (\boldsymbol{\sigma}(\vartheta + \vartheta_{\text{init}}) + \boldsymbol{\chi}(\vartheta + \vartheta_{\text{init}})) : \dot{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}}) \\ & - (\vartheta + \vartheta_{\text{init}})\, \boldsymbol{t}'(\vartheta + \vartheta_{\text{init}}) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})) - \dot{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}})) \\ & + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})).\end{aligned}$$

In Lemma 3.7 we prove the Lipschitz property of $\mathcal{R}$.

c) We next define the following three maps in order to construct the solution operator of the heat equation with right hand side $\mathcal{R}(\vartheta)$. To complete the right hand side of (3.3a), we define the affine map

$$\begin{gathered}\mathcal{F} : L^{\frac{q}{2}}(0,T; L^{\frac{p}{2}}(\Omega)) \to L^{\frac{q}{2}}(0,T; W_\diamond^{-1,v(p)}(\Omega)), \\ \mathcal{F}(f) := f + r,\end{gathered}$$

composed of an embedding plus an addition of the thermal loads.

The second map

$$\begin{aligned}\Pi : {} & L^{\frac{q}{2}}(0,T; W_\diamond^{-1,v(p)}(\Omega)) \\ & \to W_0^{1,\frac{q}{2}}(0,T; W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega)), \\ & \qquad \Pi(f) := \vartheta\end{aligned}$$

is linear and it is given by the unique solution of

$$\langle \dot{\vartheta}, z \rangle + \int_\Omega \kappa\, \nabla\vartheta \cdot \nabla z \,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\, \vartheta\, z \,\mathrm{d}s = \langle f, z \rangle \tag{3.5}$$

for all $z \in W^{1,v(p)'}(\Omega)$ and almost all $t \in (0,T)$. Here we benefit from maximal parabolic regularity results; see Lemma 3.10.

Finally, we denote by $\mathcal{E}$ the compact embedding

$$\begin{aligned}\mathcal{E} : {} & W_0^{1,\frac{q}{2}}(0,T; W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega)) \\ & \hookrightarrow\!\hookrightarrow C([0,T]; L^p(\Omega)),\end{aligned}$$

see Lemma 3.11, which imposes a lower bound on q.

d) In virtue of the above, we can define the reduced formulation of (1.1)–(1.5) in terms of the temperature alone as a fixed-point problem, $\vartheta = \Theta(\vartheta)$, where

$$\begin{gathered}\Theta : L^\infty(0,T; L^p(\Omega)) \to L^\infty(0,T; L^p(\Omega)), \\ \Theta(\vartheta) := \mathcal{E}\, \Pi\, \mathcal{F}(\mathcal{R}(\vartheta)).\end{gathered}$$

We show in Lemma 3.12 the Lipschitz continuity of Θ.

e) Unfortunately, Θ is not necessarily contractive when defined on the entire time interval $(0,T)$. We therefore split the time interval into smaller parts. The application of the concatenation technique is aggravated by the fact the Lipschitz constant of Θ depends on the initial condition and thus the lengths of the subintervals might degrade. We overcome this problem by considering (1.1)–(1.5) iteratively on a sequence of intervals $[T_{n-1}, T_n]$ of equal lengths and prepend the unique solution already established on $[0, T_{n-1}]$.

3. The fixed-point problem provides a unique solution $\vartheta \in L^\infty(0,T;L^p(\Omega))$. From there, a unique solution $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ as in Theorem 3.1 can be deduced.

The following three subsections are arranged according to the structure above.

3.1. Existence and Uniqueness of a Solution for Given Temperature Field

In order to prove the existence and uniqueness of a weak solution of (1.1)–(1.4) for a given temperature field, we reformulate (1.1)–(1.4) as an ODE and use the Picard-Lindelöf theorem, following Paoli and Petrov [2012]. We start with two lemmas which help to rephrase the balance of momentum (1.4) and the plastic flow rule (1.3).

Lemma 3.2. There exists $\hat{p} > 2$ such that for all $2 \le p \le \hat{p}$ and $\boldsymbol{F} \in \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)$, there exists a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$ of

$$\int_\Omega \gamma\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x = \langle \boldsymbol{F},\, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in \boldsymbol{W}^{1,p'}_{\mathfrak{D}}(\Omega). \tag{3.6}$$

The corresponding solution operator $\Phi^{\boldsymbol{u}} : \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega) \to \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$, $\boldsymbol{F} \mapsto \boldsymbol{u}$ is linear and bounded and satisfies the following estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} = \|\Phi^{\boldsymbol{u}}(\boldsymbol{F})\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} \le C\,\gamma^{-1} \|\boldsymbol{F}\|_{\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)}. \tag{3.7}$$

The Lipschitz constant $C\,\gamma^{-1}$ is independent of $p \in [2, \hat{p}]$.

Proof: The result follows immediately from [Herzog et al., 2011, Theorem 1.1] with $\boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u})) := \gamma\, \boldsymbol{\varepsilon}(\boldsymbol{u})$. Assumption 2.7 item 2 is used here. ☐

Remark 3.3. In the sequel, we use as $\boldsymbol{F}$

$$\langle \boldsymbol{F}(\boldsymbol{\ell}, \boldsymbol{u}, \boldsymbol{p}, \theta),\, \boldsymbol{v} \rangle := \int_\Omega \boldsymbol{\ell} \cdot \boldsymbol{v}\, \mathrm{d}x - \int_\Omega \mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)\big) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x \tag{3.8}$$

with $\boldsymbol{\ell} \in \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)$, $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$, $\boldsymbol{p} \in \boldsymbol{Q}^p(\Omega)$ and $\theta \in L^1(\Omega)$. Then $\boldsymbol{F}(\boldsymbol{\ell}, \boldsymbol{u}, \boldsymbol{p}, \theta) \in \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)$ holds and (3.6) (with $\boldsymbol{u}$ replaced by $\dot{\boldsymbol{u}}$) equals the balance of momentum (1.4) at some fixed point in time.

Next, to handle the plastic flow rule (1.3), let us consider the following variational inequality

$$\epsilon\, \boldsymbol{a} : (\boldsymbol{q} - \boldsymbol{a}) + D(\boldsymbol{q}, \theta) - D(\boldsymbol{a}, \theta) \ge \boldsymbol{f} : (\boldsymbol{q} - \boldsymbol{a}) \quad \text{for all } \boldsymbol{q} \in \mathbb{R}^{3\times 3}_{\text{dev}}. \tag{3.9}$$

and prove its solvability for every fixed right hand side $\boldsymbol{f} \in \mathbb{R}^{3\times3\,\prime}_{\text{sym}} \simeq \mathbb{R}^{3\times3}_{\text{sym}}$ and temperature $\theta \in \mathbb{R}$. Note that since $\boldsymbol{f} : \boldsymbol{q} = [\boldsymbol{f}]^D : \boldsymbol{q}$ for all $\boldsymbol{q} \in \mathbb{R}^{3\times3}_{\text{dev}}$, the solution will depend only on the deviatoric part $[\boldsymbol{f}]^D$ of the right hand side.

Lemma 3.4. For every fixed temperature $\theta \in \mathbb{R}$ and right hand side $\boldsymbol{f} \in \mathbb{R}^{3\times3}_{\text{sym}}$, there exists a unique solution $\boldsymbol{a} \in \mathbb{R}^{3\times3}_{\text{dev}}$ of (3.9) and it fulfills the inequality

$$\epsilon\,|\boldsymbol{a}| \leq |\boldsymbol{f}|. \tag{3.10}$$

Furthermore, the solution operator $\Phi^p : \mathbb{R} \times \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{dev}}$, $(\theta, \boldsymbol{f}) \mapsto \boldsymbol{a}$ is Lipschitz continuous. More precisely,

$$|\Phi^p(\theta_1, \boldsymbol{f}_1) - \Phi^p(\theta_2, \boldsymbol{f}_2)| \leq \epsilon^{-1}|\boldsymbol{f}_1 - \boldsymbol{f}_2| + \epsilon^{-1} L_{\Phi^p}|\theta_1 - \theta_2|, \tag{3.11}$$

where L_{Φ^p} just depends on the Lipschitz constant of the yield stress function σ_0.

Proof: **Existence and uniqueness:** We can use [Han and Reddy, 1999, Theorem 6.6] to obtain a unique solution for every right hand side $\boldsymbol{f} \in \mathbb{R}^{3\times3}_{\text{sym}}$ and fixed temperature θ, because $\boldsymbol{q} \mapsto D(\boldsymbol{q}, \theta) := \sqrt{2/3}\,\sigma_0(\theta)|\boldsymbol{q}|$ is proper, convex and lower semicontinuous.

Estimate: We choose $\boldsymbol{q} = \boldsymbol{0}$ in (3.9) and get

$$-\epsilon\langle \boldsymbol{a}, \boldsymbol{a}\rangle - D(\boldsymbol{a}, \theta) \geq -\langle \boldsymbol{f}, \boldsymbol{a}\rangle \qquad \text{or} \qquad \epsilon\,|\boldsymbol{a}|^2 \leq \langle \boldsymbol{f}, \boldsymbol{a}\rangle - D(\boldsymbol{a}, \theta) \leq |\boldsymbol{f}||\boldsymbol{a}|,$$

where we used that σ_0 is positive.

Lipschitz continuity: We consider $\boldsymbol{a}_1 = \Phi^p(\theta_1, \boldsymbol{f}_1)$ and $\boldsymbol{a}_2 = \Phi^p(\theta_2, \boldsymbol{f}_2)$ and get from (3.9)

$$\begin{aligned} \epsilon\langle \boldsymbol{a}_1, \boldsymbol{q} - \boldsymbol{a}_1\rangle + D(\boldsymbol{q}, \theta_1) - D(\boldsymbol{a}_1, \theta_1) &\geq \langle \boldsymbol{f}_1, \boldsymbol{q} - \boldsymbol{a}_1\rangle \quad \text{for all } \boldsymbol{q} \in \mathbb{R}^{3\times3}_{\text{dev}}, \\ \epsilon\langle \boldsymbol{a}_2, \boldsymbol{q} - \boldsymbol{a}_2\rangle + D(\boldsymbol{q}, \theta_2) - D(\boldsymbol{a}_2, \theta_2) &\geq \langle \boldsymbol{f}_2, \boldsymbol{q} - \boldsymbol{a}_2\rangle \quad \text{for all } \boldsymbol{q} \in \mathbb{R}^{3\times3}_{\text{dev}}. \end{aligned}$$

We choose $\boldsymbol{q} = \boldsymbol{a}_2$ in the first inequality and $\boldsymbol{q} = \boldsymbol{a}_1$ in the second and add both inequalities:

$$\epsilon\,|\boldsymbol{a}_1 - \boldsymbol{a}_2|^2 \leq \langle \boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{a}_1 - \boldsymbol{a}_2\rangle + \sqrt{\frac{2}{3}}\big[\sigma_0(\theta_1) - \sigma_0(\theta_2)\big]\big[|\boldsymbol{a}_1| - |\boldsymbol{a}_2|\big].$$

Using the Cauchy-Schwarz inequality and the Lipschitz continuity of σ_0 we get

$$|\Phi^p(\theta_1, \boldsymbol{f}_1) - \Phi^p(\theta_2, \boldsymbol{f}_2)| = |\boldsymbol{a}_1 - \boldsymbol{a}_2| \leq \epsilon^{-1}|\boldsymbol{f}_1 - \boldsymbol{f}_2| + \epsilon^{-1} L_{\Phi^p}|\theta_1 - \theta_2|.$$

□

Remark 3.5. The inequality (3.9) (with $\boldsymbol{a} = \dot{\boldsymbol{p}}$ and $\boldsymbol{f} = \boldsymbol{\sigma} + \boldsymbol{\chi}$) corresponds to the formulation of the plastic flow rule (1.3) for a certain point in time and space $(t, \boldsymbol{x})$. To see this, substitute $\boldsymbol{q}$ by $(\boldsymbol{q} - \dot{\boldsymbol{p}})\,\varphi + \dot{\boldsymbol{p}}$ with $\varphi \in C_0^\infty(\Omega)$ arbitrary such that $0 \leq \varphi \leq 1$ in (1.3). The fundamental lemma of calculus of variations then yields, for almost all $\boldsymbol{x} \in \Omega$,

$$\epsilon\langle \dot{\boldsymbol{p}}, \boldsymbol{q} - \dot{\boldsymbol{p}}\rangle + D(\boldsymbol{q}, \theta) - D(\dot{\boldsymbol{p}}, \theta) \geq \langle \boldsymbol{\sigma} + \boldsymbol{\chi}, \boldsymbol{q} - \dot{\boldsymbol{p}}\rangle \quad \text{for all } \boldsymbol{q} \in \mathbb{R}^{3\times3}_{\text{dev}}.$$

With the solution operators $\Phi^{\boldsymbol{u}}$ and $\Phi^{\boldsymbol{p}}$ at hand, we can now prove the existence and uniqueness of a solution of (1.1)–(1.4) for a given temperature field.

