An optimal control problem of static plasticity with linear kinematic hardening and von Mises yield condition is studied. The problem is treated in its primal formulation, where the state system is a variational inequality of the second kind. First-order necessary optimality conditions are obtained by means of an approximation by a family of control problems with state system regularized by Huber-type smoothing, and a subsequent limit analysis. The equivalence of the optimality conditions with the C-stationarity system for the equivalent dual formulation of the problem is proved. Numerical experiments are presented, which demonstrate the viability of the Huber-type smoothing approach.

**KEYWORDS:** optimal control, first-order necessary optimality conditions, mathematical program with equilibrium constraints (MPEC), variational inequality of the second kind, elastoplasticity
1 INTRODUCTION

We consider an optimal control problem for static small-strain elastoplasticity in its primal formulation. This model describes the deformation of a solid body under high loads, such that the yield stress is reached and permanent deformation ensues. Although the static variational inequality (VI) has only limited physical meaning, it can be regarded as time discretization of a corresponding quasi-static counterpart. The latter models elastoplastic deformation processes and thus appears in various industrial applications. When an instantaneous control strategy is applied to optimize or control such processes, then the static optimal control problem considered in this paper will arise.

We consider here a linear kinematic hardening model with von Mises yield condition and under the assumption of small strains. The description of the forward problem follows (Han and Reddy, 1999b, Chapter 12.2) where the quasistatic case is considered, see also Alberty et al. (1999). The solid body \( \Omega \subset \mathbb{R}^3 \) is clamped on a non-vanishing Dirichlet part \( \Gamma_D \) of its boundary \( \Gamma \), and it is subject to boundary loads on the remaining Neumann part \( \Gamma_N \). The variables of the problem are the displacement

\[
u \in V := H_D^1(\Omega; \mathbb{R}^3) = \{ u \in H^1(\Omega; \mathbb{R}^3) : u = 0 \text{ on } \Gamma_D \}
\]

and the plastic strain

\[
p \in Q := L^2(\Omega; \mathbb{S}) = \{ p \in S : \text{trace}(p) = 0 \}
\]

where \( S := L^2(\Omega; \mathbb{S}) \) and \( \mathbb{S} = \mathbb{R}^{3 \times 3}_{\text{sym}} \) are spaces of symmetric matrices. By \( Q := \{ q \in \mathbb{S} : \text{trace}(q) = 0 \} \) we denote the subspace of trace-free (deviatoric) symmetric matrices. All spaces are endowed with their natural inner products and norms. For \( \mathbb{S} \) and \( Q \), this is the Frobenius norm, denoted by \( |p| \), and corresponding inner product \( p : q \).

Given \( \ell \in V' \), the forward problem is to find \( W = (u, p) \in Z := V \times Q \) which satisfies the following variational inequality (VI) of the second kind.

\[
a(W, Y - W) + j(q) - j(p) \geq \langle \ell, v - u \rangle \quad \text{for all } Y = (v, q) \in Z. \quad (1.1)
\]

The forms \( a, j \) and \( \ell \) in (1.1) are defined as follows:

\[
a(W, Y) = \int_{\Omega} [ (\varepsilon(u) - p) : C (\varepsilon(v) - q) ] \, dx + \int_{\Omega} p : H q \, dx \quad (1.2a)
\]

\[
j(p) = \text{s.o.} \int_{\Omega} |p| \, dx \quad (1.2b)
\]

\[
\langle \ell, v \rangle = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds. \quad (1.2c)
\]

In (1.2a), \( C \) represents the material’s fourth-order elasticity tensor, see e.g. (Han and Reddy, 1999b, Chapter 2.3), and \( H \) is the hardening modulus. The constant \( \text{s.o.} > 0 \) in (1.2b) denotes the material’s yield stress. The data \( f \) and \( g \) in (1.2c) are volume and
boundary loads, respectively. We remark that the second term of the energy form $a(\cdot, \cdot)$ and the specific choice of $j(\cdot)$ are characteristic for linear kinematic hardening.

We consider the following optimal control problem with control variables $(f, g) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma^N; \mathbb{R}^3)$ and state variables $(u, p) \in V \times Q$.

$$\text{Minimize } J(u, f, g) := \frac{1}{2} \|u - u_d\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \frac{\nu_1}{2} \|f\|_{L^2(\Omega; \mathbb{R}^3)}^2 + \frac{\nu_2}{2} \|g\|_{L^2(\Gamma^N; \mathbb{R}^3)}^2$$

subject to the variational inequality (1.1). (1.3)

One can prove by standard methods that there exists at least one global optimal solution of (1.3) so we do not discuss this in detail. The derivation of optimality conditions, however, is by no means standard, since (1.3) constitutes a generalized mathematical program with equilibrium constraints (MPEC) in function space. This is due to the fact that the solution operator associated with (1.1) is in general not Gâteaux differentiable. Therefore standard techniques for smooth optimal control problems based on adjoint calculus cannot be applied directly. A well-established approach for the derivation of optimality conditions is to regularize the VI in order to obtain a smooth control-to-state mapping. An analysis of vanishing regularization parameter then yields an optimality system for the original nonsmooth problem. Depending on the particular VI, the chosen regularization technique, and the assumptions on the problem data, different optimality systems have been established in this way. In case of optimal control of VIs of the first kind, these optimality systems have been classified systematically, see Scheel and Scholtes (2000) for the finite dimensional case and Hintermüller and Kopacka (2009) for the function space setting. To the best of our knowledge, there is no such classification known for VIs of the second kind. The aim of this contribution is to make a first step towards such a classification. The problem of static elastoplasticity suits this purpose well, as it can be modeled either in primal or in dual form, corresponding to VIs of first and second kind, respectively. In this way, an optimality system for (1.3) can be transferred to an optimality system for an optimal control problem subject to a VI of first kind. This system can then be classified according to the scheme known for the latter.

Let us give some more details concerning our regularization method and the optimality conditions which are obtained via the limit analysis, as well as the comparison with the results for the dual form. To regularize the control-to-state mapping we apply a well-established Huber-type smoothing, which was already used in de los Reyes (2011, 2012). By exploiting the special structure of static elastoplasticity, the results of de los Reyes (2011, 2012) can be significantly sharpened, in particular with regard to the multipliers in the optimality system. This however requires a substantial extension of the limit analysis in de los Reyes (2011, 2012). As indicated above the optimality system in the limit can be transformed into the dual, stress-based setting, which involves a VI of first kind. It turns out that the optimality system derived in this way is equivalent to C-stationarity conditions for the dual form, which were established in Herzog et al. (2012) for arbitrary local minimizers. The concept of C-stationarity is of intermediate strength and it is generically obtained when a limit process for a sequence of regularized problems is considered. We refer to Hintermüller and Kopacka (2009) for a classification of
the various stationarity concepts for optimal control of (elliptic) VIs of the first kind. Our result thus shows that a vanishing regularization approach for problems involving VIs of the second kind may yield optimality conditions of comparable strength.

Let us emphasize that the investigation of optimal control of primal elastoplasticity is not only of interest due to the comparison with the dual formulation. The primal formulation also suggests an alternative way of smoothing (the Huber-type regularization already mentioned), which can also be used for numerical computations, see Section 8. Moreover, there are numerous constitutive laws that can be formulated only through primal variables, as for instance in case of thermoplasticity, see e.g. Bartels and Roubíček (2008). Our work thus lays the foundations for the derivation of optimality conditions for such cases.

To put our work into perspective, let us comment on existing literature for optimal control of VIs of second kind. In contrast to problems involving VIs of first kind, where the picture is much more complete, there are rather few problems treated in the literature, and each comes with its own specific set of optimality conditions, see Wenbin and Rubio (1991); Bonnans and Tiba (1991); Bonnans and Casas (1995); de los Reyes (2011, 2012). A comparison of the strengths of these conditions between problems seems to be lacking, as indicated above. The work closest to ours is de los Reyes (2011, 2012), but we sharpen here the results significantly and adopt them to the optimal control of the static elastoplasticity model (1.1).

The outline of the paper is as follows. Sections 2 and 3 collect our standing assumptions, as well as some known facts concerning the forward problem (1.1). Sections 4 and 5 are devoted to the study of regularized optimal control problems, which are obtained by approximating the VI (1.1) by an equation. A Huber-type smoothing of \( j(\cdot) \) is used for this purpose. We point out that an improved integrability result for \( (u, p) \) based on Herzog et al. (2011b) plays an essential role in the Fréchet differentiability of the regularized forward problem, see Theorem 5.2. The first-order optimality system for the regularized case is given in Theorem 6.1. In Section 6 we pass to the limit to obtain an optimality system for (1.3), cf. Theorem 6.4. The equivalence of the resulting optimality conditions with the system of C-stationarity of the problem involving the corresponding dual formulation is shown in Section 7. Finally, Section 8 reports on some numerical experiments based on the proposed regularization of (1.3).

**Notation**

We shall use the short hand notation \((\cdot, \cdot)_{\Omega}\) to denote the standard \(L^2\) inner product in spaces such as \(L^2(\Omega), L^2(\Omega; \mathbb{R}^3)\) and \(L^2(\Omega; \mathbb{S})\). Similarly, \((\cdot, \cdot)_{\Gamma_N}\) denotes the \(L^2\) inner product of functions defined on \(\Gamma_N\). Besides the space \(H^1_0(\Omega; \mathbb{R}^3)\) already defined, we will also use the more general Sobolev spaces\[ W^{1,p}_D(\Omega; \mathbb{R}^3) = \{ u \in W^{1,p}(\Omega; \mathbb{R}^3) : u = 0 \text{ on } \Gamma_D \}\]
for values of \( p \in [1, \infty) \). We will denote the conjugate exponent of \( p \) by \( p' \) and write \( W_{D}^{1,p}(\Omega; \mathbb{R}^{3}) \) to denote the dual space of \( W_{D}^{1,p}(\Omega; \mathbb{R}^{3}) \). The dual of a normed linear space \( X \) is denoted as \( X' \) and the duality pairing by \( \langle \cdot, \cdot \rangle \).

## 2 Standing Assumptions

Our first assumption concerns the domain \( \Omega \).

**Assumption 2.1** (Domain and its boundary).

(a) The boundary \( \Gamma \) of the domain \( \Omega \subset \mathbb{R}^{3} \) is Lipschitz, i.e., the boundary consists of a finite number of local graphs of Lipschitz maps, see, e.g., (Grisvard, 1985, Definition 1.2.1.1).

(b) Moreover, the boundary is assumed to consist of two disjoint measurable parts \( \Gamma_{N} \) and \( \Gamma_{D} \) such that \( \Gamma = \Gamma_{N} \cup \Gamma_{D} \). While \( \Gamma_{N} \) is relatively open, \( \Gamma_{D} \) is a relatively closed subset of \( \Gamma \). Furthermore \( \Gamma_{D} \) is assumed to have positive measure.

(c) In addition, the set \( \Omega \cup \Gamma_{N} \) is regular in the sense of Gröger (1989).

The class of domains fulfilling Assumption 2.1 covers a wide range of geometries. Loosely speaking, condition (c) means that the hypersurface separating \( \Gamma_{N} \) and \( \Gamma_{D} \) is Lipschitz. We refer to (Haller-Dintelmann et al., 2009, Section 5) for more details. We make this assumption in order to apply the integrability results in Herzog et al. (2011b) pertaining to systems of nonlinear elasticity, which leads to Theorem 5.2 and Corollary 5.3, i.e., \((u, p) \in W_{D}^{1,p}(\Omega; \mathbb{R}^{3}) \times L^{p}(\Omega; \mathbb{Q}) \) holds for some \( p > 2 \).

**Assumption 2.2** (Elasticity and hardening tensors). The tensor-valued functions \( C \) and \( H \) are elements of \( L^{\infty}(\Omega; \mathcal{L}(\mathfrak{S})) \), where \( \mathcal{L}(\mathfrak{S}) \) denotes the space of linear operators \( \mathfrak{S} \rightarrow \mathfrak{S} \). Both \( C(x) \) and \( H(x) \) are assumed to be uniformly bounded and coercive with coercivity constants \( \varepsilon > 0 \) and \( h > 0 \), respectively. That is, for all \( \varepsilon \in \mathfrak{S} \) and almost all \( x \in \Omega \) there holds \( \varepsilon : C(x) \varepsilon \geq \varepsilon |e|^{2} \) and \( p : H(x) p \geq h |p|^{2} \). Moreover, we assume as usual that \( C \) and \( H \) are symmetric, i.e.,

\[
C_{ijkl} = C_{jikl} = C_{klij}. \tag{2.1}
\]

and similarly for \( H \).

In what follows, we abbreviate \( \|C\|_{L^{\infty}(\Omega; \mathcal{L}(\mathfrak{S}))} =: \tau \) and \( \|H\|_{L^{\infty}(\Omega; \mathcal{L}(\mathfrak{S}))} =: \nu \). In homogeneous isotropic materials, \( C \) is given by \( C_{ijkl} = \lambda_{L} \delta_{ij} \delta_{kl} + \mu_{L} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \) with Lamé constants satisfying \( \mu_{L} > 0 \) and \( 3 \lambda_{L} + 2 \mu_{L} > 0 \). In this case, we have \( \varepsilon = \min(3 \lambda_{L} + 2 \mu_{L}, 2 \mu_{L}) \). A standard example for the hardening modulus is \( H = k_{1} \mathbb{I} \), with the fourth-order identity tensor \( \mathbb{I} \) and a constant \( k_{1} > 0 \), hence \( \nu = k_{1} \) holds.

As in Herzog and Meyer (2011), we introduce the linear and compact operator \( R : L^{2}(\Omega; \mathbb{R}^{3}) \times L^{2}(\Gamma_{N}; \mathbb{R}^{3}) \rightarrow V' \) by

\[
\langle R(f, g), \varphi \rangle := \langle \ell, \varphi \rangle.
\]
3 Known Results

First we address the solvability of (1.1). Note that $a(\cdot, \cdot)$ is clearly bounded and also coercive on $Z$, since Young’s inequality implies

$$a(W, W) = \int_\Omega [(\varepsilon(u) - p) : C(\varepsilon(u) - p) + p : H p] \, dx$$

$$\geq \frac{c}{\zeta} \|\varepsilon(u) - p\|^2_2 + \frac{h}{\zeta} \|p\|^2_2 \geq \frac{c}{1 - \kappa} \|\varepsilon(u)\|^2_2 + \left(\frac{c}{1 - 1/\kappa} + h\right) \|p\|^2_2$$  \hfill (3.1)

for any $\kappa > 0$. Any choice of $\kappa$ subject to $c / (c + h) < \kappa < 1$, together with Korn’s inequality (see (Temam, 1983, Proposition 1.1)) then gives the ellipticity of $a$:

$$a(W, W) \geq g (\|u\|^2_V + \|p\|^2_S)$$  \hfill (3.2)

for some positive constant $g$. Moreover, in the considered case of linear kinematic hardening, $j$ is convex and finite on all of $Z = V \times Q$. Therefore, existence and uniqueness for (1.1) follow by standard arguments, see e.g. (Han and Reddy, 1999a, Theorem 6.6):

**Lemma 3.1.** For every $\ell \in V'$, there is a unique solution $(u, p) \in V \times Q$ of (1.1).

While we treat primarily the primal formulation of static plasticity in this paper, it is useful to recall the dual formulation as well. In place of the plastic strain $p$, the dual formulation uses the stress and backstress $(\sigma, \chi)$, which are confined to a feasible set $K$. In our case of the von Mises yield condition, $K$ is defined as

$$K := \left\{ (\tau, \mu) \in S \times S : (\tau(x), \mu(x)) \in K \text{ a.e. in } \Omega \right\}$$

and

$$K := \left\{ (\tau, \mu) \in \bar{S} \times \bar{S} : |\tau^D + \mu^D| \leq \sigma_0 \right\}.$$  \hfill (3.3)

Here and throughout,

$$\tau^D = \tau - (1/3) \text{ trace}(\tau) I$$  \hfill (3.4)

denotes the deviatoric (trace-free) part of the matrix $\tau \in \mathbb{R}^{3 \times 3}$, and $I$ is the identity matrix.