Proposition 3.6 (Existence and uniqueness for given temperature field)**.** Let $\theta \in L^1(0,T;L^1(\Omega))$, $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$, $\boldsymbol{p}_0 \in \boldsymbol{Q}^p(\Omega)$ and $\boldsymbol{\ell} \in L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega))$ be given; cf. Assumption 2.4. Then there exists a unique weak solution

$$(\boldsymbol{u},\boldsymbol{p},\boldsymbol{\sigma},\boldsymbol{\chi}) \in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega)) \times \left[W^{1,q}(0,T;\boldsymbol{L}^p(\Omega))\right]^2$$

of (1.1)–(1.4) in the sense of Definition 2.6 with $p \in [2,\hat{p}]$ and $1 < q < \infty$ where $\hat{p} > 2$ is determined by Lemma 3.2. Furthermore, the solution operator $\Lambda : \theta \mapsto (\boldsymbol{u},\boldsymbol{p},\boldsymbol{\sigma},\boldsymbol{\chi})$ fulfills the following two properties.

1. The solution operator $\Lambda|_{L^q(0,T;L^p(\Omega))}$ is Lipschitz continuous with Lipschitz constant $L_\Lambda(T)$.
2. The image of Λ is bounded by $C(T,\boldsymbol{u}_0,\boldsymbol{p}_0)$ independently of the temperature θ.

Proof: **Existence:** We can rewrite the balance of momentum (1.4) and the plastic flow rule (1.3) by means of the solution operators $\Phi^{\boldsymbol{u}}$ and $\Phi^{\boldsymbol{p}}$ defined in Lemma 3.2 and Lemma 3.4, respectively, cf. Remark 3.3 and Remark 3.5. Therefore we obtain the Banach space-valued ODE system

$$\begin{pmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} \Phi^{\boldsymbol{u}}(\boldsymbol{F}(\boldsymbol{\ell},\boldsymbol{u},\boldsymbol{p},\theta)) \\ \Phi^{\boldsymbol{p}}(\theta,\Phi^{\boldsymbol{\sigma}}(\boldsymbol{u},\boldsymbol{p},\theta)+\Phi^{\boldsymbol{\chi}}(\boldsymbol{u},\boldsymbol{p},\theta)) \end{pmatrix} =: \Phi^{\boldsymbol{up}}(\boldsymbol{u},\boldsymbol{p}). \tag{3.12}$$

The maps $\Phi^{\boldsymbol{\sigma}}$ and $\Phi^{\boldsymbol{\chi}}$ are defined by the algebraic relations (1.1) and (1.2), respectively. Note that the right hand side is non-autonomous since $\boldsymbol{\ell}$ and θ depend on time. It follows from (3.7) and the pointwise estimate of (3.10) that $\Phi^{\boldsymbol{up}}$ maps $L^q(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^q(0,T;\boldsymbol{Q}^p(\Omega))$ into itself.

To apply the Picard-Lindelöf theorem we show that $\Phi^{\boldsymbol{up}}$ is Lipschitz continuous uniformly in time. More precisely, we show the estimate

$$\begin{aligned}\|\Phi^{\boldsymbol{up}}(\boldsymbol{u}_1,\boldsymbol{p}_1)-\Phi^{\boldsymbol{up}}(\boldsymbol{u}_2,\boldsymbol{p}_2)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}& \\ \leq C\,\|(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{p}_1-\boldsymbol{p}_2)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}&\end{aligned}$$

for all $\boldsymbol{u}_1,\boldsymbol{u}_2 \in \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$ and $\boldsymbol{p}_1,\boldsymbol{p}_2 \in \boldsymbol{Q}^p(\Omega)$, where C is independent of the time t. We calculate

$$\begin{aligned}&\|\Phi^{\boldsymbol{up}}(\boldsymbol{u}_1,\boldsymbol{p}_1)-\Phi^{\boldsymbol{up}}(\boldsymbol{u}_2,\boldsymbol{p}_2)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)} \\ &\quad= \|\Phi^{\boldsymbol{u}}(\boldsymbol{F}(\boldsymbol{\ell},\boldsymbol{u}_1,\boldsymbol{p}_1,\theta))-\Phi^{\boldsymbol{u}}(\boldsymbol{F}(\boldsymbol{\ell},\boldsymbol{u}_2,\boldsymbol{p}_2,\theta))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} \\ &\qquad+\|\Phi^{\boldsymbol{p}}(\theta,\Phi^{\boldsymbol{\sigma}}(\boldsymbol{u}_1,\boldsymbol{p}_1,\theta)+\Phi^{\boldsymbol{\chi}}(\boldsymbol{u}_1,\boldsymbol{p}_1,\theta)) \\ &\qquad\quad-\Phi^{\boldsymbol{p}}(\theta,\Phi^{\boldsymbol{\sigma}}(\boldsymbol{u}_2,\boldsymbol{p}_2,\theta)+\Phi^{\boldsymbol{\chi}}(\boldsymbol{u}_2,\boldsymbol{p}_2,\theta))\|_{\boldsymbol{Q}^p(\Omega)}\end{aligned}$$

$$\overset{(3.7),(3.11)}{\leq} C\,\gamma^{-1}\|\mathbb{C}(\varepsilon(\boldsymbol{u}_1)-\boldsymbol{p}_1)-\mathbb{C}(\varepsilon(\boldsymbol{u}_2)-\boldsymbol{p}_2)\|_{L^p(\Omega)}$$
$$+\,\epsilon^{-1}\|\Phi^\sigma(\boldsymbol{u}_1,\boldsymbol{p}_1,\theta)+\Phi^\chi(\boldsymbol{u}_1,\boldsymbol{p}_1,\theta)-\Phi^\sigma(\boldsymbol{u}_2,\boldsymbol{p}_2,\theta)-\Phi^\chi(\boldsymbol{u}_2,\boldsymbol{p}_2,\theta)\|_{L^p(\Omega)}.$$

Now we use the properties of $\mathbb{C},\mathbb{H}$ (Assumption 2.1) and the definitions of Φ^σ and Φ^χ to get

$$\|\Phi^{\boldsymbol{up}}(\boldsymbol{u}_1,\boldsymbol{p}_1)-\Phi^{\boldsymbol{up}}(\boldsymbol{u}_2,\boldsymbol{p}_2)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}$$
$$\leq C(\gamma^{-1}+\epsilon^{-1})\|(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{p}_1-\boldsymbol{p}_2)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}.$$

Therefore, we obtain from the Picard-Lindelöf theorem (see [Gajewski et al., 1974, Chapter V, Lemma 1.5] for the case $p=2$) a unique solution $(\Lambda^{\boldsymbol{u}}(\theta),\Lambda^{\boldsymbol{p}}(\theta))=(\boldsymbol{u}(\theta),\boldsymbol{p}(\theta))\in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))\times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))$. For the remaining two components we set

$$\Lambda^\sigma(\theta):=\Phi^\sigma(\boldsymbol{u}(\theta),\boldsymbol{p}(\theta),\theta)=\mathbb{C}\big(\varepsilon(\boldsymbol{u}(\theta))-\boldsymbol{p}(\theta)-\boldsymbol{t}(\theta)\big),$$
$$\Lambda^\chi(\theta):=\Phi^\chi(\boldsymbol{u}(\theta),\boldsymbol{p}(\theta),\theta)=-\mathbb{H}\,\boldsymbol{p}(\theta)$$

and define the solution operator Λ of (1.1)–(1.4) as $\Lambda=(\Lambda^{\boldsymbol{u}},\Lambda^{\boldsymbol{p}},\Lambda^\sigma,\Lambda^\chi)$.

Lipschitz continuity: Let $\theta_i\in L^q(0,T;L^p(\Omega))$ and $(\boldsymbol{u}_i,\boldsymbol{p}_i,\sigma_i,\chi_i):=\Lambda(\theta_i)$ for $i=1,2$. First we integrate (3.12) and calculate with the same argument as above

$$\|(\boldsymbol{u}_1(t),\boldsymbol{p}_1(t))-(\boldsymbol{u}_2(t),\boldsymbol{p}_2(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}$$
$$\leq\int_0^t\|\Phi^{\boldsymbol{u}}(\boldsymbol{F}(\boldsymbol{\ell},\boldsymbol{u}_1,\boldsymbol{p}_1,\theta_1))-\Phi^{\boldsymbol{u}}(\boldsymbol{F}(\boldsymbol{\ell},\boldsymbol{u}_2,\boldsymbol{p}_2,\theta_2))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)}\,\mathrm{d}s$$
$$+\int_0^t\|\Phi^{\boldsymbol{p}}(\theta_1,\sigma_1+\chi_1)-\Phi^{\boldsymbol{p}}(\theta_2,\sigma_2+\chi_2)\|_{\boldsymbol{Q}^p(\Omega)}\,\mathrm{d}s$$
$$\leq C(\gamma^{-1}+\epsilon^{-1})\int_0^t\|(\boldsymbol{u}_1,\boldsymbol{p}_1)-(\boldsymbol{u}_2,\boldsymbol{p}_2)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}\,\mathrm{d}s$$
$$+\,C(\gamma^{-1}+\epsilon^{-1})\int_0^t\|\theta_1-\theta_2\|_{L^p(\Omega)}\,\mathrm{d}s.$$

Next we obtain from Gronwall's lemma and Hölder's inequality

$$\|(\boldsymbol{u}_1(t),\boldsymbol{p}_1(t))-(\boldsymbol{u}_2(t),\boldsymbol{p}_2(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}$$
$$\leq C(\gamma^{-1}+\epsilon^{-1})\int_0^t\|\theta_1-\theta_2\|_{L^p(\Omega)}\,\mathrm{d}s\,\mathrm{e}^{C(\gamma^{-1}+\epsilon^{-1})\,t}$$
$$\leq C(\gamma^{-1}+\epsilon^{-1})\|\theta_1-\theta_2\|_{L^q(0,t;L^p(\Omega))}\,t^{\frac{q-1}{q}}\,\mathrm{e}^{C(\gamma^{-1}+\epsilon^{-1})\,t}.$$

Further, we infer again as above

$$\|(\dot{\boldsymbol{u}}_1(t),\dot{\boldsymbol{p}}_1(t))-(\dot{\boldsymbol{u}}_2(t),\dot{\boldsymbol{p}}_2(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}$$
$$\leq C(\gamma^{-1}+\epsilon^{-1})\|(\boldsymbol{u}_1(t),\boldsymbol{p}_1(t))-(\boldsymbol{u}_2(t),\boldsymbol{p}_2(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times\boldsymbol{Q}^p(\Omega)}$$

$$+ C(\gamma^{-1} + \epsilon^{-1})\|\theta_1(t) - \theta_2(t)\|_{L^p(\Omega)}$$

and together we conclude

$$\begin{aligned}
&\|(\Lambda^{\boldsymbol{u}}(\theta_1), \Lambda^{\boldsymbol{p}}(\theta_1)) - (\Lambda^{\boldsymbol{u}}(\theta_2), \Lambda^{\boldsymbol{p}}(\theta_2))\|_{W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))\times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))} \\
&\quad = \|(\boldsymbol{u}_1, \boldsymbol{p}_1) - (\boldsymbol{u}_2, \boldsymbol{p}_2)\|_{W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))\times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))} \\
&\quad \le C(T, q, \gamma^{-1}, \epsilon^{-1})\|\theta_1 - \theta_2\|_{L^q(0,T;L^p(\Omega))}.
\end{aligned}$$

The Lipschitz continuity of Λ^{σ} and Λ^{χ} is clear.

Boundedness: Finally, we have to show that the image of Λ is bounded. We prove this with the same techniques as above. Let $\theta \in L^1(0,T;L^1(\Omega))$ and $(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{\sigma}, \boldsymbol{\chi}) := \Lambda(\theta)$. First we integrate (3.12) and calculate

$$\begin{aligned}
&\|(\boldsymbol{u}(t), \boldsymbol{p}(t)) - (\boldsymbol{u}_0, \boldsymbol{p}_0)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{Q}^p(\Omega)} \\
&\quad \le \int_0^t \|\Phi^{\boldsymbol{u}}(\boldsymbol{F}(\boldsymbol{\ell}, \boldsymbol{u}, \boldsymbol{p}, \theta))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} \,\mathrm{d}s + \int_0^t \|\Phi^{\boldsymbol{p}}(\theta, \boldsymbol{\sigma} + \boldsymbol{\chi})\|_{\boldsymbol{Q}^p(\Omega)} \,\mathrm{d}s \\
&\quad \le C(\gamma^{-1} + \epsilon^{-1}) \int_0^t \|(\boldsymbol{u}, \boldsymbol{p})\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} \,\mathrm{d}s + C\gamma^{-1} \int_0^t \|\boldsymbol{\ell}\|_{\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)} \,\mathrm{d}s \\
&\quad\quad + C(\gamma^{-1} + \epsilon^{-1})\, t,
\end{aligned}$$

where we used the estimates (3.7), (3.10) and the boundedness of the thermal strain $\boldsymbol{t}$. It follows from Gronwall's lemma that

$$\begin{aligned}
&\|(\boldsymbol{u}(t), \boldsymbol{p}(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{Q}^p(\Omega)} \\
&\quad \le \left[C\gamma^{-1} \int_0^t \|\boldsymbol{\ell}\|_{\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)} \,\mathrm{d}s + C(\gamma^{-1} + \epsilon^{-1})\, t \right] \mathrm{e}^{C(\gamma^{-1}+\epsilon^{-1})\, t} \\
&\quad\quad + \|(\boldsymbol{u}_0, \boldsymbol{p}_0)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{Q}^p(\Omega)} \,\mathrm{e}^{C(\gamma^{-1}+\epsilon^{-1})\, t}
\end{aligned}$$

holds. Therefore, we conclude that $(\boldsymbol{u}, \boldsymbol{p})$ is bounded in $L^\infty(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^\infty(0,T;\boldsymbol{Q}^p(\Omega))$ independently of $\theta \in L^1(0,T;L^1(\Omega))$. In the second step we use again (3.12) and calculate for almost all $t \in (0,T)$ with the same techniques as above

$$\begin{aligned}
\|(\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{p}}(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{Q}^p(\Omega)} &\le C(\gamma^{-1} + \epsilon^{-1})\|(\boldsymbol{u}(t), \boldsymbol{p}(t))\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{Q}^p(\Omega)} \\
&\quad + C\gamma^{-1}\|\boldsymbol{\ell}(t)\|_{\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)} + C(\gamma^{-1} + \epsilon^{-1}).
\end{aligned}$$

Now we conclude that $(\dot{\boldsymbol{u}}, \dot{\boldsymbol{p}})$ is bounded in $L^q(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^q(0,T;\boldsymbol{Q}^p(\Omega))$ independently of $\theta \in L^1(0,T;L^1(\Omega))$ and together with the first step we obtain the boundedness of $(\Lambda^{\boldsymbol{u}}(\theta), \Lambda^{\boldsymbol{p}}(\theta)) = (\boldsymbol{u}(\theta), \boldsymbol{p}(\theta))$ in $W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))$. The boundedness of Λ^{σ} and Λ^{χ} is clear. □

3.2. Results for the Reduced Model

In this subsection we prove the Lipschitz continuity and the contractivity (on small time intervals) of the fixed point mapping Θ in order to apply the Banach fixed point theorem. A subsequent concatenation argument then yields a unique weak solution (1.1)–(1.5) on the entire time interval. As was already mentioned in the roadmap, the fixed point mapping $\Theta = \mathcal{E}\,\Pi\,\mathcal{F}\mathcal{R}$ is a combination of four individual mappings and therefore we start with proving some properties for these. For the purpose of the concatenation argument, we will consider the fixed point operator Θ and the operators involved in its definition on different time intervals of the form $[0, S]$ with $S \in (0, T]$. These operators will be denoted by the subscript S.