**Lemma 3.2.** Problem (1.1) is equivalent to the dual problem of the following VI of first kind in mixed form: given $\ell \in V'$, find $u \in V$ and $(\sigma, \chi) \in K$ such that

$$\int_\Omega \sigma : C^{-1}(\tau - \sigma) \, dx + \int_\Omega \chi : H^{-1}(\mu - \chi) \, dx$$

$$- \int_\Omega \varepsilon(u) : (\tau - \sigma) \, dx \geq 0 \quad \text{for all } (\tau, \mu) \in K$$  \hfill (3.5a)

$$\int_\Omega \varepsilon(v) : \sigma \, dx = \langle \ell, v \rangle \quad \text{for all } v \in V.$$  \hfill (3.5b)

The equivalence holds in the following sense: Problem (3.5) admits a unique solution $(\sigma, \chi, u) \in S \times S \times V$. It is related to the unique solution $(u, p)$ of (1.1) via

$$\sigma = C(\varepsilon(u) - p) \quad \text{and} \quad \chi = -H p.$$
4 Regularized control problems

Following the road map in de los Reyes (2011) we consider the following family of regularized optimal control problems:

\[
\text{Minimize } \frac{1}{2} \| u - u_d \|_{L^2(\Omega; \mathbb{R}^3)}^2 + \frac{v_1}{2} \| f \|_{L^2(\Omega; \mathbb{R}^3)}^2 + \frac{v_2}{2} \| g \|_{L^2(\Gamma_N; \mathbb{R}^3)}^2 \\
\text{s.t. } a(W, Y) + \int_\Omega h_\gamma(p) : q \, dx = \langle \ell, v \rangle \text{ for all } Y = (v, q) \in Z.
\] (4.1a)

The function

\[
h_\gamma(p) := s. o. \frac{p}{m_\gamma(|p|)}
\] (4.2)

is the derivative of a Huber-type regularization (see (Huber, 1981, eq. (7.14)) or (Huber, 1973, eq. (1.6))) of $j$ with the following local smoothing of the function $\mathbb{R} \ni p \mapsto \max(0, p) \in \mathbb{R}$, parametrized by $\gamma > 0$:

\[
m_\gamma(p) := \begin{cases} 
\gamma p, & \text{if } \gamma p \geq s. o. + \frac{1}{2\gamma} \\
s. o. + \gamma \left(\gamma p - s. o. + \frac{1}{2\gamma}\right)^2, & \text{if } |\gamma p - s. o.| \leq \frac{1}{2\gamma} \\
s. o., & \text{if } \gamma p \leq s. o. - \frac{1}{2\gamma},
\end{cases}
\] (4.3)

The definition of $m_\gamma$ implies that the Nemytskii operators associated with $m_\gamma$ and $m'_\gamma$ map $L^p(\Omega)$ into $L^p(\Omega)$ for every $p \in [1, \infty]$. Furthermore, since $h_\gamma : \mathbb{R} \to \mathbb{R}$ is bounded, it follows immediately that the associated Nemytskii operator maps $\mathbb{S} = L^2(\Omega; \mathbb{S})$ into $L^\infty(\Omega; \mathbb{S})$. To simplify notation, we denote this Nemytskii operator by the same symbol. It is easy to see that $h_\gamma : \mathbb{S} \to \mathbb{S}$ is monotone and globally Lipschitz, see also the preprint (de los Reyes et al., 2013, Lemma 4.1). By standard arguments, this Nemytskii operator inherits these properties from $h_\gamma : \mathbb{S} \to \mathbb{S}$. We refer to Goldberg et al. (1992) or (Tröltzsch, 2010, Section 4.3). Thus the Browder-Minty theorem immediately gives the following result.

**Theorem 4.1.** For each $\ell \in V'$ the regularized equation (4.1b) admits a unique solution $W_\gamma = (u_\gamma, p_\gamma) \in Z$. The associated solution operator $G_\gamma : V' \ni \ell \mapsto W_\gamma \in Z$ is Lipschitz continuous with a Lipschitz constant independent of $\gamma$.

The following theorem addresses the convergence of the regularization and uses arguments analogous to (de los Reyes, 2012, Theorem 3.3 and Theorem 3.5) so its proof is omitted here.
Theorem 4.2. (a) Let $\ell \in V'$ be given and denote by $W_{\gamma} = (u_{\gamma}, p_{\gamma}) \in Z$ the unique solution of (4.1b). The sequence $\{W_{\gamma}\}_{\gamma > 0}$ converges strongly in $Z$ to the unique solution $W$ of (1.1) as $\gamma \to \infty$.

(b) For each $\gamma > 0$, there exists a globally optimal solution for problem (4.1). Moreover, every sequence $\{(f_{\gamma}, g_{\gamma})\}_{\gamma > 0}$ of global solutions to (4.1) contains a weakly convergent subsequence. Any weak accumulation point is a globally optimal solution for (1.3).

(c) In case the global solution $(f^*, g^*) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ of (1.3) is unique, then $(f_{\gamma}, g_{\gamma}) \to (f^*, g^*)$ strongly in $L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ as $\gamma \to \infty$.

Remark 4.3. It is well known that with a slight modification of the regularized problems (4.1), the result of Theorem 4.2 can be sharpened so that every local minimum of (1.3) can be approximated. To be more precise, let $(f^*, g^*)$ be an arbitrary local minimum of (1.3). When the term

$$ r \frac{1}{2} \|f - f^*\|^2_{L^2(\Omega; \mathbb{R}^3)} + r \frac{1}{2} \|g - g^*\|^2_{L^2(\Gamma_N; \mathbb{R}^3)} $$

(4.4)

with sufficiently large $r > 0$ is added to the objective in (4.1a), then it can be shown that a sequence of solutions of these modified regularized problems exists which converges strongly in $L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ to $(f^*, g^*)$, cf. also Barbu (1984) and Mignot and Puel (1984).

5 Differentiability of the Regularized Solution Operator

This section addresses the Fréchet differentiability of the solution map of (4.1b). In fact, we prove this result in a slightly more general setting of the form

$$ a(W, Y) + \int_{\Omega} h(p) : q \, dx = \langle \ell, v \rangle \quad \text{for all } Y = (v, q) \in Z, $$

(5.1)

with unknown $W = (u, p) \in Z$ and a general nonlinear Nemytskii operator $h$. Assumptions on $h$ are given below, and they admit $h_{\gamma}$ from (4.2) as a special case. Equation (5.1) with its general nonlinearity is of independent interest, since it comprises, for example, models of static viscoplasticity if $h : \mathbb{S} \to \mathbb{S}$ is properly chosen.

The main step in proving the differentiability of the solution map $\ell \mapsto W$ of (5.1) is to establish an integrability result for the solution $W$, i.e., to show that $(u, p) \in W_{1,p}^0(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{Q})$. This is achieved in Theorem 5.2 and Corollary 5.3. The main result of this section in the abstract setting is Theorem 5.5. Lemma 5.6 shows its applicability to the regularized state equation (4.1b), and Corollary 5.7 summarizes the differentiability result for the regularized state equation (4.1b).
In this section, we work with the following assumption for the nonlinearity \( h \) in order to achieve the higher integrability result. An additional assumption will be added later (Assumption 5.4).

**Assumption 5.1 (Nonlinearity \( h \)).** The function \( h : \mathbb{S} \to \mathbb{S} \) is continuously differentiable and its derivative satisfies

\[
\|h'(p)\|_{\mathcal{L}(\mathbb{S})} \leq L_h \quad \text{for all } p \in \mathbb{S},
\]

(5.2)

\[
q : h'(p) q \geq 0 \quad \text{for all } p, q \in \mathbb{S}
\]

(5.3)

with a constant \( L_h > 0 \). For every \( p \in \mathbb{S} \), the operator \( h'(p) \in \mathcal{L}(\mathbb{S}) \) is self-adjoint, i.e. \( h'(p) \) is a symmetric fourth-order tensor in the sense of (2.1). For simplicity, we also assume \( h(0) = 0 \).

The conditions in Assumption 5.1 clearly imply that \( h : \mathbb{S} \to \mathbb{S} \) is monotone and globally Lipschitz continuous with Lipschitz constant \( L_h \). These properties carry over to the Nemytskii operator associated with \( h \), which maps \( L^p(\Omega; \mathbb{S}) \) into \( L^p(\Omega; \mathbb{S}) \), for any \( p \in [1, \infty] \). For simplicity, we denote this Nemytskii operator by the same symbol.

Next we define the nonlinear map \( \mathcal{N} : Z \to Z' \) by the left hand side of (5.1), i.e.,

\[
\langle \mathcal{N} W, Y \rangle = a(W, Y) + (h(p), q)_{\Omega}.
\]

(5.4)

We know from (3.2) and the monotonicity of \( h \) that \( \mathcal{N} \) is strongly monotone, i.e.,

\[
\langle \mathcal{N}(u_1 - u_2, p_1 - p_2), (u_1 - u_2, p_1 - p_2) \rangle \geq a \left( \|u_1 - u_2\|^2_V + \|p_1 - p_2\|^2_S \right).
\]

Moreover \( \mathcal{N} \) is clearly continuous. The Browder-Minty theorem implies that (5.1) admits a unique solution \( W = (u, p) \in V \times Q \) for every \( \ell \in V' \).

We now transform (5.1) into an equation in the displacement \( u \) only. Let us briefly sketch the approach. We borrow ideas from linear saddle-point systems and apply them to

\[
\begin{bmatrix}
\varepsilon(v)
\end{bmatrix}^T \begin{bmatrix}
C(x) & -C(x) \Pi^D \\
-\Pi^D C(x) & \Pi^D \left[ C(x) + \mathbb{H}(x) + h(\cdot) \right] \Pi^D
\end{bmatrix} \begin{bmatrix}
\varepsilon(u)
\end{bmatrix}^T,
\]

which is a pointwise representation of the terms appearing on the left hand side of (5.1). The symbol \( \Pi^D : \mathbb{S} \to Q \) denotes the orthogonal projection to the deviatoric part, \( \Pi^D \tau := \tau^D \) for \( \tau \in \mathbb{S} \), see (3.4). Note that the (2,2) block is nonlinear.

To achieve the reduction to \( u \), we test (5.1) with \( v = 0 \) and \( q \in Q \) arbitrary and arrive at

\[
\Pi^D \left[ C(x) p(x) + \mathbb{H}(x) p(x) + h(p(x)) \right] = \Pi^D C(x) \varepsilon(u(x)) \quad \text{a.e. in } \Omega.
\]

(5.5)

Note that \( Q \) consists of all elements of \( S \) with vanishing trace so that only the deviatoric parts show up in (5.5). Let us denote the left hand side in (5.5) by \( \mathcal{F} : Q \to Q \), i.e. \( \mathcal{F}(p) := \Pi^D \left[ C p + \mathbb{H} p + h(p) \right] \). Of course \( \mathcal{F} \) depends on \( x \), since \( C \) and \( \mathbb{H} \) need
not to be constant, but we suppress this dependence in the following for the sake of
readability. Thanks to $\Pi^D \tau : q = \tau : q$ for all $\tau \in \mathbb{S}$ and $q \in Q$, the monotonicity of $h$
yields
\[
(F(q_1) - F(q_2)) : (q_1 - q_2) \geq (\varepsilon + h) |q_1 - q_1|^2 \quad \text{for all } q_1, q_2 \in Q, \tag{5.6}
\]
where $\varepsilon$ and $h$ are the coercivity constants from Assumption 2.2. Moreover, as $\Pi^D$ is
linear and bounded with constant one, Assumptions 2.2 and 5.1 imply
\[
|F(q_1) - F(q_2)| \leq (\varepsilon + h) |q_1 - q_2| \quad \text{for all } q_1, q_2 \in Q. \tag{5.7}
\]
As the constants in (5.6) and (5.7) are independent of $x$, the Browder-Minty theorem
implies that $F(x, \cdot) : Q \to Q$ is continuously invertible for almost all $x \in \Omega$ such that
(5.5) gives
\[
p(x) = F^{-1}(x, \Pi^D C(x) \varepsilon (u(x))) \quad \text{a.e. in } \Omega. \tag{5.8}
\]
Moreover, since $F$ is strongly monotone and globally Lipschitz, the pointwise inverse
$F^{-1}(x, \cdot) : Q \to Q$ is strongly monotone and Lipschitz continuous with constants inde-
pendent of $x$. This also follows from the Browder-Minty theorem. Arguing as in (Betz and
Meyer, 2012, Theorem 2.4), one can show in addition that $x \mapsto F^{-1}(x, q)$ is mea-
surable for every $q \in Q$. Thus the Nemitskii operator associated with $F$ satisfies the
Carathéodory condition and it maps $L^p(\Omega; Q)$ into $L^p(\Omega; Q)$ for every $p \in [1, \infty]$. We
will denote this operator by the same symbol. Moreover, according to Goldberg et al.
(1992) or (Tröltzsch, 2010, Section 4.3), the Lipschitz continuity of $F^{-1}(x, \cdot)$ carries over
to its Nemitskii operator.

We now test (5.1) with $q = 0$ and $v \in V$ and eliminate $p$ by (5.8). This yields the desired
reduced formulation:
\[
\int_{\Omega} b(x, \varepsilon(u)) : \varepsilon(v) \, dx = (\ell, v) \quad \text{for all } v \in V, \tag{5.9}
\]
where we are using the abbreviation
\[
b(x, \cdot) : \mathbb{S} \ni \varepsilon \mapsto C(x) \left( \varepsilon - F^{-1}(x, \Pi^D C(x) \varepsilon) \right) \in \mathbb{S}. \tag{5.10}
\]
The above derivation shows that, if $(u, p) \in V \times Q$ solves (5.1), then $u$ is a solution of
(5.9). On the other hand it is easily seen that, if $u \in V$ solves (5.9), then $(u, p)$ with $p$
as defined in (5.8) is a solution of (5.1). Thus (5.1) and (5.9) are indeed equivalent. The
unique solvability of (5.1) implies that, for every $\ell \in V'$, (5.9) admits a unique solution
$u \in V$.

As indicated at the beginning of this section, we wish to prove higher integrability of $u$
provided that $\ell$ is more regular than just an element of $V'$. To this end we aim to apply
(Herzog et al., 2011b, Theorem 1.1) to the reduced problem (5.9). This requires
us to verify that $b(x, \cdot)$ is strongly monotone and Lipschitz continuous with constants
independent of $x$. The uniform Lipschitz continuity of $b$ follows immediately from the
uniform boundedness of $C$ and the uniform Lipschitz continuity of $F^{-1}(x, \cdot)$. To
prove the strong monotonicity of $b$, let us define, for any $x \in \Omega$, the pointwise map $\mathcal{M}(x, \cdot, \cdot): \mathbb{S} \times \mathbb{Q} \to \mathbb{S} \times \mathbb{Q}$ by

$$
\mathcal{M}(x, \epsilon, p) := \begin{bmatrix} C(x)(\epsilon - p) \\ F(x, p) - \Pi \mathcal{C}(x) \epsilon \end{bmatrix}.
$$

(5.11)

Arguing as in (3.1) and using again the monotonicity of $h$, we infer that $\mathcal{M}(x, \cdot, \cdot)$ is strongly monotone:

$$
(\mathcal{M}(x, \epsilon_1, p_1) - \mathcal{M}(x, \epsilon_2, p_2)) : (\epsilon_1 - \epsilon_2, p_1 - p_2) \geq m (|\epsilon_1 - \epsilon_2|^2 + |p_1 - p_2|^2) \quad (5.12)
$$

with some $m > 0$. Due to the uniform coercivity of $\mathcal{C}$ and $\mathcal{H}$, the constant $m$ is independent of $x$. Now let $\epsilon_1, \epsilon_2 \in \mathbb{S}$ be arbitrary and let $p_1 = F^{-1}(x, \Pi \mathcal{C}(x)(\epsilon_1 - \epsilon_2))$ and $p_2 = 0$. Inserting these into (5.12) yields

$$
m |\epsilon_1 - \epsilon_2|^2 \leq m (|\epsilon_1 - \epsilon_2|^2 + |p_1|^2) \\
\leq (\mathcal{M}(\epsilon_1, p_1) - \mathcal{M}(\epsilon_2, 0)) : (\epsilon_1 - \epsilon_2, p_1) \\
= \left( \mathcal{C}(\epsilon_1 - \epsilon_2) - \mathcal{C} F^{-1}(\Pi \mathcal{C}(\epsilon_1 - \epsilon_2)) \right) : (\epsilon_1 - \epsilon_2) - F(0) : p_1 \\
= b(\cdot, \epsilon_1 - \epsilon_2) : (\epsilon_1 - \epsilon_2) - F(0) : p_1,
$$

where again we suppress the dependence on $x$. Because of Assumption 5.1 we have $F(0) = 0$ so that (5.10) gives

$$
m |\epsilon_1 - \epsilon_2|^2 \leq b(x, \epsilon_1 - \epsilon_2) : (\epsilon_1 - \epsilon_2),
$$

uniformly for $x \in \Omega$. In addition, because of $h(0) = 0$, we have $b(0) = 0 \in L^\infty(\Omega; \mathbb{S})$. Since $x \mapsto b(x, \epsilon)$ is also measurable for every $\epsilon \in \mathbb{S}$ — thanks to the measurability of $F^{-1}$ mentioned above — the operator $b$ satisfies the conditions in (Herzog et al., 2011b, Assumption 1.5(2)). Taking into account Assumption 2.1, (Herzog et al., 2011b, Theorem 1.1) is applicable and it yields the following higher integrability result:

**Theorem 5.2.** There exists an index $p > 2$ such that for every $\ell \in W^{1-p}_D(\Omega; \mathbb{R}^3) = W^{1,p}_D(\Omega; \mathbb{R}^3)^*$, the equation (5.9) admits a unique solution $u \in W^{1,p}_D(\Omega; \mathbb{R}^3)$. Moreover, the associated solution mapping is globally Lipschitz, i.e., there exists $L > 0$ such that

$$
\|u_1 - u_2\|_{W^{1,p}(\Omega; \mathbb{R}^3)} \leq L \|\ell_1 - \ell_2\|_{W^{1,p}(\Omega; \mathbb{R}^3)}
$$

holds for all $\ell_1, \ell_2 \in W^{1,p}(\Omega; \mathbb{R}^3)$.

Due to the trace theorem and Sobolev embeddings, an inhomogeneity of the form

$$
\langle \ell, v \rangle = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds = \langle R(f, g), v \rangle
$$

(5.13)
with \( f \in L^2(\Omega; \mathbb{R}^3) \) and \( g \in L^2(\Gamma_N; \mathbb{R}^3) \) does represent an element of \( W^{-1,p}(\Omega; \mathbb{R}^3) \) for every \( p < 4 \).