We begin with the right hand side $\mathcal{R}$ of the homogeneous part of the temperature equation (1.5), see (3.3).

Lemma 3.7. Suppose $2 < p \le \hat{p}$ (determined by Lemma 3.2), $2 < q < \infty$, and $S \in (0, T]$. Then the mapping $\mathcal{R}_S : L^q(0, S; L^p(\Omega)) \to L^{\frac{q}{2}}(0, S; L^{\frac{p}{2}}(\Omega))$, defined by

$$\begin{aligned}\mathcal{R}_S(\vartheta) := {}& (\boldsymbol{\sigma}(\vartheta + \vartheta_{\text{init}}) + \boldsymbol{\chi}(\vartheta + \vartheta_{\text{init}})) : \dot{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}}) \\ & - (\vartheta + \vartheta_{\text{init}})\, \boldsymbol{t}'(\vartheta + \vartheta_{\text{init}}) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})) - \dot{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}})) \\ & + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}}))\end{aligned}$$

is Lipschitz continuous. Moreover, the Lipschitz constant $L_{\mathcal{R}}(\boldsymbol{u}_0, \boldsymbol{p}_0)$ can be chosen independently of S.

Proof: Take $\vartheta_1, \vartheta_2 \in L^q(0, S; L^p(\Omega))$ and set $\theta_1 = \vartheta_1 + \vartheta_{\text{init}}$ and $\theta_2 = \vartheta_2 + \vartheta_{\text{init}}$ respectively. Notice that, according to (3.4) and Corollary A.7, ϑ_{init} satisfies

$$\vartheta_{\text{init}} \in W^{1,\infty}(0, S; W_\diamond^{-1,v(p)}(\Omega)) \cap L^\infty(0, S; W^{1,v(p)}(\Omega)) \hookrightarrow C([0, S]; L^p(\Omega)).$$

We calculate

$$\begin{aligned}\|\mathcal{R}_S(\vartheta_1) - \mathcal{R}_S(\vartheta_2)\|_{L^{\frac{q}{2}}(0,S;L^{\frac{p}{2}}(\Omega))} \\ \le{}& \|\boldsymbol{\sigma}(\theta_1) - \boldsymbol{\sigma}(\theta_2) + \boldsymbol{\chi}(\theta_1) - \boldsymbol{\chi}(\theta_2)\|_{L^q(0,S;L^p(\Omega))} \|\dot{\boldsymbol{p}}(\theta_1)\|_{L^q(0,S;L^p(\Omega))} \\ & + \|\boldsymbol{\sigma}(\theta_2) + \boldsymbol{\chi}(\theta_2)\|_{L^q(0,S;L^p(\Omega))} \|\dot{\boldsymbol{p}}(\theta_1) - \dot{\boldsymbol{p}}(\theta_2)\|_{L^q(0,S;L^p(\Omega))} \\ & + \|\theta_1 \boldsymbol{t}'(\theta_1) - \theta_2 \boldsymbol{t}'(\theta_2)\|_{L^q(0,S;L^p(\Omega))} \|\mathbb{C}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_1)) - \dot{\boldsymbol{p}}(\theta_1)\big)\|_{L^q(0,S;L^p(\Omega))} \\ & + \|\theta_2 \boldsymbol{t}'(\theta_2)\|_{L^q(0,S;L^p(\Omega))} \|\mathbb{C}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_1)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_2))\big)\|_{L^q(0,S;L^p(\Omega))} \\ & + \|\theta_2 \boldsymbol{t}'(\theta_2)\|_{L^q(0,S;L^p(\Omega))} \|\mathbb{C}\big(\dot{\boldsymbol{p}}(\theta_1) - \dot{\boldsymbol{p}}(\theta_2)\big)\|_{L^q(0,S;L^p(\Omega))} \\ & + \gamma\, \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_1)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_2))\|_{L^q(0,S;L^p(\Omega))} \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_1))\|_{L^q(0,S;L^p(\Omega))} \\ & + \gamma\, \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_2))\|_{L^q(0,S;L^p(\Omega))} \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_1)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\theta_2))\|_{L^q(0,S;L^p(\Omega))} \\ \le{}& C(S, q, \gamma^{-1}, \epsilon^{-1}, \boldsymbol{u}_0, \boldsymbol{p}_0) \|\theta_1 - \theta_2\|_{L^q(0,S;L^p(\Omega))}\end{aligned}$$

$$=: L_{\mathcal{R}}(\boldsymbol{u}_0, \boldsymbol{p}_0; S) \|\vartheta_1 - \vartheta_2\|_{L^q(0,S;L^p(\Omega))}.$$

Here we have used the Lipschitz continuity and boundedness of the solution operator Λ (see Proposition 3.6) and of the mapping $\theta\, \boldsymbol{t}'(\theta)$ (see Assumption 2.1). Moreover, the proof of Proposition 3.6 shows that the Lipschitz constant of Λ increases monotonically with S. Therefore, $L_{\mathcal{R}}(\boldsymbol{u}_0, \boldsymbol{p}_0; S)$ can be bounded by a Lipschitz constant $L_{\mathcal{R}}(\boldsymbol{u}_0, \boldsymbol{p}_0)$ independent of S. □

Remark 3.8. We point out that the Lipschitz constant $L_{\mathcal{R}}$ depends on $\boldsymbol{u}_0$ and $\boldsymbol{p}_0$. As already indicated in item 2e of the roadmap, this issue aggravates the application of the concatenation argument in the proof of Proposition 3.15 below.

Furthermore, we have the following properties for the other required mappings, which are easy to verify.

Lemma 3.9. Suppose $2 < p, q < \infty$ and $S \in (0, T]$. Then the affine mapping

$$\mathcal{F}_S : L^{\frac{q}{2}}(0, S; L^{\frac{p}{2}}(\Omega)) \to L^{\frac{q}{2}}(0, S; W_\diamond^{-1,v(p)}(\Omega)), \quad f \mapsto f + r$$

(defined via the embedding) is Lipschitz continuous with some Lipschitz constant $L_{\mathcal{F}}$ independent of S.

Proof: see Remark 2.5 □

Next we consider the Lipschitz continuity of the solution operator Π of the heat equation (3.5) with a general right hand side. Notice that we benefit from maximal parabolic regularity results at this point.

Lemma 3.10. Suppose $2 < p < \infty$ such that $v(p) \le \hat{v}$ (determined by Lemma A.4) and $2 < q < \infty$ hold. Moreover, let $S \in (0, T]$. Then the solution operator

$$\Pi_S : L^{\frac{q}{2}}(0, S; W_\diamond^{-1,v(p)}(\Omega)) \to W_0^{1,\frac{q}{2}}(0, S; W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0, S; W^{1,v(p)}(\Omega))$$
$$f \mapsto \vartheta$$

related to (3.5), considered as an equation on $[0, S]$, is linear and bounded, i.e., it satisfies the following estimate,

$$\|\vartheta\|_{W_0^{1,\frac{q}{2}}(0,S;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,S;W^{1,v(p)}(\Omega))} \le L_\Pi \|f\|_{L^{\frac{q}{2}}(0,S;W_\diamond^{-1,v(p)}(\Omega))}. \tag{3.13}$$

The Lipschitz constant L_Π can be chosen independently of S.

Proof: For the first statement we benefit from Assumption 2.7 item 3; cf. Remark 2.8. Notice that Assumption 2.7 item 3 is satisfied in the case $\hat{v}' \le v(p) \le \hat{v}$ anyway. Only in the case $\frac{3}{2} < v(p) < \hat{v}' < 2$ does it constitute an additional assumption. The estimate follows easily from the closed graph theorem since the operator A related to (3.5) is closed.

To verify the independence of L_Π of the length of the time interval, let $0 < S' < S$ and $f \in L^{\frac{q}{2}}(0, S'; W_\diamond^{-1,v(p)}(\Omega))$ be given. Furthermore, define the shifted (right-aligned) extension by zero,

$$(If)(t) := \begin{cases} 0 & \text{for } 0 \le t \le S - S', \\ f(t-(S-S')) & \text{for } S - S' < t \le S. \end{cases}$$

Using the identity $\Pi_S \, If = I \, \Pi_{S'} f$, one sees that the Lipschitz constant does not increase when the interval length shrinks so that L_Π is uniformly bounded in S by the constant associated with T; compare [Hömberg et al., 2009/10, Lemma 3.16 (i)]. □

The next result concerns the embedding into spaces of continuous functions in time.

Lemma 3.11. Suppose $2 < p < \infty$ and $2 < q$ (depending on p) sufficiently large, and let $S \in (0, T]$ be arbitrary. Then the embedding

$$\mathcal{E}_S : W_0^{1,\frac{q}{2}}(0, S; W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0, S; W^{1,v(p)}(\Omega)) \to C([0, S]; L^p(\Omega))$$
$$f \mapsto f$$

is continuous. Its Lipschitz constant $L_\mathcal{E}$ can be chosen independently of S.

Proof: We refer the reader to Corollary A.7 for the embedding and the precise link between p and q. Similarly to the proof of Lemma 3.10, one shows that the Lipschitz constant does not increase when the interval length shrinks by employing the shifted extension by zero, cf. [Hömberg et al., 2009/10, Lemma 3.16 (ii)]. □

With these lemmas we are now able to prove the Lipschitz continuity of the fixed point operator Θ.

Lemma 3.12. Suppose $2 < p \le \hat{p}$ (determined by Lemma 3.2) such that $v(p) \le \hat{v}$ (determined by Lemma A.4) and $2 < q < \infty$ (depending on p) sufficiently large. Let moreover $S \in (0, T]$ be arbitrary. Then the mapping $\Theta_S : L^\infty(0, S; L^p(\Omega)) \to L^\infty(0, S; L^p(\Omega))$, $\Theta_S = \mathcal{E}_S \, \Pi_S \, \mathcal{F}_S \mathcal{R}_S$, is Lipschitz continuous, and it satisfies

$$\|\Theta_S(\vartheta_1) - \Theta_S(\vartheta_2)\|_{L^\infty(0,S;L^p(\Omega))} \le L_\Theta \, S^{\frac{1}{q}} \, \|\vartheta_1 - \vartheta_2\|_{L^\infty(0,S;L^p(\Omega))}$$

for all $\vartheta_1, \vartheta_2 \in L^\infty(0, S; L^p(\Omega))$. Hence the Lipschitz constant becomes arbitrarily small for sufficiently small $S > 0$.

Proof: Choose $\vartheta_1, \vartheta_2 \in L^\infty(0, S; L^p(\Omega))$. We use Lemma 3.7 through Lemma 3.11 and the Hölder inequality to obtain the following estimate.

$$\|\Theta_S(\vartheta_1) - \Theta_S(\vartheta_2)\|_{L^\infty(0,S;L^p(\Omega))} \le L_\mathcal{E} L_\Pi L_\mathcal{F} L_\mathcal{R} \|\vartheta_1 - \vartheta_2\|_{L^q(0,S;L^p(\Omega))}$$

$$\leq \underbrace{L_{\mathcal{E}} L_{\Pi} L_{\mathcal{F}} L_{\mathcal{R}}}_{=:L_\Theta} S^{\frac{1}{q}} \|\vartheta_1 - \vartheta_2\|_{L^\infty(0,S;L^p(\Omega))}$$

with $2 < q < \infty$ sufficiently large such that the embedding $\mathcal{E}$ is valid, cf. Corollary A.7. Notice that L_Θ can be chosen independently of S, compare Lemma 3.7, Lemma 3.9, Lemma 3.10 and Lemma 3.11. □

As an immediate consequence we obtain the following

Corollary 3.13. Suppose $2 < p \leq \hat{p}$ (determined by Lemma 3.2) such that $v(p) \leq \hat{v}$ (determined by Lemma A.4). Then the mapping $\Theta_S : L^\infty(0,S;L^p(\Omega)) \to L^\infty(0,S;L^p(\Omega))$ is contractive for S sufficiently small.