In order to transfer the result to the original problem (5.1), we exploit that we can recover the plastic strain \( p \) from \( u \) by the pointwise relation (5.8). Since \( F^{-1} \) is globally Lipschitz from \( L^p(\Omega; \mathbb{S}) \) to \( L^p(\Omega; \mathbb{Q}) \) as seen above, Assumption 2.2 implies the same for \( F^{-1} \circ \Pi_{DC} \) and we can conclude the following result.

**Corollary 5.3.** There exists an index \( p \in (2, 4) \) such that for every \( f \in L^2(\Omega; \mathbb{R}^3) \) and \( g \in L^2(\Gamma_N; \mathbb{R}^3) \), the equation (5.1) with \( \ell \) as in (5.13) admits a unique solution \((u, p) \in W^{1,p}_D(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{Q})\). Moreover, the associated solution mapping \( \mathcal{G} : L^2(\Omega; \mathbb{R}^3) \to L^2(\Omega; \mathbb{R}^3) \to W^{1,p}_D(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{Q}) \) is globally Lipschitz continuous, i.e., there exists \( L > 0 \) such that
\[
\|u_1 - u_2\|_{W^{1,p}(\Omega; \mathbb{R}^3)} + \|p_1 - p_2\|_{L^p(\Omega; \mathbb{Q})} \leq L \left( \|f_1 - f_2\|_{L^2(\Omega; \mathbb{R}^3)} + \|g_1 - g_2\|_{L^2(\Gamma_N; \mathbb{R}^3)} \right)
\]
holds for all \( f_1, f_2 \in L^2(\Omega; \mathbb{R}^3) \) and \( g_1, g_2 \in L^2(\Gamma_N; \mathbb{R}^3) \).

Based on this integrability result, we are now in the position to prove the differentiability of \( \mathcal{G} \). An additional assumption is needed.

**Assumption 5.4.** Assume that the Nemytskii operator associated with \( h \) is Fréchet differentiable from \( L^p(\Omega; \mathbb{S}) \) to \( L^p(\Omega; \mathbb{Q}) \) with \( p > 2 \) as in Corollary 5.3.

This assumption will be verified for the particular nonlinearity \( h_\gamma \) from (4.2) in Lemma 5.6 below.

**Theorem 5.5.** Under Assumptions 5.1 and 5.4, the mapping \( \mathcal{G} \) is Fréchet differentiable from \( L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3) \) to \( Z \), and the derivative \( \delta W = (\delta u, \delta p) = \mathcal{G}'(f, g)(\delta f, \delta g) \) at \((f, g)\) in the direction \((\delta f, \delta g)\) is given by the unique solution \( \delta W = (\delta u, \delta p) \in Z \) of the linearized equation
\[
a(\delta W, Y) + \int_\Omega h'(p) \delta p : q \, dx = \int_\Omega \delta f \cdot v \, dx + \int_{\Gamma_N} \delta g \cdot v \, ds \quad \text{for all } Y = (v, q) \in Z,
\]
where \((u, p) = \mathcal{G}(f, g)\).

**Proof.** Thanks to the Lipschitz continuity in Corollary 5.3, the result follows from (Wachsmuth, 2014, Theorem 2.1) by setting \( Y^0 = Z, Z^0 = Z', U = L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3), \) and \( Y^+ = W^{1,p}_D(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{Q}) \) there.
Lemma 5.6. For every $\gamma > 0$, the function $h_\gamma$ defined in (4.2) satisfies the conditions in Assumption 5.1. Moreover, the associated Nemytskii operator is Fréchet differentiable from $L^p(\Omega; \mathbb{S})$ to $L^2(\Omega; \mathbb{S})$ for every $p > 2$, so that Assumption 5.4 holds as well. The derivative of $h_\gamma$ at $p \in L^p(\Omega; \mathbb{S})$ is given by

$$
(h_\gamma'(p) \delta p)(x) = \text{s.o. } \gamma \left( \frac{\delta p(x)}{m_\gamma(|p(x)|)} \right) - m_\gamma'(|p(x)|) \left( \frac{p(x)}{m_\gamma(|p(x)|)^2} \right)
$$

with $m_\gamma : \mathbb{R} \to \mathbb{R}$ as defined in (4.3) and $m_\gamma'$ given by

$$
m_\gamma'(p) = \begin{cases} 
\gamma, & \text{if } \gamma p \geq \text{s.o.} + \frac{1}{\varepsilon_\gamma}, \\
\gamma^2 (\gamma |p - \text{s.o.}| + \frac{1}{\varepsilon_\gamma}), & \text{if } \gamma |p - \text{s.o.}| \leq \frac{1}{\varepsilon_\gamma}, \\
0, & \text{if } \gamma p \leq \text{s.o.} - \frac{1}{\varepsilon_\gamma}.
\end{cases}
$$

Proof. The first assertion follows from elementary calculations, see the preprint (de los Reyes et al., 2013, Lemma 4.1) for details. To show the differentiability property of the Nemytskii operator associated with $h_\gamma$, let $p > 2$ be given and define $r = 2p/(p - 2) < \infty$. Thanks to (5.2) we have that $S \ni p \mapsto h_\gamma'(p) \in L^\infty(\Omega; L(\mathbb{S})) \hookrightarrow L^r(\Omega; L(\mathbb{S}))$. Since moreover $h_\gamma$ maps $S$ and thus also $L^p(\Omega; \mathbb{S})$ into $L^\infty(\Omega; \mathbb{S}) \hookrightarrow S$, the desired Fréchet differentiability follows from abstract results for Nemytskii operators, see Goldberg et al. (1992) or (Tröltzsch, 2010, Section 4.3).

Corollary 5.7. For any $\gamma > 0$, there exists $p \in (2, 4)$ such that the solution operator $G_\gamma$ of (4.1b), with $\ell = R(f, g)$ as in (5.13), maps $L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ into $W^{1,p}_D(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{Q})$, and it is Fréchet differentiable from $L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ to $Z$. The derivative $\delta W = (\delta u, \delta p) = G_\gamma'(f, g)(\delta f, \delta g)$ at $(f, g)$ in the direction $(\delta f, \delta g)$ is given by the unique solution $\delta W = (\delta u, \delta p) \in Z$ of the linearized equation (5.14) with $h_\gamma$ in place of $h$.

Proof. Lemma 5.6 shows that the nonlinearity $h_\gamma$ from (4.2) verifies Assumptions 5.1 and 5.4. Consequently, Corollary 5.3 and Theorem 5.5 apply for this particular choice of $h$. Thus the solution operator $G_\gamma$ of (4.1b) maps $L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ to $W^{1,p}_D(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{Q})$ for some $p \in (2, 4)$. The Fréchet derivative is given by (5.14) with $h_\gamma'$ in place of $h'$.

6 Optimality System

Based on the differentiability result established in Theorem 5.5, we can now derive first-order necessary optimality conditions for the regularized control problem (4.1) by
standard arguments. The main endeavor is to pass to the limit $\gamma \to \infty$ to obtain the optimality system for the original problem (1.3) stated in Theorem 6.4 below.

**Theorem 6.1 (Regularized optimality system).** Suppose that $(f^\gamma, g^\gamma) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ is a locally optimal solution of the regularized problem (4.1) with associated state $W^\gamma = (u^\gamma, p^\gamma) \in Z$. Then there exists an adjoint state $Z^\gamma = (w^\gamma, r^\gamma) \in Z$ and multipliers $\varrho^\gamma \in Q$ and $\pi^\gamma \in Q$ such that the following optimality system is satisfied:

\begin{align}
    a(W^\gamma, Y) + \int_\Omega q^\gamma : q \, dx &= \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds \quad \text{for all } Y = (v, q) \in Z \tag{6.1a} \\
    q^\gamma &= s \circ \gamma \frac{\rho^\gamma}{m^\gamma(|p^\gamma|)} \quad \text{a.e. in } \Omega \tag{6.1b} \\
    a(Y, Z^\gamma) + \int_\Omega \pi^\gamma : q \, dx &= -\int_\Omega (u^\gamma - u_d) \cdot v \, dx \quad \text{for all } Y = (v, q) \in Z \tag{6.1c} \\
    \pi^\gamma &= s \circ \gamma \left( \frac{r^\gamma}{m^\gamma(|p^\gamma|)} - m'(|p^\gamma|) \frac{r^\gamma : p^\gamma}{m^\gamma(|p^\gamma|)^2 |p^\gamma|} \right) \quad \text{a.e. in } \Omega \tag{6.1d} \\
    \nu_1 f^\gamma - w^\gamma &= 0 \quad \text{a.e. in } \Omega, \quad \nu_2 g^\gamma - w^\gamma = 0 \quad \text{a.e. in } \Gamma_N. \tag{6.1e}
\end{align}

As already indicated the above optimality system can be derived by standard techniques, see e.g. (Tröltzsch, 2010, Chapter 4). We note that (6.1a)–(6.1b) represents the state equation (4.1). We introduced the term $q^\gamma$ through (6.1b) in order to facilitate the passage to the limit in Proposition 6.2 below. Equations (6.1c)–(6.1d) are the adjoint equation, while (6.1e) represents the stationarity condition w.r.t. the controls. It will be convenient to refer to (6.1) as the regularized optimality system (rather than the optimality system of the regularized control problem), and to (6.1c)–(6.1d) as the regularized adjoint equation, etc. These slightly imprecise terms will not give rise to confusion.

We now pass to the limit in the regularized optimality system (6.1). As an intermediate step in the proof of Theorem 6.4, we obtain in Proposition 6.2 a preliminary version of first-order necessary optimality conditions for the original problem (1.3). This result will be refined in Lemma 6.3 below, which subsequently leads to the proof of Theorem 6.4. The arguments underlying the proof of Proposition 6.2 are similar to (de los Reyes, 2011, Theorem 5.1) with the following modifications: the multiplier $\pi$ has more regularity. It is a function in $Q$ in contrast to its counterpart $\xi$ in de los Reyes (2011), which is only a distribution. Due to this fact, the complementarity in (6.2e) and the sign condition in (6.2f) hold pointwise, which requires further modifications. We refer to the preprint de los Reyes et al. (2013) for the full proof.

**Proposition 6.2 (Preliminary optimality system).** Let $(f, g)$ be a locally optimal solution for (1.3) with associated state $(u, p) \in Z$. Then there exists an adjoint state $Z = (w, r) \in Z$ such that \( a(W, Y) + \int_\Omega q : q \, dx = \int_\Omega f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds \quad \text{for all } Y = (v, q) \in Z \) and

\begin{align}
    q &= s \circ \gamma \frac{\rho}{m(|p|)} \quad \text{a.e. in } \Omega \tag{6.2a} \\
    a(Y, Z) + \int_\Omega \pi : q \, dx &= -\int_\Omega (u - u_d) \cdot v \, dx \quad \text{for all } Y = (v, q) \in Z \tag{6.2b} \\
    \pi &= s \circ \gamma \left( \frac{r}{m(|p|)} - m'(|p|) \frac{r : p}{m(|p|)^2 |p|} \right) \quad \text{a.e. in } \Omega \tag{6.2c} \\
    \nu_1 f - w &= 0 \quad \text{a.e. in } \Omega, \quad \nu_2 g - w = 0 \quad \text{a.e. in } \Gamma_N. \tag{6.2d}
\end{align}
Z and multipliers \( q \in Q \) and \( \pi \in Q \) such that the following optimality system is satisfied:

\[
\begin{align*}
 a(W, Y) + \int_{\Omega} q : q \, dx &= \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds \quad \text{for all } Y = (v, q) \in Z, \\
 q : p &= s.o. \quad |q| \leq s.o. \quad \text{a.e. in } \Omega, \\
 a(Y, Z) + \int_{\Omega} \pi : q \, dx &= -\int_{\Omega} (u - u_d) \cdot v \, dx \quad \text{for all } Y = (v, q) \in Z, \\
 v_1 f - w &= 0 \quad \text{a.e. in } \Omega, \\
 v_2 g - w &= 0 \quad \text{a.e. in } \Gamma_N,
\end{align*}
\]

as well as

\[
\begin{align*}
 \pi : p &= 0 \quad \text{a.e. in } \Omega, \\
 \pi : r &\geq 0 \quad \text{a.e. in } \Omega, \\
 r &= 0 \quad \text{a.e. in } \mathcal{I} = \{ x \in \Omega : |q(x)| < s.o. \}.
\end{align*}
\]

Note that (6.2b) is equivalent to \( q \in s.o. \partial |p| \), the subdifferential of the Frobenius norm.

By further exploiting the structure of the problem at hand, it turns out that we can refine the result of Proposition 6.2 by introducing a new scalar valued multiplier \( \vartheta \), which gives more structure to the adjoint plastic strain \( r \) and leads to a strengthening of conditions (6.2f) and (6.2g), so that we finally arrive at the optimality system presented in Theorem 6.4, eq. (6.10). This step is essential in proving the equivalence of (6.10) to the C-stationarity system for the equivalent dual formulation of the problem in Section 7.

Lemma 6.3. Under the conditions of Proposition 6.2, there exists a multiplier \( \vartheta \in L^2(\Omega) \) such that the adjoint plastic strain can be decomposed as follows:

\[
 r = \frac{1}{s.o.} (|p| \pi + \vartheta q). 
\]

Proof. From (6.1b) and (6.1d) it follows that \( r_\gamma \) satisfies

\[
r_\gamma = \frac{1}{s.o.} \left( \frac{m_\gamma(|p_\gamma|)}{\gamma} \pi_\gamma + \frac{m_\gamma'(|p_\gamma|)}{\gamma} \frac{r_\gamma \cdot p_\gamma}{|p_\gamma|} q_\gamma \right).
\]

We introduce the approximate multiplier

\[
\vartheta_\gamma := \frac{m_\gamma'(|p_\gamma|)}{\gamma} \frac{r_\gamma \cdot p_\gamma}{|p_\gamma|}.
\]

From (5.16) it follows that \( 0 \leq m_\gamma'(p) \leq \gamma \) for all \( p \in \mathbb{R} \). Thus the boundedness of \( r_\gamma \) (see step 4 in the proof of Proposition 6.2 in de los Reyes et al. (2013) for details) implies
that
\[ \|\vartheta_\gamma\|_{L^2(\Omega)} \leq \|r_\gamma\|_S \leq K < \infty. \]
Consequently, up to a subsequence, \( \vartheta_\gamma \rightarrow \vartheta \) in \( L^2(\Omega) \) for some \( \vartheta \in L^2(\Omega) \).

Let us now define the auxiliary quantity \( \alpha_\gamma \in L^2(\Omega) \) by
\[
\alpha_\gamma = \frac{m_\gamma(|p_\gamma|)}{\gamma} = \begin{cases} 
\frac{s.o.}{\gamma} & \text{a.e. in } A_\gamma, \\
\frac{1}{2}(\gamma|p_\gamma| - s.o. + \frac{1}{2\gamma})^2 & \text{a.e. in } S_\gamma, \\
0 & \text{a.e. in } I_\gamma,
\end{cases}
\]
where \( m_\gamma \) was defined in (4.3) and \( A_\gamma, S_\gamma, \) and \( I_\gamma \) are
\[
A_\gamma = \{ x \in \Omega : \gamma|p_\gamma| \geq s.o. + \frac{1}{2\gamma} \}, \\
S_\gamma = \{ x \in \Omega : \gamma|p_\gamma| - s.o. \leq \frac{1}{2\gamma} \}, \\
I_\gamma = \Omega \setminus (A_\gamma \cup S_\gamma) = \{ x \in \Omega : \gamma|p_\gamma| \leq s.o. - \frac{1}{2\gamma} \}.
\]

We now verify that \( \alpha_\gamma \to |p| \) strongly in \( L^2(\Omega) \). First of all, Young’s inequality yields
\[
\int_\Omega (\alpha_\gamma - |p|)^2 \, dx \leq 2 \int_\Omega \left( \frac{m_\gamma(|p_\gamma|)}{\gamma} - |p_\gamma| \right)^2 \, dx + 2 \||p_\gamma| - |p||^2_{L^2(\Omega)}.
\]
From the representation (6.5) we get
\[
\frac{m_\gamma(|p_\gamma|)}{\gamma} - |p_\gamma| = \begin{cases} 
0 & \text{a.e. in } A_\gamma, \\
\frac{s.o}{\gamma} - |p_\gamma| & \text{a.e. in } I_\gamma.
\end{cases}
\]
On \( S_\gamma \) we have by (6.5) and Young’s inequality
\[
\left( \frac{m_\gamma(|p_\gamma|)}{\gamma} - |p_\gamma| \right)^2 \leq 2 \left( \frac{s.o.}{\gamma} - |p_\gamma| \right)^2 + \frac{1}{2} \left( \gamma|p_\gamma| - s.o. + \frac{1}{2\gamma} \right)^4 \text{ a.e. in } S_\gamma.
\]
Moreover the definitions of \( S_\gamma \) and \( I_\gamma \), respectively, immediately yield
\[
\left| \gamma|p_\gamma| - s.o. + \frac{1}{2\gamma} \right| \leq \left| \gamma|p_\gamma| - s.o. \right| + \frac{1}{2\gamma} \leq \frac{1}{\gamma} \text{ a.e. in } S_\gamma
\]
and
\[
\left| \frac{s.o.}{\gamma} - |p_\gamma| \right| \leq \frac{s.o.}{\gamma} + |p_\gamma| \leq \frac{2s.o.}{\gamma} - \frac{1}{2\gamma^2} \leq \frac{2s.o.}{\gamma} \text{ a.e. in } I_\gamma.
\]