The Banach fixed point theorem, together with a careful concatenation argument, shows the main result of item 2 of the roadmap, as we will see in the sequel. For this purpose, let us define the following: given an interval $J \subseteq [0,S]$ with $0 < S \leq T$ we denote by

$$\chi_J : L^\infty(0,S;L^p(\Omega)) \to L^\infty(J;L^p(\Omega))$$

the restriction of a function in $L^\infty(0,S;L^p(\Omega))$ to the interval J. Then, by the semi-group properties of the involved operators, we have the following elementary result:

Lemma 3.14. Let $0 < T_1 < T_2 \leq T$ be given. Then, for every $\vartheta \in L^\infty(0,T_2;L^p(\Omega))$, it holds

$$\chi_{[0,T_1]}\Theta_{T_2}(\vartheta) = \Theta_{T_1}(\chi_{[0,T_1]}\vartheta).$$

Proposition 3.15. Suppose $2 < p \leq \hat{p}$ (determined by Lemma 3.2) such that $v(p) \leq \hat{v}$ (determined by Lemma A.4) holds. Then the mapping

$$\Theta : L^\infty(0,T;L^p(\Omega)) \to L^\infty(0,T;L^p(\Omega)), \quad \Theta := \Theta_T = \mathcal{E}_T \Pi_T \mathcal{F}_T \mathcal{R}_T$$

has a unique fixed point.

Proof: We begin by fixing a value of q satisfying the conditions of Lemma 3.11.

Existence of a fixed point on a small time interval $[0,T_1]$**:** On account of the Banach fixed point theorem, Corollary 3.13 yields a unique fixed point $\vartheta_{T_1} \in L^\infty(0,T_1;L^p(\Omega))$ of Θ_{T_1}, provided that T_1 is sufficiently small.

Concatenation argument: We split the time interval $[0,T]$ into N parts of equal length T_1 and define $T_n := nT_1$ for $n = 1,\ldots,N$ where $T_N = T$. (It is clear that T/T_1 can be made integer by slighting reducing T_1 if necessary.)

We use an induction argument to conclude the existence of a unique fixed point ϑ_{T_n} for Θ_{T_n}, provided that the existence of a unique fixed point $\vartheta_{T_{n-1}}$ for $\Theta_{T_{n-1}}$ has already been established. In what follows, we denote by $f * g$ the concatenation of the functions f and g defined on neighboring time intervals.

Let $\vartheta_{T_{n-1}}$ be the unique fixed point for $\Theta_{T_{n-1}}$. Consider the mapping

$$\begin{aligned}\mathcal{K}_n &: L^\infty(T_{n-1}, T_n; L^p(\Omega)) \to L^\infty(T_{n-1}, T_n; L^p(\Omega)),\\ \mathcal{K}_n &: f \mapsto \chi_{[T_{n-1},T_n]}\Theta_{T_n}(\vartheta_{T_{n-1}} * f).\end{aligned}$$

The mapping $\mathcal{K}_n$ is contractive because we obtain, for $f_1, f_2 \in L^\infty(T_{n-1}, T_n; L^p(\Omega))$, with calculations similar as in the proof of Lemma 3.12, the estimate

$$\begin{aligned}&\|\mathcal{K}_n(f_1) - \mathcal{K}_n(f_2)\|_{L^\infty(T_{n-1},T_n;L^p(\Omega))}\\ &\quad\le \|\Theta_{T_n}(\vartheta_{T_{n-1}} * f_1) - \Theta_{T_n}(\vartheta_{T_{n-1}} * f_2)\|_{L^\infty(0,T_n;L^p(\Omega))}\\ &\quad\le L_\mathcal{E} L_\Pi L_\mathcal{F} L_\mathcal{R} \|\vartheta_{T_{n-1}} * f_1 - \vartheta_{T_{n-1}} * f_2\|_{L^q(0,T_n;L^p(\Omega))}\\ &\quad= L_\mathcal{E} L_\Pi L_\mathcal{F} L_\mathcal{R} \|f_1 - f_2\|_{L^q(T_{n-1},T_n;L^p(\Omega))}\\ &\quad\le \underbrace{L_\mathcal{E} L_\Pi L_\mathcal{F} L_\mathcal{R} T_1^q}_{<1 \text{ by assumption}} \|f_1 - f_2\|_{L^\infty(T_{n-1},T_n;L^p(\Omega))}.\end{aligned}$$

Therefore this auxiliary mapping has a unique fixed point $f \in L^\infty(T_{n-1}, T_n; L^p(\Omega))$ and we define $\vartheta_{T_n} := \vartheta_{T_{n-1}} * f \in L^\infty(0, T_n; L^p(\Omega))$.

It remains to show that ϑ_{T_n}, obtained by concatenation, is indeed the unique fixed point of Θ_{T_n}. Using the induction hypothesis, Lemma 3.14, and the result above, we find

$$\begin{aligned}\Theta_{T_n}(\vartheta_{T_n}) &= \begin{cases}\Theta_{T_{n-1}}(\vartheta_{T_{n-1}}) & \text{for } t \in [0, T_{n-1}],\\ \Theta_{T_n}(\vartheta_{T_{n-1}} * f) & \text{for } t \in [T_{n-1}, T_n]\end{cases}\\ &= \begin{cases}\vartheta_{T_{n-1}} & \text{for } t \in [0, T_{n-1}],\\ f & \text{for } t \in [T_{n-1}, T_n].\end{cases}\end{aligned}$$

This shows that ϑ_{T_n} is a fixed point of Θ_{T_n}. The uniqueness follows from the uniqueness on both subintervals, and the induction step is complete.

When $n = N$ is reached, the assertion is proved since $\Theta = \Theta_{T_N}$.

Note that, thanks to Lemma 3.14, the unique solution pertaining to a particular partitioning of $[0, T]$ is also the unique solution on any refinement of that partitioning. Since any two partitions of $[0, T]$ can always be refined to form a common partition, it follows that the solution must actually be independent of the choice of the subintervals.

□

Remark 3.16. Note that the above proof differs from the classical concatenation argument in the proof of the theorem of Picard-Lindelöf, since the Lipschitz constant of $\mathcal{R}$ and thus L_Θ in Lemma 3.12 depend on the initial values $(\boldsymbol{u}_0, \boldsymbol{p}_0)$, cf. Remark 3.8. This is the reason why we define the operator $\mathcal{K}_n$ by starting the time evolution in $t = 0$.

3.3. Proof of the Main Theorem

Now we are in the position to prove our main Theorem 3.1. From now on, we consider Θ and the operators involved in its definition only on $[0, T]$ so that the subscript T can be left out.

Proof of Theorem 3.1: First we set $\bar{p} := \max\{p \leq \hat{p} : v(p) \leq \hat{v}\}$, where $\hat{p}$ and $\hat{v}$ are determined by Lemma 3.2 and Lemma A.4, respectively. We collect the results proven in Section 3.1 and Section 3.2 in order to show item 3 in the roadmap given at the beginning of Section 3.

So far (at the end of item 2e) in the roadmap, we have established the existence of a unique fixed point ϑ of $\Theta = \mathcal{E}\,\Pi\,\mathcal{F}\mathcal{R}$. Leaving out the embedding $\mathcal{E}$, we obtain $\vartheta = \Pi\,\mathcal{F}(\mathcal{R}(\vartheta))$, which ensures the desired regularity for the homogeneous part of the temperature. Finally, we use Proposition 3.6 to define a solution in the following way,

$$\begin{aligned}
&(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi}) \\
&\quad := (\Lambda^{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}}), \Lambda^{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}}), \vartheta + \vartheta_{\text{init}}, \Lambda^{\boldsymbol{\sigma}}(\vartheta + \vartheta_{\text{init}}), \Lambda^{\boldsymbol{\chi}}(\vartheta + \vartheta_{\text{init}})) \\
&\quad \in W^{1,q}(0, T; \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times W^{1,q}(0, T; \boldsymbol{Q}^p(\Omega)) \\
&\qquad \times W^{1,\frac{q}{2}}(0, T; W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0, T; W^{1,v(p)}(\Omega)) \\
&\qquad \times W^{1,q}(0, T; \boldsymbol{L}^p(\Omega)) \times W^{1,q}(0, T; \boldsymbol{L}^p(\Omega)).
\end{aligned}$$

Note that by (3.4), ϑ_{init} has the regularity required. Furthermore, the solution given above is unique, since every other solution $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{p}}, \hat{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\chi}})$ of (1.1)–(1.5) can equivalently be written with Proposition 3.6 as $(\Lambda^{\boldsymbol{u}}(\hat{\theta}), \Lambda^{\boldsymbol{p}}(\hat{\theta}), \hat{\theta}, \Lambda^{\boldsymbol{\sigma}}(\hat{\theta}), \Lambda^{\boldsymbol{\chi}}(\hat{\theta}))$. Therefore, in view of the definition of $\mathcal{R}$, $\hat{\vartheta} := \hat{\theta} - \vartheta_{\text{init}}$ is a fixed point of Θ, which however is unique by Proposition 3.15, giving in turn $\hat{\theta} \equiv \theta$. Consequently, the remaining components of the state vector are also unique. □

Remark 3.17 (Bounds for $\bar{p}$ and $\bar{q}$)**.**

1. The spatial p-integrability of the displacement $\boldsymbol{u}$ and plastic strain $\boldsymbol{p}$ is limited by $\bar{p}$. This follows from [Herzog et al., 2011, Theorem 1.1], which was used to prove Proposition 3.6, together with Lemma A.4, which was needed to ensure maximal parabolic regularity for the operator related to the heat equation (3.3).
2. The q-integrability in time of the displacement $\boldsymbol{u}$ and the plastic strain $\boldsymbol{p}$ has to be larger than $\bar{q}$ (in dependence of p) to ensure that the embedding $\mathcal{E}$ is valid. Corollary A.7 gives the precise link between p and $\bar{q}$ as follows. Fix $p > 2$ and $v(p)$ by (2.1), then choose $\bar{q}$ as
   a) for $p < 6$: $\quad \bar{q} > \frac{2}{a} \quad$ with $\quad 0 < a < \begin{cases} 1 - \frac{3}{2p} & \text{if } p < 3 \\ \frac{1}{2} & \text{otherwise,} \end{cases}$
   b) for $p \geq 6$: $\quad \bar{q} > \frac{2}{a} \quad$ with $\quad 0 < a < \begin{cases} 1 - \frac{3}{2v(p)} + \frac{3}{2p} & \text{if } v(p) < p \\ \frac{1}{2} & \text{otherwise.} \end{cases}$

Furthermore, we obtain the following property for the solution operator of (1.1)–(1.5) which will be essential in proving the existence of a global minimizer in Section 4.

Lemma 3.18 (Boundedness of the solution operator)**.** Under the assumptions of Theorem 3.1, the mapping $(\ell, r) \mapsto (\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ from $L^q(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega)) \times L^{\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega))$ into the spaces for $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ as in Theorem 3.1 is bounded, i.e., the images of bounded sets are bounded.

Proof: Suppose $\boldsymbol{B} \subset L^q(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega)) \times L^{\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega))$ is a bounded set. Consider the image $(\boldsymbol{u}(\boldsymbol{B}), \boldsymbol{p}(\boldsymbol{B}), \theta(\boldsymbol{B}), \boldsymbol{\sigma}(\boldsymbol{B}), \boldsymbol{\chi}(\boldsymbol{B}))$. The proof of Proposition 3.6 shows that $\boldsymbol{u}(\boldsymbol{B})$, $\boldsymbol{p}(\boldsymbol{B})$, $\boldsymbol{\sigma}(\boldsymbol{B})$ and $\boldsymbol{\chi}(\boldsymbol{B})$ are bounded.

The boundedness of the temperatures $\theta(\boldsymbol{B}) = \vartheta(\boldsymbol{B}) + \vartheta_{\text{init}}$ can be shown using first the embedding according to Lemma A.6, then estimate (3.13) and finally Gronwall's lemma. We choose $(\ell, r) \in \boldsymbol{B}$ and calculate

$$
\begin{aligned}
\|\vartheta(t)\|_{L^p(\Omega)} \le \|\vartheta\|_{L^\infty(0,t;L^p(\Omega))} &\le C\|\vartheta\|_{W_0^{1,\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,t;W^{1,v(p)}(\Omega))} \\
&\le C\|f\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))}
\end{aligned}
$$

where $f \in L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))$ is defined as

$$
\begin{aligned}
\langle f, z\rangle := \langle r, z\rangle &+ \int_\Omega (\boldsymbol{\sigma}(\vartheta + \vartheta_{\text{init}}) + \boldsymbol{\chi}(\vartheta + \vartheta_{\text{init}})) : \dot{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}})\, z \,\mathrm{d}\boldsymbol{x} \\
&- \int_\Omega (\vartheta + \vartheta_{\text{init}})\, \boldsymbol{t}'(\vartheta + \vartheta_{\text{init}}) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})) - \dot{\boldsymbol{p}}(\vartheta + \vartheta_{\text{init}}))\, z \,\mathrm{d}\boldsymbol{x} \\
&+ \gamma \int_\Omega \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}})) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(\vartheta + \vartheta_{\text{init}}))\, z \,\mathrm{d}\boldsymbol{x}
\end{aligned}
$$

for $z \in L^{\frac{q}{q-2}}(0,t;W^{1,v(p)'}(\Omega))$. Since $\boldsymbol{u}(\boldsymbol{B})$ and $\boldsymbol{p}(\boldsymbol{B})$ are bounded in the desired spaces and ϑ_{init} is fixed with regularity as in (3.4) we can estimate in the following way,

$$
\|\vartheta(t)\|_{L^p(\Omega)} \le C + C\|\vartheta\|_{L^q(0,t;L^p(\Omega))}.
$$

By the convexity of $z \mapsto z^q$ for $z \ge 0$, we obtain the following estimate

$$
\|\vartheta(t)\|_{L^p(\Omega)}^q \le C + C\int_0^t \|\vartheta\|_{L^p(\Omega)}^q
$$

and by Gronwall's lemma

$$
\|\vartheta(t)\|_{L^p(\Omega)}^q \le C \qquad \text{for all } t \in [0,T].
$$

This means that ϑ is bounded in $L^\infty(0,T;L^p(\Omega))$. Next we can again use the estimate (3.13) to show that ϑ is bounded in the desired space. Together with the regularity of ϑ_{init} (see (3.4)) we obtain the assertion. □

4. Optimal Control Problem

In this section we present an optimal control problem governed by the thermoviscoplastic model (1.1)–(1.5), where the controls consist of boundary forces and surface tractions $\boldsymbol{\ell}$, and heat sources r. The aim is to prove the existence of a global minimizer by way of weak continuity of the control-to-state mapping; see Proposition 4.8. As before, p' and q' denote the conjugate indices of p and q, respectively.

Problem 4.1. Find optimal controls

$$\boldsymbol{\ell}^* \in W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)), \qquad r^* \in L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega))$$

and corresponding states

$$\boldsymbol{u}^* \in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)), \qquad \boldsymbol{p}^* \in W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega)),$$
$$\theta^* \in W^{1,\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))$$

which minimize

$$F(\boldsymbol{\ell},r,\boldsymbol{u},\boldsymbol{p},\theta) := \Psi(\boldsymbol{u},\boldsymbol{p},\theta) + \beta_1\|\boldsymbol{\ell}\|^{b_1}_{W^{1,q}(0,T;\boldsymbol{L}^p(\Omega))} + \beta_2\|r\|^{b_2}_{L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega))}$$

subject to (1.1)–(1.5) and $|\theta(t,\boldsymbol{x})| \le M$ almost everywhere in $(0,T)\times\Omega$.

Note that the state constraints are sensible from a physical point of view to avoid destruction of the material, and they also can be useful to obtain a thermodynamically consistent model, cf. Remark 2.9.

The following assumptions are imposed.

Assumption 4.2.

1. The function $\Psi : W^{1,2}(0,T;\boldsymbol{W}^{1,2}_{\mathfrak{D}}(\Omega)) \times L^2(0,T;\boldsymbol{Q}^2(\Omega)) \times L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ is weakly sequentially lower semi-continuous and bounded from below.
2. Cost parameters β_1, β_2 are positive, i.e. $\beta_1, \beta_2 > 0$.
3. The exponents b_1 and b_2 satisfy $1 < b_1, b_2 < \infty$.
4. The bound M is positive and there exists at least one feasible control such that the associated state fulfills the state contraints $|\theta(t,\boldsymbol{x})| \le M$.

Remark 4.3 (Feasible set)**.** The question of existence of a feasible control is non-trivial and strongly depends on the given data. Assume for instance that the initial temperature θ_0 equals the constant reference temperature at which no thermal strains exist, i.e., $\boldsymbol{t}(\theta_0) = \boldsymbol{0}$. Suppose moreover that this temperature is consistent with the Robin boundary conditions of the heat equation, i.e., it equals the surrounding temperature of the work piece. Then it is easy to see that, for thermal and mechanical loads equal to zero, $\boldsymbol{u} \equiv \boldsymbol{0}$, $\boldsymbol{p} = \boldsymbol{\sigma} = \boldsymbol{\chi} \equiv \boldsymbol{0}$, and $\theta \equiv \theta_0$ is a solution of the thermoviscoplastic system. Thus, if $|\theta_0| \leq M$, then $(\boldsymbol{\ell}, r) = (\boldsymbol{0}, 0)$ is a feasible control.

There are many possibilities to create a suitable objective function. For instance it could be of interest to optimize the displacement, the residual stress or the plastic strain, see the following example.

Example 4.4 (Possible choices for Ψ)**.**

- Let $\tilde{\boldsymbol{u}}$ be a desired displacement. Then the term $\Psi(\boldsymbol{u}) = \frac{1}{2}\int_\Omega |\boldsymbol{u}(T) - \tilde{\boldsymbol{u}}|^2 \, \mathrm{d}\boldsymbol{x}$ is a classical tracking-type objective for the terminal displacement.
- The objective $\Psi(\boldsymbol{u}, \boldsymbol{p}, \theta) = \frac{1}{2}\int_\Omega |\mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}(T)) - \boldsymbol{p}(T) - \boldsymbol{t}(\theta(T)))|^2 \, \mathrm{d}\boldsymbol{x}$ seeks to minimize the terminal residual stress.

Both examples are more meaningful when a cooling phase is appended to the end of the control horizon $[0, T]$, which is easily accounted for by bound constraints for the controls, cf. Remark 4.7.

Theorem 4.5 (Existence of an optimal control)**.** Under the assumptions of Theorem 3.1 and Assumption 4.2, there exists at least one global minimizer $(\boldsymbol{\ell}^*, r^*, \boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$ of Problem 4.1 such that

$$
\begin{aligned}
\boldsymbol{\ell}^* &\in W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)), & r^* &\in L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega)),\\
\boldsymbol{u}^* &\in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)), & \boldsymbol{p}^* &\in W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega)),\\
\theta^* &\in W^{1,\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)). & &
\end{aligned}
$$

Remark 4.6. With Theorem 3.1 at hand we can define the control-to-state mapping $\mathcal{G} : (\boldsymbol{\ell}, r) \mapsto (\boldsymbol{u}, \boldsymbol{p}, \theta)$ for the thermoviscoplastic model (1.1)–(1.5).

Proof of Theorem 4.5: The proof follows standard arguments so we can be brief. First we use the control-to-state-map $\mathcal{G} : (\boldsymbol{\ell}, r) \mapsto (\boldsymbol{u}, \boldsymbol{p}, \theta)$ to define the reduced functional $f(\boldsymbol{\ell}, r) := F(\boldsymbol{\ell}, r, \mathcal{G}(\boldsymbol{\ell}, r))$. The reduced objective f is bounded from below by Assumption 4.2, we get the existence of an infimum z,

$$z := \inf f(\boldsymbol{\ell}, r) \in \mathbb{R}.$$

Let $\{(\boldsymbol{\ell}_n, r_n)\}_{n\in\mathbb{N}}$ be a minimizing sequence with $\lim_{n\to\infty} f(\boldsymbol{\ell}_n, r_n) = z$. Because Ψ is bounded from below and the cost parameters are positive, we get the following bound for the control

$$\|\boldsymbol{\ell}_n\|_{W^{1,q}(0,T;\boldsymbol{L}^p(\Omega))} + \|r_n\|_{L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega))} \leq C.$$

Therefore, there exists a control

$$(\boldsymbol{\ell}^*, r^*) \in W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)) \times L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega))$$

and a subsequence (again denoted with n) such that for $n \to \infty$

$$\begin{aligned} \boldsymbol{\ell}_n &\rightharpoonup \boldsymbol{\ell}^* \qquad \text{weakly in } W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)), \\ r_n &\rightharpoonup r^* \qquad \text{weakly in } L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega)). \end{aligned}$$

The functional f is weakly sequentially lower semi-continuous, because $\mathcal{G}$ is weakly continuous, see Proposition 4.8, and Ψ and the norms are weakly sequentially lower semi-continuous. Therefore we get

$$z = \lim_{n\to\infty} F(\boldsymbol{\ell}_n, r_n, \mathcal{G}(\boldsymbol{\ell}_n, r_n)) = \lim_{n\to\infty} f(\boldsymbol{\ell}_n, r_n) \geq f(\boldsymbol{\ell}^*, r^*) \geq z,$$

and the proof of Proposition 4.8 shows also that θ^* defined by $\mathcal{G}(\boldsymbol{\ell}^*, r^*) = (\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$ fulfills $|\theta^*(t,\boldsymbol{x})| \leq M$ almost everywhere in $(0,T)\times\Omega$ due to the weak sequential closedness of the set of feasible temperatures. Therefore $(\boldsymbol{\ell}^*, r^*)$ is a global minimizer. □

Remark 4.7 (Additional constraints and objectives)**.** Theorem 4.5 remains true when the controls $(\boldsymbol{\ell}, r)$ are restricted to a convex closed subset of their respective spaces, as described for instance by pointwise bounds. Furthermore, the stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\chi}$ can be also included in the objective.

Weak continuity of the control-to-state mapping

In this section we provide the remaining proof of the control-to-state mapping's weak sequential continuity.

Proposition 4.8. Under the assumptions of Theorem 3.1, and provided that $\boldsymbol{\ell} \in W^{1,q'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ holds, the control-to-state mapping $(\boldsymbol{\ell}, r) \mapsto (\boldsymbol{u}, \boldsymbol{p}, \theta)$ from $L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)) \cap W^{1,q'}(0,T;\boldsymbol{L}^{p'}(\Omega)) \times L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega))$ into the spaces for $(\boldsymbol{u}, \boldsymbol{p}, \theta)$ as in Theorem 3.1, is weakly sequentially continuous.

Remark 4.9 (Additional regularity for controls)**.** Notice that the controls in Theorem 4.5 satisfy the additional regularity assumptions of Proposition 4.8 according to the choice of the norms in the objective of our optimization Problem 4.1 and the embedding $W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)) \hookrightarrow W^{1,q'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ for any $p, q > 2$. Furthermore, note that for $p \leq 3$, the embedding $W^{1,q'}(0,T;\boldsymbol{L}^{p'}(\Omega)) \hookrightarrow L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega))$ holds.

At first glance it may seem surprising that the passage to the limit for weakly convergent sequences is possible for each of the equations (1.1)–(1.5). It turns out that the passage to the limit for the equations (1.4) and (1.3) can be easily done using a reformulation, cf. [Han and Reddy, 1999, Section 7.2] or Bartels and Roubíček [2008], and that the second and third term of the right hand side of the heat equation,

$$\dot\theta - \operatorname{div}(\kappa\,\nabla\theta) = r + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} - \theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}}),$$

cause the most difficulties due to their nonlinearities. In order to handle them we would expect to need some terms strongly convergent in suitable spaces, such as for example $\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})$. To overcome these difficulties, we adapt a technique from Bartels and Roubíček [2008], which encompasses a joint treatment of the two terms in question rather than considering the limits individually.

We begin with the reformulation of the balance of momentum (1.4) and plastic flow rule (1.3), see Lemma 4.10, and subsequently give the proof of Proposition 4.8. For that we define the following variational inequality for a given temperature $\theta \in L^1(0,T;L^1(\Omega))$,

$$\begin{aligned} J(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{v},\boldsymbol{q}) :=& \\ &\epsilon\int_Q \dot{\boldsymbol{p}} : (\boldsymbol{q} - \dot{\boldsymbol{p}})\,\mathrm{d}(\boldsymbol{x},t) \\ &\quad + \int_Q \mathbb{C}\,[\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)] : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{q} - (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}}))\,\mathrm{d}(\boldsymbol{x},t) \\ &\quad + \int_Q \mathbb{H}\,\boldsymbol{p} : (\boldsymbol{q} - \dot{\boldsymbol{p}})\,\mathrm{d}(\boldsymbol{x},t) + \gamma\int_Q \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))\,\mathrm{d}(\boldsymbol{x},t) \\ &\quad + \int_Q D(\boldsymbol{q},\theta)\,\mathrm{d}(\boldsymbol{x},t) - \int_Q D(\dot{\boldsymbol{p}},\theta)\,\mathrm{d}(\boldsymbol{x},t) - \int_0^T \langle \boldsymbol{\ell},\, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle\,\mathrm{d}t \ge 0 \end{aligned} \tag{4.1}$$

for $(\boldsymbol{v},\boldsymbol{q}) \in L^{q'}(0,T;\boldsymbol{W}^{1,p'}_{\mathfrak{D}}(\Omega)) \times L^q(0,T;\boldsymbol{Q}^p(\Omega))$. This inequality is related to the thermoviscoplastic system in the following way.

Lemma 4.10. Let $p,q \ge 2$, $\boldsymbol{u} \in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))$ and $\boldsymbol{p} \in W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))$. Then a solution $(\boldsymbol{u},\boldsymbol{p})$ of the variational inequality (4.1) is also a solution of (1.1)–(1.4) in the weak sense according to Definition 2.6, and vice versa.

Proof: "⇐" Insert (1.1) and (1.2) into (1.3) and (1.4), respectively. Now substitute $\boldsymbol{v}$ by $\boldsymbol{v} - \dot{\boldsymbol{u}}$ in (1.4), add (1.3) and (1.4) and integrate over time to obtain (4.1).

"⇒" First choose $\boldsymbol{v} = \dot{\boldsymbol{u}}$ in (4.1) to get

$$\begin{aligned} \epsilon\int_Q \dot{\boldsymbol{p}} : (\boldsymbol{q} - \dot{\boldsymbol{p}})\,\mathrm{d}(\boldsymbol{x},t) - \int_Q \mathbb{C}\,[\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)] : (\boldsymbol{q} - \dot{\boldsymbol{p}})\,\mathrm{d}(\boldsymbol{x},t) \\ + \int_Q \mathbb{H}\,\boldsymbol{p} : (\boldsymbol{q} - \dot{\boldsymbol{p}})\,\mathrm{d}(\boldsymbol{x},t) + \int_Q D(\boldsymbol{q},\theta)\,\mathrm{d}(\boldsymbol{x},t) - \int_Q D(\dot{\boldsymbol{p}},\theta)\,\mathrm{d}(\boldsymbol{x},t) \ge 0 \end{aligned} \tag{4.2}$$

for all $\boldsymbol{q} \in L^q(0,T;\boldsymbol{Q}^p(\Omega))$. Next choose $\boldsymbol{q} = \dot{\boldsymbol{p}}$ and substitute $\boldsymbol{v}$ by $\pm\boldsymbol{v} + \dot{\boldsymbol{u}}$ in (4.1) to get

$$\int_Q \mathbb{C}\,[\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)] : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}(\boldsymbol{x},t) + \gamma \int_Q \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}(\boldsymbol{x},t) \\ = \int_0^T \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle\,\mathrm{d}t \qquad \text{for all } \boldsymbol{v} \in L^{q'}(0,T;\boldsymbol{W}^{1,p'}_{\mathfrak{D}}(\Omega)). \tag{4.3}$$

Finally, substitute $\boldsymbol{q}$ by $(\boldsymbol{q} - \dot{\boldsymbol{p}})\,\varphi + \dot{\boldsymbol{p}}$ with $\varphi \in C_0^\infty(0,T)$ and $0 \le \varphi \le 1$ and $\boldsymbol{v}$ by $\varphi\,\boldsymbol{v}$ with $\varphi \in C_0^\infty(0,T)$ in (4.2) and (4.3), respectively, and use the fundamental lemma of the calculus of variations to get (1.1)–(1.4). □

Notice the structural advantages of this formulation. In the proof of Proposition 4.8 we benefit from the quadratic structure of several terms since we can exploit the lower semicontinuity in $L^2(0,T;\boldsymbol{L}^2(\Omega))$ to handle them.

Proof of Proposition 4.8: Let us consider sequences $\{(\boldsymbol{\ell}_n, r_n)\}$

$$\begin{aligned} \boldsymbol{\ell}_n &\rightharpoonup \boldsymbol{\ell}^* && \text{weakly in } W^{1,q'}(0,T;\boldsymbol{L}^{p'}(\Omega)),\\ \boldsymbol{\ell}_n &\rightharpoonup \boldsymbol{\ell}^* && \text{weakly in } L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)),\\ r_n &\rightharpoonup r^* && \text{weakly in } L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \end{aligned}$$

and define $(\boldsymbol{u}_n, \boldsymbol{p}_n, \theta_n) := \mathcal{G}(\boldsymbol{\ell}_n, r_n)$. We have to show that

$$(\boldsymbol{u}_n, \boldsymbol{p}_n, \theta_n) = \mathcal{G}(\boldsymbol{\ell}_n, r_n) \rightharpoonup \mathcal{G}(\boldsymbol{\ell}^*, r^*) =: (\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*).$$

Definition of a candidate $(\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$: The displacements $\{\boldsymbol{u}_n\}$, the plastic strains $\{\boldsymbol{p}_n\}$ and the temperatures $\{\theta_n\}$ are, by Lemma 3.18, bounded independently of n in the following sense:

$$\begin{aligned} \|\boldsymbol{u}_n\|_{W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))} + \|\boldsymbol{p}_n\|_{W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))} &\le C,\\ \|\theta_n\|_{W^{1,\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))} &\le C. \end{aligned}$$

Therefore, there exist $\boldsymbol{u}^* \in W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))$, $\boldsymbol{p}^* \in W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))$, and $\theta^* \in W^{1,\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))$ and a subsequence (denoted by n again) such that

$$\begin{aligned} \boldsymbol{u}_n &\rightharpoonup \boldsymbol{u}^* && \text{weakly in } W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)),\\ \boldsymbol{p}_n &\rightharpoonup \boldsymbol{p}^* && \text{weakly in } W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega)),\\ \theta_n &\rightharpoonup \theta^* && \text{weakly in } W^{1,\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)),\\ \theta_n &\to \theta^* && \text{strongly in } C([0,T];L^p(\Omega)) \qquad \text{(use Corollary A.7)}. \end{aligned}$$

Candidate is admissible, i.e., $\mathcal{G}(\boldsymbol{\ell}^*, r^*) = (\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$: The idea is to show that $(\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$ fulfills the inequality (4.1) (which is equivalent to (1.1)–(1.4) by Lemma 4.10) and the heat equation (1.5). In order to do this we prove for $n \to \infty$ and for arbitrary $\boldsymbol{q} \in L^q(0,T;\boldsymbol{Q}^p(\Omega))$, $\boldsymbol{v} \in L^{q'}(0,T;\boldsymbol{W}^{1,p'}(\Omega))$ and $\varphi \in L^{\frac{q}{q-2}}(0,T;W^{1,v(p)'}(\Omega))$ the following items.

1. $\lim\limits_{n\to\infty} \int_Q \dot{\boldsymbol{p}}_n : \boldsymbol{q}\,\mathrm{d}(\boldsymbol{x},t) = \int_Q \dot{\boldsymbol{p}}^* : \boldsymbol{q}\,\mathrm{d}(\boldsymbol{x},t)$
2. $\liminf\limits_{n\to\infty} \int_Q \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n\,\mathrm{d}(\boldsymbol{x},t) \ge \int_Q \dot{\boldsymbol{p}}^* : \dot{\boldsymbol{p}}^*\,\mathrm{d}(\boldsymbol{x},t)$
3. $\lim\limits_{n\to\infty} \int_Q \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \boldsymbol{p}_n) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{q})\,\mathrm{d}(\boldsymbol{x},t)$
   $= \int_Q \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^*) - \boldsymbol{p}^*) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{q})\,\mathrm{d}(\boldsymbol{x},t)$
4. $\liminf\limits_{n\to\infty} \int_Q \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \boldsymbol{p}_n) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n)\,\mathrm{d}(\boldsymbol{x},t)$
   $\ge \int_Q \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^*) - \boldsymbol{p}^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - \dot{\boldsymbol{p}}^*)\,\mathrm{d}(\boldsymbol{x},t)$
5. $\lim\limits_{n\to\infty} \int_Q \mathbb{C}\,\boldsymbol{t}(\theta_n) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{q})\,\mathrm{d}(\boldsymbol{x},t) = \int_Q \mathbb{C}\,\boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{q})\,\mathrm{d}(\boldsymbol{x},t)$
6. $\lim\limits_{n\to\infty} \int_Q \mathbb{C}\,\boldsymbol{t}(\theta_n) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n)\,\mathrm{d}(\boldsymbol{x},t) = \int_Q \mathbb{C}\,\boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - \dot{\boldsymbol{p}}^*)\,\mathrm{d}(\boldsymbol{x},t)$
7. $\lim\limits_{n\to\infty} \int_Q \mathbb{H}\,\boldsymbol{p}_n : \boldsymbol{q}\,\mathrm{d}(\boldsymbol{x},t) = \int_Q \mathbb{H}\,\boldsymbol{p}^* : \boldsymbol{q}\,\mathrm{d}(\boldsymbol{x},t)$
8. $\liminf\limits_{n\to\infty} \int_Q \mathbb{H}\,\boldsymbol{p}_n : \dot{\boldsymbol{p}}_n\,\mathrm{d}(\boldsymbol{x},t) \ge \int_Q \mathbb{H}\,\boldsymbol{p}^* : \dot{\boldsymbol{p}}^*\,\mathrm{d}(\boldsymbol{x},t)$
9. $\lim\limits_{n\to\infty} \int_Q \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}(\boldsymbol{x},t) = \int_Q \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}(\boldsymbol{x},t)$
10. $\liminf\limits_{n\to\infty} \int_Q \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\,\mathrm{d}(\boldsymbol{x},t) \ge \int_Q \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)\,\mathrm{d}(\boldsymbol{x},t)$
11. $\lim\limits_{n\to\infty} \int_Q D(\boldsymbol{q},\theta_n)\,\mathrm{d}(\boldsymbol{x},t) = \int_Q D(\boldsymbol{q},\theta^*)\,\mathrm{d}(\boldsymbol{x},t)$
12. $\liminf\limits_{n\to\infty} \int_Q D(\dot{\boldsymbol{p}}_n,\theta_n)\,\mathrm{d}(\boldsymbol{x},t) \ge \int_Q D(\dot{\boldsymbol{p}}^*,\theta^*)\,\mathrm{d}(\boldsymbol{x},t)$
13. $\lim\limits_{n\to\infty} \int_0^T \langle \boldsymbol{\ell}_n,\, \boldsymbol{v}\rangle\,\mathrm{d}t = \int_0^T \langle \boldsymbol{\ell}^*,\, \boldsymbol{v}\rangle\,\mathrm{d}t$
14. $\lim\limits_{n\to\infty} \int_0^T \langle \boldsymbol{\ell}_n,\, \dot{\boldsymbol{u}}_n\rangle\,\mathrm{d}t = \int_0^T \langle \boldsymbol{\ell}^*,\, \dot{\boldsymbol{u}}^*\rangle\,\mathrm{d}t$
15. $\lim\limits_{n\to\infty} \int_0^T \langle \dot{\theta}_n,\, \varphi\rangle\,\mathrm{d}t = \int_0^T \langle \dot{\theta}^*,\, \varphi\rangle\,\mathrm{d}t$
16. $\lim\limits_{n\to\infty} \int_Q \kappa\,\nabla\theta_n \cdot \nabla\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\,\theta_n\,\varphi\,\mathrm{d}s = \int_Q \kappa\,\nabla\theta^* \cdot \nabla\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\,\theta^*\,\varphi\,\mathrm{d}s$
17. $\lim\limits_{n\to\infty} \int_Q r_n\varphi\,\mathrm{d}(\boldsymbol{x},t) = \int_Q r^*\varphi\,\mathrm{d}(\boldsymbol{x},t)$
18. $\lim\limits_{n\to\infty} \int_Q \theta_n \boldsymbol{t}'(\theta_n) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n)\,\varphi\,\mathrm{d}(\boldsymbol{x},t)$
    $= \int_Q \theta^*\,\boldsymbol{t}'(\theta^*) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - \dot{\boldsymbol{p}}^*)\,\varphi\,\mathrm{d}(\boldsymbol{x},t)$
19. $\lim\limits_{n\to\infty} \int_Q (\boldsymbol{\sigma}_n + \boldsymbol{\chi}_n) : \dot{\boldsymbol{p}}_n\varphi + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\,\varphi\,\mathrm{d}(\boldsymbol{x},t)$
    $= \int_Q (\boldsymbol{\sigma}^* + \boldsymbol{\chi}^*) : \dot{\boldsymbol{p}}^*\varphi + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)\,\varphi\,\mathrm{d}(\boldsymbol{x},t)$,
    where $\boldsymbol{\sigma}_n := \mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \boldsymbol{p}_n - \boldsymbol{t}(\theta_n)\big)$, $\boldsymbol{\chi}_n := -\mathbb{H}\,\boldsymbol{p}_n$ and analogously $\boldsymbol{\sigma}^*$, $\boldsymbol{\chi}^*$ are defined.

From item 1 to item 14 we can conclude that $(\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$ fulfills (4.1). Indeed, since $(\boldsymbol{u}_n, \boldsymbol{p}_n, \theta_n)$ verifies (4.1) we can write this as $0 \le J(\boldsymbol{u}_n, \boldsymbol{p}_n, \theta_n; \boldsymbol{v}, \boldsymbol{q})$ for all $(\boldsymbol{v}, \boldsymbol{q})$. Now we use item 1 to item 14 to get

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \ J(\boldsymbol{u}_n, \boldsymbol{p}_n, \theta_n; \boldsymbol{v}, \boldsymbol{q}) \\ &= -\liminf_{n\to\infty} \ -J(\boldsymbol{u}_n, \boldsymbol{p}_n, \theta_n; \boldsymbol{v}, \boldsymbol{q}) \le J(\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*; \boldsymbol{v}, \boldsymbol{q}). \end{aligned}$$

Using item 15 to item 19 we conclude that $(\boldsymbol{u}^*, \boldsymbol{p}^*, \theta^*)$ fulfills the heat equation (1.5).

Let us prove the items above. First of all, item 1, item 3, item 7, item 9, item 13, item 15, item 16 and item 17 are clear. Concerning item 2 (and similarly item 10) we use the weak sequentially lower semi-continuity of the norm in $L^2(0, T; \boldsymbol{L}^2(\Omega))$ to get

$$\liminf_{n\to\infty} \int_Q \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n \, \mathrm{d}(\boldsymbol{x}, t) \ge \int_Q \dot{\boldsymbol{p}}^* : \dot{\boldsymbol{p}}^* \, \mathrm{d}(\boldsymbol{x}, t).$$

Concerning item 4 (and similarly item 8) we benefit from the property $\boldsymbol{u} : \dot{\boldsymbol{u}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u} : \boldsymbol{u})$ and the weak sequential lower semi-continuity of the norm in $\boldsymbol{L}^2(\Omega)$ to calculate

$$\begin{aligned} \liminf_{n\to\infty} & \int_Q \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \boldsymbol{p}_n) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n) \, \mathrm{d}(\boldsymbol{x}, t) \\ &= \liminf_{n\to\infty} \frac{1}{2} \int_\Omega \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_n(T)) - \boldsymbol{p}_n(T)) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_n(T)) - \boldsymbol{p}_n(T)) \, \mathrm{d}\boldsymbol{x} \\ &\quad - \frac{1}{2} \int_\Omega \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_n(0)) - \boldsymbol{p}_n(0)) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_n(0)) - \boldsymbol{p}_n(0)) \, \mathrm{d}\boldsymbol{x} \\ &\ge \frac{1}{2} \int_\Omega \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^*(T)) - \boldsymbol{p}^*(T)) : (\boldsymbol{\varepsilon}(\boldsymbol{u}^*(T)) - \boldsymbol{p}^*(T)) \, \mathrm{d}\boldsymbol{x} \\ &\quad - \frac{1}{2} \int_\Omega \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}_0^*) - \boldsymbol{p}_0^*) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_0^*) - \boldsymbol{p}_0^*) \, \mathrm{d}\boldsymbol{x} \\ &= \int_Q \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^*) - \boldsymbol{p}^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - \dot{\boldsymbol{p}}^*) \, \mathrm{d}(\boldsymbol{x}, t). \end{aligned}$$

Notice that the weak convergence of the sequences $\{\boldsymbol{\varepsilon}(\boldsymbol{u}_n(T))\}$ and $\{\boldsymbol{p}_n(T)\}$ in $\boldsymbol{L}^2(\Omega)$ follows from the continuity of the embedding $W^{1,q}(0, T; \boldsymbol{L}^p(\Omega))$ into $C([0, T]; \boldsymbol{L}^p(\Omega))$.

Concerning item 6 (and similarly item 5 and item 18) we calculate

$$\begin{aligned} \lim_{n\to\infty} & \int_Q \mathbb{C}\,\boldsymbol{t}(\theta_n) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n) \pm \mathbb{C}\,\boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n) \\ &\qquad - \mathbb{C}\,\boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})^* - \dot{\boldsymbol{p}}^*) \, \mathrm{d}(\boldsymbol{x}, t) \\ &\le C \lim_{n\to\infty} \|\theta_n - \theta^*\|_{L^{q'}(0,T;L^{p'}(\Omega))} \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n\|_{L^q(0,T;\boldsymbol{L}^p(\Omega))} \\ &\quad + \int_Q \mathbb{C}\,\boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - (\dot{\boldsymbol{p}}_n - \dot{\boldsymbol{p}}^*)) \, \mathrm{d}(\boldsymbol{x}, t) = 0. \end{aligned}$$

For the last term we use that $\boldsymbol{m} \mapsto \mathbb{C}\, \boldsymbol{t}(\theta^*) : \boldsymbol{m}$ with $L^p(0,T;\boldsymbol{L}^q(\Omega)) \to \mathbb{R}$ is a linear and continuous. Therefore, it is an element of $L^{p'}(0,T;\boldsymbol{L}^{q'}(\Omega))$ and we get with the weak convergence of $\boldsymbol{\varepsilon}(\boldsymbol{u}_n)$ and $\boldsymbol{p}_n$ that

$$\int_Q \mathbb{C}\, \boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - (\dot{\boldsymbol{p}}_n - \dot{\boldsymbol{p}}^*))\, \mathrm{d}(\boldsymbol{x},t) = 0.$$

Concerning item 11 and item 12 we can use the same arguments as above in combination with the Lipschitz continuity and positivity of σ_0.

Concerning item 14 we have the compact embedding $\boldsymbol{L}^{p'}(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{W}_{\mathfrak{D}}^{-1,p'}(\Omega)$ and therefore the embedding $W^{1,q'}(0,T;\boldsymbol{L}^{p'}(\Omega)) \hookrightarrow\hookrightarrow L^{q'}(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p'}(\Omega))$ is also compact, see [Simon, 1986, Theorem 3, (6.5)]. That means that there exists a subsequence $\boldsymbol{\ell}_n \to \boldsymbol{\ell}^*$ in $L^{q'}(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p'}(\Omega))$ and

$$\begin{aligned}\lim_{n\to\infty} &\int_0^T \langle \boldsymbol{\ell}_n, \dot{\boldsymbol{u}}_n\rangle - \langle \boldsymbol{\ell}^*, \dot{\boldsymbol{u}}^*\rangle\, \mathrm{d}t \\ &\leq \lim_{n\to\infty} \|\boldsymbol{\ell}_n - \boldsymbol{\ell}^*\|_{L^{q'}(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p'}(\Omega))} \|\dot{\boldsymbol{u}}_n\|_{L^q(0,T;\boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega))} \\ &\quad + \int_0^T \langle \boldsymbol{\ell}^*, \dot{\boldsymbol{u}}_n - \dot{\boldsymbol{u}}^*\rangle\, \mathrm{d}t = 0\end{aligned}$$

follows.

Concerning item 19 (following [Bartels and Roubíček, 2008, Proof of Proposition 4.6]) we test the plastic flow rule (1.3) by $\boldsymbol{q} = \boldsymbol{0}$ and $\boldsymbol{q} = 2\dot{\boldsymbol{p}}$ and get

$$D(\dot{\boldsymbol{p}}, \theta) = (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} - \epsilon\, \dot{\boldsymbol{p}} : \dot{\boldsymbol{p}}.$$

Therefore, we can rephrase the term in item 19 as

$$\begin{aligned}&\int_Q (\boldsymbol{\sigma}_n + \boldsymbol{\chi}_n) : \dot{\boldsymbol{p}}_n \varphi + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\, \varphi\, \mathrm{d}(\boldsymbol{x},t) \\ &\quad = \int_Q D(\dot{\boldsymbol{p}}_n, \theta_n)\, \varphi + \epsilon\, \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n \varphi + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\, \varphi\, \mathrm{d}(\boldsymbol{x},t)\end{aligned}$$

Since $\{D(\dot{\boldsymbol{p}}_n, \theta_n)\}$ is bounded in $L^q(0,T;L^p(\Omega))$, there exists $\xi_1 \in L^q(0,T;L^p(\Omega))$ and a subsequence such that

$$D(\dot{\boldsymbol{p}}_n, \theta_n) \rightharpoonup \xi_1.$$

Similarly $\{\epsilon\, \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n\}$ and $\{\gamma\, \dot{\boldsymbol{u}}_n : \dot{\boldsymbol{u}}_n\}$ are bounded in $L^{\frac{p}{2}}(0,T;L^{\frac{q}{2}}(\Omega))$ and there exist $\xi_2, \xi_3 \in L^{\frac{p}{2}}(0,T;L^{\frac{q}{2}}(\Omega))$ and subsequences such that

$$\epsilon\, \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n \rightharpoonup \xi_2 \qquad \text{and} \qquad \gamma\, \dot{\boldsymbol{u}}_n : \dot{\boldsymbol{u}}_n \rightharpoonup \xi_3.$$

We use item 12 and the weak sequential lower semicontinuity of the norm in $L^2(0,T;\boldsymbol{L}^2(\Omega))$ to calculate

$$
\begin{aligned}
&\int_Q \boldsymbol{\sigma}^* : \dot{\boldsymbol{p}}^* + \boldsymbol{\chi}^* : \dot{\boldsymbol{p}}^* + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)\, \mathrm{d}(\boldsymbol{x},t) \\
&\quad = \int_Q D(\theta^*, \dot{\boldsymbol{p}}^*) + \epsilon\, \dot{\boldsymbol{p}}^* : \dot{\boldsymbol{p}}^* + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)\, \mathrm{d}(\boldsymbol{x},t) \\
&\quad \le \liminf_{n\to\infty} \int_Q D(\theta_n, \dot{\boldsymbol{p}}_n) + \epsilon\, \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\, \mathrm{d}(\boldsymbol{x},t) \\
&\quad \le \limsup_{n\to\infty} \int_Q D(\theta_n, \dot{\boldsymbol{p}}_n) + \epsilon\, \dot{\boldsymbol{p}}_n : \dot{\boldsymbol{p}}_n + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\, \mathrm{d}(\boldsymbol{x},t) \\
&\quad = \limsup_{n\to\infty} \int_Q \boldsymbol{\sigma}_n : \dot{\boldsymbol{p}}_n + \boldsymbol{\chi}_n : \dot{\boldsymbol{p}}_n + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\, \mathrm{d}(\boldsymbol{x},t) \\
&\qquad\quad - \int_Q (\boldsymbol{\sigma}_n + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n)\, \mathrm{d}(\boldsymbol{x},t) + \int_0^T \langle \boldsymbol{\ell}_n, \dot{\boldsymbol{u}}_n \rangle\, \mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big[\text{by setting } \boldsymbol{v} = \dot{\boldsymbol{u}}_n \text{ in (1.4)}\big] \\
&\quad \le \limsup_{n\to\infty} \int_Q -\mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \boldsymbol{p}_n\big) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n) - \mathbb{H}\boldsymbol{p}_n : \dot{\boldsymbol{p}}_n\, \mathrm{d}(\boldsymbol{x},t) \\
&\qquad\quad + \int_Q \mathbb{C}\, \boldsymbol{t}(\theta_n) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_n) - \dot{\boldsymbol{p}}_n)\ \mathrm{d}(\boldsymbol{x},t) + \int_0^T \langle \boldsymbol{\ell}_n, \dot{\boldsymbol{u}}_n \rangle\, \mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big[\text{by using (1.1) and (1.2)}\big] \\
&\quad = \int_Q -\mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^*) - \boldsymbol{p}^*\big) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - \dot{\boldsymbol{p}}^*) - \mathbb{H}\boldsymbol{p}^* : \dot{\boldsymbol{p}}^*\, \mathrm{d}(\boldsymbol{x},t) \\
&\qquad\quad + \int_Q \mathbb{C}\, \boldsymbol{t}(\theta^*) : (\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) - \dot{\boldsymbol{p}}^*)\ \mathrm{d}(\boldsymbol{x},t) + \int_0^T \langle \boldsymbol{\ell}^*, \dot{\boldsymbol{u}}^* \rangle\, \mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad \big[\text{by using item 4, item 6, item 8 and item 14}\big] \\
&\quad = \int_Q \boldsymbol{\sigma}^* : \dot{\boldsymbol{p}}^* + \boldsymbol{\chi}^* : \dot{\boldsymbol{p}}^* + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)\, \mathrm{d}(\boldsymbol{x},t) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big[\text{by setting } \boldsymbol{v} = \dot{\boldsymbol{u}}^* \text{ in (1.4)}\big].
\end{aligned}
$$

Therefore, all inequalities are in fact equalities. Now we use that if

$$\lim_{n\to\infty} \int_Q a_n\, \mathrm{d}(\boldsymbol{x},t) = \int_Q a\, \mathrm{d}(\boldsymbol{x},t) \qquad \text{with } a_n, a \ge 0,$$

then we get for arbitrary $\varphi \in L^\infty(0,T;L^\infty(\Omega))$

$$\lim_{n\to\infty} \int_Q (a_n - a)\, \varphi\, \mathrm{d}(\boldsymbol{x},t) \le \operatorname{ess\,sup} \varphi \cdot \lim_{n\to\infty} \int_Q (a_n - a)\, \mathrm{d}(\boldsymbol{x},t) = 0.$$

Hence we conclude that

$$\boldsymbol{\sigma}_n : \dot{\boldsymbol{p}}_n + \boldsymbol{\chi}_n : \dot{\boldsymbol{p}}_n + \gamma\, \dot{\boldsymbol{u}}_n : \dot{\boldsymbol{u}}_n \rightharpoonup \boldsymbol{\sigma}^* : \dot{\boldsymbol{p}}^* + \boldsymbol{\chi}^* : \dot{\boldsymbol{p}}^* + \gamma\, \dot{\boldsymbol{u}}^* : \dot{\boldsymbol{u}}^* \text{ in } L^1(0,T;L^1(\Omega)).$$

Since the weak limit is unique, we get

$$\sigma_n : \dot{p}_n + \chi_n : \dot{p}_n + \gamma\, \dot{u}_n : \dot{u}_n \rightharpoonup \sigma^* : \dot{p}^* + \chi^* : \dot{p}^* + \gamma\, \dot{u}^* : \dot{u}^* (= \xi_1 + \xi_2 + \xi_3)$$

in $L^{\frac{q}{2}}(0, T; W_\diamond^{-1,v(p)}(\Omega))$.

Convergence $(u_n, p_n, \theta_n) \rightharpoonup (u^*, p^*, \theta^*)$ **for the entire sequence:** In the step above we have shown that $(u_n, p_n, \theta_n) \rightharpoonup (u^*, p^*, \theta^*)$ for a subsequence. With the arguments above we can prove that every subsequence has a subsequence converging to the same (u^*, p^*, θ^*); see Theorem 3.1. Therefore, the entire sequence (u_n, p_n, θ_n) converges to (u^*, p^*, θ^*). □

Acknowledgment

The authors would like to thank Joachim Rehberg (WIAS Berlin) and Hannes Meinlschmidt (TU Darmstadt) for fruitful discussions about maximal parabolic regularity theory.

A. Appendix

A.1. Semigroup theory

Lemma A.1. The solution ϑ_{init} of (3.2) satisfies

$$\vartheta_{\text{init}} \in W^{1,\infty}(0, T; W_\diamond^{-1,v(p)}(\Omega)) \cap L^\infty(0, T; W^{1,v(p)}(\Omega)).$$

Proof: Notice, that the semigroup $(T(t))_t : W_\diamond^{-1,v(p)}(\Omega) \to W_\diamond^{-1,v(p)}(\Omega)$ with domain $W^{1,v(p)}$ of the operator $-A$ related to (3.2) is analytic and the solution is given by $\vartheta_{\text{init}}(t) = T(t)\theta_0$. We can estimate using the properties of an analytic semigroup in the following way,

$$\begin{aligned}
\|-A\vartheta_{\text{init}}(t)\|_{W_\diamond^{-1,v(p)}(\Omega)} &= \|-AT(t)\theta_0\|_{W_\diamond^{-1,v(p)}(\Omega)} \\
&= \|T(t)(-A)\theta_0\|_{W_\diamond^{-1,v(p)}(\Omega)} \\
&\le \|T(t)\|_{W_\diamond^{-1,v(p)}(\Omega)\to W_\diamond^{-1,v(p)}} \|-A\theta_0\|_{W_\diamond^{-1,v(p)}(\Omega)}.
\end{aligned}$$

Using the equivalence of the graph norm $\|-A\cdot\|_{W_\diamond^{-1,v(p)}(\Omega)} + \|\cdot\|_{W_\diamond^{-1,v(p)}(\Omega)}$ and the norm of the space $W^{1,v(p)}(\Omega)$ we infer that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\vartheta_{\text{init}}(t)\|_{W^{1,v(p)}(\Omega)}$$

$$\leq C \sup_{t\in[0,T]} \|T(t)\|_{W_\diamond^{-1,v(p)}(\Omega)\to W_\diamond^{-1,v(p)}(\Omega)} \|\theta_0\|_{W^{1,v(p)}(\Omega)} < C.$$

In combination with $\dot{\vartheta}_{\text{init}} = -A\vartheta_{\text{init}} \in L^\infty(0,T;W_\diamond^{-1,v(p)}(\Omega))$ we obtain the regularity

$$\vartheta_{\text{init}} \in W^{1,\infty}(0,T;W_\diamond^{-1,v(p)}(\Omega)) \cap L^\infty(0,T;W^{1,v(p)}(\Omega)).$$

□

A.2. Maximal Parabolic Regularity

Definition A.2 (Maximal parabolic regularity)**.** Let X be a Banach space and A a closed operator with dense domain $\mathcal{D} \subseteq X$. Suppose $0 < T < \infty$. Then the operator A satisfies maximal parabolic regularity in X iff there exists $r \in (1,\infty)$ such that for any $f \in L^r(0,T;X)$ there is a unique function $w \in W_0^{1,r}(0,T;X) \cap L^r(0,T;\mathcal{D})$ which fulfills

$$\dot{w} + Aw = f.$$

Remark A.3. It is well known that the property of maximal parabolic regularity of an operator A is independent of $r \in (1,\infty)$ and the choice of the time interval $[0,T]$; cf. [Dore, 1993, Theorem 4.2, Theorem 2.5].

Lemma A.4. Under Assumption 2.7 item 2, there exists $\hat{v} > 2$ such that for every $v \in [2,\hat{v}]$ the operator related to (3.2) satisfies maximal parabolic regularity in $W_\diamond^{-1,v}(\Omega)$ independently of the time interval and the time integrability $r \in (1,\infty)$.

Proof: See [Gröger, 1989, Theorem 1 and Remark 5] and Remark A.3. □

Lemma A.5. Suppose A is a closed densely defined operator with domain $\mathcal{D}$ satisfying maximal parabolic regularity in X. Then its adjoint operator A^* satisfies maximal parabolic regularity in $\mathcal{D}'$.

Proof: By Definition A.2, an operator A satisfies maximal parabolic regularity in X iff there exists $r \in (1,\infty)$ such that the mapping

$$\partial_t + A : W_0^{1,r}(0,T;X) \cap L^r(0,T;\mathcal{D}) \to L^r(0,T;X)$$

is an isomorphism, where ∂_t denotes the weak time derivative. Then the adjoint operator is also an isomorphism

$$(\partial_t + A)^* : L^{r'}(0,T;X') \to (W_0^{1,r}(0,T;X) \cap L^r(0,T;\mathcal{D}))',$$

i.e. for all $g \in (W_0^{1,r}(0,T;X) \cap L^r(0,T;\mathcal{D}))'$, there exists a unique $\psi \in L^{r'}(0,T;X')$ such that

$$(\partial_t + A)^*\psi = \partial_t^*\psi + A^*\psi = g.$$

Furthermore, we obtain the following equation for given $g \in L^{r'}(0,T;\mathcal{D}') \subseteq (W_0^{1,r}(0,T;X) \cap L^r(0,T;\mathcal{D}))'$ and for all $\xi \in C_c^\infty(0,T;\mathcal{D})$ satisfying $\xi(t) = v(t)\,u$, where $v \in C_c^\infty(0,T)$ and $u \in \mathcal{D}$:

$$\int_0^T \langle(\partial_t + A)^*\psi, \xi\rangle_{\mathcal{D}}\,\mathrm{d}t = \int_0^T \langle\partial_t^*\psi, \xi\rangle_{\mathcal{D}}\,\mathrm{d}t + \int_0^T \langle A^*\psi, \xi\rangle_{\mathcal{D}}\,\mathrm{d}t = \int_0^T \langle g, \xi\rangle_{\mathcal{D}}\,\mathrm{d}t.$$

This means that

$$\begin{aligned}\langle u, \int_0^T v'\psi\,\mathrm{d}t\rangle_{\mathcal{D}} = \int_0^T \langle\psi, \partial_t\xi\rangle_{\mathcal{D}}\,\mathrm{d}t &= \int_0^T \langle\partial_t^*\psi, \xi\rangle_{\mathcal{D}}\,\mathrm{d}t \\ &= \int_0^T \langle g - A^*\psi, \xi\rangle_{\mathcal{D}}\,\mathrm{d}t \\ &= \langle u, \int_0^T (g - A^*\psi)\,v\,\mathrm{d}t\rangle_{\mathcal{D}}.\end{aligned}$$

Since the equation above is satisfied for all $u \in \mathcal{D}$ we obtain

$$\int_0^T v'\psi\,\mathrm{d}t = \int_0^T (g - A^*\psi)\,v\,\mathrm{d}t$$

for all $v \in C_c^\infty(0,T)$, and the regularity of g implies that the distributional time derivative of ψ is regular and satisfies

$$-\partial_t\psi = g - A^*\psi \in L^{r'}(0,T;\mathcal{D}'). \tag{A.1}$$

Therefore we have $\psi \in W^{1,r'}(0,T;\mathcal{D}') \cap L^{r'}(0,T;X')$. Now we use integration by parts [Amann, 2005, Proposition 5.1] to get, for all $\varphi \in W_0^{1,r}(0,T;X) \cap L^r(0,T;\mathcal{D})$,

$$\begin{aligned}\int_0^T \langle g, \varphi\rangle_{\mathcal{D}}\,\mathrm{d}t &= \int_0^T \langle\partial_t^*\psi, \varphi\rangle_{\mathcal{D}}\,\mathrm{d}t + \int_0^T \langle A^*\psi, \varphi\rangle_{\mathcal{D}}\,\mathrm{d}t \\ &= \int_0^T \langle\psi, \partial_t\varphi\rangle_X\,\mathrm{d}t + \int_0^T \langle A^*\psi, \varphi\rangle_{\mathcal{D}}\,\mathrm{d}t \\ &= \int_0^T \langle-\partial_t\psi, \varphi\rangle_{\mathcal{D}}\,\mathrm{d}t + \int_0^T \langle\psi(T), \varphi(T)\rangle_{(X,\mathcal{D})_{1/r',r}}\,\mathrm{d}t \\ &\quad + \int_0^T \langle A^*\psi, \varphi\rangle_{\mathcal{D}}\,\mathrm{d}t,\end{aligned}$$

where $(X,\mathcal{D})_{1/r',r}$ denotes the real interpolation space. Using (A.1) and the fact that φ was arbitrary, we obtain $\psi(T) = 0$ in $(X,\mathcal{D})'_{1/r',r} = (\mathcal{D}',X')_{1/r,r'} \hookrightarrow \mathcal{D}'$. Therefore for all $g \in L^{r'}(0,T;\mathcal{D}')$, there exists a unique $\psi \in W^{1,r'}(0,T;\mathcal{D}') \cap L^{r'}(0,T;X')$ such that

$$-\partial_t\psi + A^*\psi = g \qquad \text{and} \qquad \psi(T) = 0 \quad \text{in } \mathcal{D}'$$

hold. Finally we transform the time variable $s \to T - s$ and see that A^* satisfies maximal parabolic regularity in $\mathcal{D}'$. □

A.3. Embeddings

We require the following results for embeddings in our analysis.

Lemma A.6. Let $0 < a < \min\left\{\frac{1}{2}, \frac{1}{2} - \frac{3}{2y} + \frac{3}{2z}\right\}$ and $\begin{cases} \frac{3z}{3+z} < y < \frac{3z}{3-z} & \text{if } p < 3 \\ \frac{3z}{3+z} < y < \infty, & \text{if } p = 3 \\ \frac{3z}{3+z} < y \leq \infty, & \text{otherwise.} \end{cases}$

1. For $aq < 2$ and $w < \frac{q}{2-aq}$ there is the compact embedding

$$W^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,y}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,y}(\Omega)) \hookrightarrow\hookrightarrow L^w(0,T;L^z(\Omega)).$$

2. For $aq > 2$ there is the compact embedding

$$W^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,y}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,y}(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^z(\Omega)).$$

Proof: The embeddings follow with Corollary 8 of Simon [1986]. Check all the assumptions therein by using Lemma 12 in Simon [1986]. □

Corollary A.7. Fix $p > 2$.

1. Let $p < 6$ and thus $v(p) = 3p/(6-p)$; cf. (2.1).

   Then for $q > \frac{2}{a}$ and $0 < a < \begin{cases} 1 - \frac{3}{2p} & \text{if } p < 3 \\ \frac{1}{2} & \text{otherwise} \end{cases}$ the following embeddings are compact:

   a) $W^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^p(\Omega))$,

   b) $W_0^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^p(\Omega))$.

2. Let $p \geq 6$ and thus $v(p) \in (\frac{3p}{3+p}, \infty)$; cf. (2.1).

   Then for $q > \frac{2}{a}$ and $0 < a < \begin{cases} 1 - \frac{3}{2v(p)} + \frac{3}{2p} & \text{if } v(p) < p \\ \frac{1}{2} & \text{otherwise} \end{cases}$ the following embeddings are compact:

   a) $W^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^p(\Omega))$,

   b) $W_0^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^p(\Omega))$.

Proof: The embeddings item 1a and item 2a follow directly from Lemma A.6. The embeddings item 1b and item 2b are the restriction of the embeddings from item 1a and item 2a, respectively, to the subspace $\{\psi : \psi(0) = 0\}$. □

References

H. Amann. Nonautonomous parabolic equations involving measures. *Journal of Mathematical Sciences*, 130:4780–8002, 2005.

Sören Bartels and Tomáš Roubíček. Thermoviscoplasticity at small strains. *ZAMM. Zeitschrift für Angewandte Mathematik und Mechanik. Journal of Applied Mathematics and Mechanics*, 88(9):735–754, 2008. ISSN 0044-2267. doi: 10.1002/zamm.200800042.

Sören Bartels and Tomáš Roubíček. Thermo-visco-elasticity with rate-independent plasticity in isotropic materials undergoing thermal expansion. *ESAIM. Mathematical Modelling and Numerical Analysis*, 45(3):477–504, 2011. ISSN 0764-583X. doi: 10.1051/m2an/2010063.

K. Chełmiński and R. Racke. Mathematical analysis of thermoplasticity with linear kinematic hardening. *Journal of Applied Analysis*, 12(1):37–57, 2006. ISSN 1425-6908. doi: 10.1515/JAA.2006.37.

Giovanni Dore. L^p regularity for abstract differential equations. In *Functional Analysis and Related Topics, 1991*, pages 25–38. Springer, 1993.

H. Gajewski, K. Gröger, and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Akademie-Verlag, Berlin, 1974.

P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985.

K. Gröger. A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. *Mathematische Annalen*, 283:679–687, 1989. doi: 10.1007/BF01442860.

R. Haller-Dintelmann, C. Meyer, J. Rehberg, and A. Schiela. Hölder continuity and optimal control for nonsmooth elliptic problems. *Applied Mathematics and Optimization*, 60:397–428, 2009. doi: 10.1007/s00245-009-9077-x.

W. Han and B. D. Reddy. *Plasticity*. Springer, New York, 1999.

R. Herzog, C. Meyer, and G. Wachsmuth. Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions. *Journal of Mathematical Analysis and Applications*, 382(2):802–813, 2011. doi: 10.1016/j.jmaa.2011.04.074.

D. Hömberg, C. Meyer, J. Rehberg, and W. Ring. Optimal control for the thermistor problem. *SIAM Journal on Control and Optimization*, 48(5):3449–3481, 2009/10. ISSN 0363-0129. doi: 10.1137/080736259.

N.S. Ottosen and M. Ristinmaa. *The Mechanics of Constitutive Modeling*. Elsevier, 2005.

Laetitia Paoli and Adrien Petrov. Global existence result for thermoviscoelastic problems with hysteresis. *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, 13(2):524–542, 2012. ISSN 1468-1218. doi: 10.1016/j.nonrwa.2011.07.018.

R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. ISBN 0-8218-0500-2.

J. Simon. Compact sets in the space $L^p(0, T; B)$. *Annali di Matematica Pura ed Applicata*, 146:65–96, 1986. doi: 10.1007/BF01762360.