Inserting the estimates (6.7) and (6.8) into (6.6) and applying (6.9a) and (6.9b) to the resulting estimate yields
\[
\int_\Omega (\alpha_\gamma - |p|)^2 \, dx \leq 4 \int_{S_\gamma} \left( \frac{s.o.}{\gamma} - |p_\gamma| \right)^2 \, dx + \int_{S_\gamma} \left( \gamma|p_\gamma| - s.o. + \frac{1}{2\gamma} \right)^4 \, dx \\
+ 2 \int_{I_\gamma} \left( \frac{s.o.}{\gamma} - |p_\gamma| \right)^2 \, dx + 2 \|p_\gamma - p\|_S^2 \\
\leq c \|\Omega\| \left( \frac{1}{\gamma^4} + \frac{1}{\gamma^2} \right) + 2 \|p_\gamma - p\|^2_5 \to 0 \text{ as } \gamma \to \infty.
\]
In the course of the proof of Proposition 6.2 one shows $\varrho_\gamma \to \varrho$ strongly in $Q$ and $\pi_\gamma \to \pi$ weakly in $S$, see de los Reyes et al. (2013). Moreover, $\vartheta_\gamma \to \vartheta$ weakly in $L^2(\Omega)$ holds, see the proof of lemma 6.3, and $\alpha_\gamma \to |p|$ strongly in $L^2(\Omega)$ has just been shown. We therefore finally obtain

\[
s.o.r_\gamma = \alpha_\gamma \pi_\gamma + \vartheta_\gamma \varrho_\gamma \Rightarrow |p| \pi + \vartheta q \quad \text{in } L^1(\Omega; \mathcal{S})
\]

so that $s.o. r = |p| \pi + \vartheta q$ holds as claimed.

With the help of the above lemma, we can now prove one of our main results, namely a stationarity condition for the original non-regularized problem (1.3):

**Theorem 6.4**. Let $(f, g) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ be a locally optimal solution for (1.3) with associated state $W = (u, p) \in \mathcal{Z}$. Then there exists an adjoint state $Z = (w, r) \in \mathcal{Z}$ and multipliers $q \in Q, \pi \in Q$ and $\vartheta \in L^2(\Omega)$ such that the following optimality system is satisfied:

\[
a(W, Y) + \int_\Omega q : q \, dx = \langle \ell, v \rangle \quad \text{for all } Y = (v, q) \in \mathcal{Z} \quad (6.10a)
\]

\[
q : p = s.o. |p|, \quad |q| \leq s.o. \quad \text{a.e. in } \Omega \quad (6.10b)
\]

\[
a(Y, Z) + \int_\Omega \pi : q \, dx = -\int_\Omega (u - u_d) \cdot v \, dx \quad \text{for all } Y = (v, q) \in \mathcal{Z} \quad (6.10c)
\]

\[
v_1 f - w = 0 \quad \text{a.e. in } \Omega, \quad v_2 g - w = 0 \quad \text{a.e. in } \Gamma_N \quad (6.10d)
\]

as well as

\[
\pi : p = 0 \quad \text{a.e. in } \Omega \quad (6.10e)
\]

\[
s.o. r = |p| \pi + \vartheta q \quad \text{a.e. in } \Omega \quad (6.10f)
\]

\[
\vartheta q : \pi \geq 0 \quad \text{a.e. in } \Omega \quad (6.10g)
\]

\[
\vartheta = 0 \quad \text{a.e. in } \mathcal{I} = \{ x \in \Omega : |q(x)| < s.o. \} \quad (6.10h)
\]

**Proof**. Let $(f, g) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)$ be a locally optimal solution for (1.3) with associated state $W = (u, p) \in \mathcal{Z}$. In view of Proposition 6.2, the preliminary optimality system (6.2) holds. A comparison between (6.2) and (6.10) shows that we only have to verify (6.10f)–(6.10h).

First we note that (6.2b) implies

\[
q(x) : p(x) = |q(x)||p(x)|, \quad (6.11)
\]

i.e., the alignment of $q$ and $p$, which will be useful in the course of the proof.
Lemma 6.3 gives the existence of \( \tilde{\vartheta} \in L^2(\Omega) \) such that \( s.o. \, r = |p| \pi + \tilde{\vartheta} q \) holds. If we define
\[
\vartheta(x) := \begin{cases} 
\tilde{\vartheta}(x), & \text{if } q(x) \neq 0, \\
0, & \text{if } q(x) = 0,
\end{cases} \tag{6.12}
\]
then we obtain \( \vartheta \in L^2(\Omega) \) and still \( s.o. \, r = |p| \pi + \vartheta q \) holds. If we define
\[
\vartheta(x) := \begin{cases} 
\tilde{\vartheta}(x), & \text{if } q(x) \neq 0, \\
0, & \text{if } q(x) = 0,
\end{cases} \tag{6.12}
\]
then we obtain \( \vartheta \in L^2(\Omega) \) and still \( s.o. \, r = |p| \pi + \vartheta q \) holds, so (6.10f) is shown.

Next we show that \( \vartheta = 0 \) holds on the set \( I \) (where \( |q| < s.o. \)), which is (6.10h). Equation (6.2b) implies \( |p| = 0 \) a.e. in \( I \), while (6.2g) shows \( r = 0 \) on \( I \). Now (6.10f) implies \( \vartheta q = 0 \) a.e. in \( I \). Due to (6.12) this gives (6.10h).

It remains to prove (6.10g), i.e., \( \vartheta q : \pi \geq 0 \). If \( p(x) \neq 0 \), then (6.11) gives that \( q(x) = \kappa p(x) \) with some \( \kappa \in \mathbb{R} \) and we obtain from (6.2e) that \( \pi(x) = 0 \) and thus in particular
\[
\vartheta(x) q(x) : \pi(x) = \vartheta(x) \kappa p(x) : \pi(x) = 0
\]
holds. If, on the other hand, \( p(x) = 0 \) holds, then (6.10f) and (6.2f) imply
\[
\vartheta(x) q(x) : \pi(x) = s.o. \, r(x) : \pi(x) \geq 0.
\]
Thus in any case (6.10g) holds, which concludes the proof.

**Remark 6.5.** Assume that (6.10) holds. Then (6.10b) implies \( p = 0 \) a.e. in \( I \). Hence (6.10h) gives \( r = 0 \) a.e. in \( I \), which is (6.2g). Moreover, by (6.10g) we obtain \( \pi : r \geq 0 \), which is (6.2f). Thus (6.10) implies (6.2).

## 7 Equivalence to C-stationarity

As was mentioned in the introduction, the strength of the optimality system obtained in Theorem 6.4 is not obvious to gauge. This is due to the lack of a classification scheme for optimality conditions of MPECs which involve variational inequalities of the second kind. We prove in this section that the result of Theorem 6.4 is in fact equivalent to C-stationarity of another optimal control problem closely related and equivalent to (1.3). That problem is obtained when the primal formulation of elastoplasticity (1.1) is replaced by the equivalent dual formulation (3.5), a variational inequality of first kind, see Lemma 3.2. To our best knowledge, this is the first time that an optimality system for an MPEC with a VI of the second kind is being classified in this sense.

By the replacement of the primal by the dual VI, we obtain the following optimal control problem
\[
\text{Minimize } J(u, f, g) \quad \text{s.t. } (3.5), \tag{7.1}
\]
which is equivalent to (1.3). Since (3.5) is a VI of first kind, it can be reformulated by means of a complementarity system involving a Lagrange multiplier \( \lambda \in L^2(\Omega) \), the so-called plastic multiplier, see (7.2a)–(7.2c) below. For a rigorous proof of the existence
and uniqueness of \( \lambda \), we refer to (Herzog et al., 2011a, Theorem 1.4 and Corollary 1.2). Problem (7.1) can thus be considered an MPCC (mathematical programs with complementarity constraints). The following first-order optimality conditions for (7.1) are proved in (Herzog et al., 2012, Theorem 3.16) by means of a Moreau-Yosida based regularization of (3.5) and a subsequent limit analysis:

**Theorem 7.1** (C-stationarity, see (Herzog et al., 2012, Theorem 3.16)). Let \((u, \sigma, \chi, \lambda, f, g) \in V \times S \times S \times L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)\) be a locally optimal solution of (7.1). Then there exist adjoint stresses \((\zeta, \psi) \in S \times S\) and displacement \(w \in V\), and a multiplier \(\theta \in L^2(\Omega)\) such that the following optimality system is satisfied:

\[
\begin{align*}
\int_{\Omega} \sigma : C^{-1} \tau \, dx + \int_{\Omega} \chi : H^{-1} \mu \, dx - \int_{\Omega} \varepsilon(u) : \tau \, dx \\
+ \int_{\Omega} \lambda (\sigma^D + \chi^D) : (\tau^D + \mu^D) \, dx &= 0 \quad \text{for all } \tau, \mu \in S \\
\int_{\Omega} \sigma : \varepsilon(v) \, dx &= \int_{\Omega} f \cdot v \, dx \int_{\Omega} g \cdot v \, ds \quad \text{for all } v \in V \\
0 &\leq \lambda \perp |\sigma^D + \chi^D|^2 - s \cdot o^2 \leq 0 \quad \text{a.e. in } \Omega
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega} \zeta : C^{-1} \tau \, dx + \int_{\Omega} \psi : H^{-1} \mu \, dx - \int_{\Omega} \varepsilon(w) : \tau \, dx \\
+ \int_{\Omega} \lambda (\zeta^D + \psi^D) : (\tau^D + \mu^D) \, dx \\
+ \int_{\Omega} \theta (\sigma^D + \chi^D) : (\tau^D + \mu^D) \, dx &= 0 \quad \text{for all } \tau, \mu \in S \\
- \int_{\Omega} \zeta : \varepsilon(v) \, dx &= + \int_{\Omega} (u - u_d) \cdot v \, dx \quad \text{for all } v \in V \\
v_1 f - w &= 0 \quad \text{a.e. in } \Omega, \quad v_2 g - w = 0 \quad \text{a.e. on } \Gamma_N
\end{align*}
\]

\[
\begin{align*}
\lambda (\sigma^D + \chi^D) : (\zeta^D + \psi^D) &= 0 \quad \text{a.e. in } \Omega \\
\theta (|\sigma^D + \chi^D|^2 - s \cdot o^2) &= 0 \quad \text{a.e. in } \Omega \\
\theta (\sigma^D + \chi^D) : (\zeta^D + \psi^D) &\geq 0 \quad \text{a.e. in } \Omega.
\end{align*}
\]

The notation for the variables pertaining to the dual formulation is the same as in Herzog et al. (2012) with the minor exception that here, the multiplier \(\mu\) associated with the non-negativity constraint for the plastic multiplier is not introduced as an extra variable. Instead, it has been replaced by its governing equation, \(\mu = (\sigma^D + \chi^D) : (\zeta^D + \psi^D)\). Moreover, in order to comply with the sign of the adjoint displacement \(w\) in (6.10), we needed to change the sign of all adjoint states and multipliers \((\zeta, \psi, w\) and \(\theta)\) appearing in (Herzog et al., 2012, Theorem 3.16).

The optimality system in (7.2) is known as the system of C-stationarity. Note that, as
usual for MPCCs, the optimality system does not involve a multiplier for the complementarity relation in (7.2c). Moreover, it is characteristic for C-stationarity that a sign condition is only known for the product in (7.2i), and not for each term individually. In the following we will show that the optimality systems (6.10) and (7.2) are indeed equivalent.

**Theorem 7.2.** Let \((u, p, f, g) \in V \times Q \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)\) satisfy the primal optimality system (6.10) with multipliers \((w, r, q, \pi, \theta) \in V \times Q^3 \times L^2(\Omega)\). Define

\[
\sigma = C(\varepsilon(u) - p), \quad \chi = -\mathbb{H} p, \quad \lambda = \frac{|p|}{s.o.}, \quad (7.3a)
\]

\[
\zeta = C(\varepsilon(w) - r), \quad \psi = -\mathbb{H} r, \quad \theta = \frac{\varrho}{s.o.}, \quad (7.3b)
\]

Then \((u, \sigma, \chi, \lambda, f, g, w, \zeta, \psi, \theta)\) fulfill the C-stationarity conditions in (7.2).

Let, on the other hand, \((u, \sigma, \chi, \lambda, f, g) \in V \times S \times L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Gamma_N; \mathbb{R}^3)\) fulfill (7.2) with multipliers \((w, \zeta, \psi, \theta) \in V \times S \times L^2(\Omega)\). If we define

\[
p = \varepsilon(u) - C^{-1}\sigma, \quad q = \sigma^D + \chi^D, \quad (7.4a)
\]

\[
r = \varepsilon(w) - C^{-1}\zeta, \quad \pi = \zeta^D + \psi^D, \quad \vartheta = s.o. \theta, \quad (7.4b)
\]

then \((u, p, f, g, w, r, q, \pi, \vartheta)\) satisfies (6.10).

**Proof:** We first assume that (6.10) holds and show (7.2).

Step 1. The state system (7.2a)–(7.2c):

Employing the definition of \(\sigma\) in (7.3a) and testing (6.10a) with \((0, v), v \in V\) arbitrary, immediately gives (7.2b). Taking \(v = 0\) and \(q \in Q\) arbitrary in (6.10a), the definition of \(\sigma\) and \(\chi\) in (7.3a) yields

\[
q = q^D = \sigma^D + \chi^D, \quad (7.5)
\]

since \(Q\) consists of all trace-free (purely deviatoric) tensor functions in \(S\). Thus (6.10b) yields \(|\sigma^D + \chi^D| \leq s.o. a.e. in \Omega\). Next we show

\[
\lambda (\sigma^D + \chi^D) = p \quad a.e. in \Omega. \quad (7.6)
\]

Thanks to the definition of \(\lambda\) in (7.3a) this is obviously true for \(p(x) = 0\). To show the relation where \(p(x) \neq 0\), employ the definition of \(\lambda\), (7.5), and (6.11) to obtain , which give

\[
\lambda (\sigma^D + \chi^D) = \frac{|p|}{s.o.} q = \frac{|p|}{|q|} q = p \quad a.e. in \{x \in \Omega: p(x) \neq 0\}.
\]

Using (7.6) and the definition of \(\lambda\) immediately gives \(\lambda(|\sigma^D + \chi^D| - s.o.) = 0\) so that the complementarity system in (7.2c) is verified. Note that the sign condition on \(\lambda\) follows from its definition in (7.3c).
By solving the definitions of $\sigma$ and $\chi$ in (7.3a) for $p$, we find $p = \varepsilon(u) - C^{-1}\sigma$ and $p = -H^{-1}\chi$. By multiplying the first equation with an arbitrary $\tau \in S$ and the second with an arbitrary $\mu \in S$, integrating over $\Omega$, and using that $p$ is trace-free, we arrive at

$$
\int_{\Omega} \sigma : C^{-1}\tau \, dx + \int_{\Omega} \chi : H^{-1}\mu \, dx - \int_{\Omega} \varepsilon(u) : \tau \, dx + \int_{\Omega} p : (\tau^D + \mu^D) \, dx = 0
$$

for all $\tau, \mu \in S$. Inserting (7.6) then yields (7.2a).

Step 2. The adjoint equation (7.2d) and (7.2e):
Similarly to above, we choose $q = 0$ in (6.10c) so that the definition of $\zeta$ in (7.3b) yields (7.2e). Choosing $v = 0$ in (6.10c), we obtain completely analogously to (7.5) that

$$
\pi = \zeta^D + \psi^D. \quad (7.7)
$$

If we solve the definitions of $\zeta$ and $\psi$ for $r$ and test the arising equations with $\tau, \mu \in S$, then we arrive at

$$
\int_{\Omega} \zeta : C^{-1}\tau \, dx + \int_{\Omega} \psi : H^{-1}\mu \, dx - \int_{\Omega} \varepsilon(w) : \tau \, dx + \int_{\Omega} r : (\tau^D + \mu^D) \, dx = 0 \quad \text{for all } \tau, \mu \in S.
$$

Together with (6.10f), (7.7), and (7.5), the definitions of $\lambda$ and $\theta$ give (7.2d).

Step 3. The complementarity relations (7.2g)–(7.2i):
Thanks to (7.6) and (7.7), (7.2g) follows immediately from (6.10e). Similarly, (7.2i) is a direct consequence of (6.10g) together with (7.5), (7.7), and the definition of $\theta$ in (7.3b). The complementarity relation in (7.2h) follows from (6.10h) and the definition of $\theta$, which imply that

$$
\theta = 0 \quad \text{a.e. in } \{x \in \Omega : |\sigma^D(x) + \chi^D(x)| < s.o.\}.
$$

Since the gradient equations in (7.2f) coincide with these in (6.10d), this ends the first part of the proof.

To show the reverse direction, assume that (7.2) holds.

Step 1. The state system (6.10a) and (6.10b):
If one tests (7.2a) with $(\tau, -\tau)$ with $\tau \in S$ arbitrary, then $\varepsilon(u) - C^{-1}\sigma + H^{-1}\chi = 0$ is obtained. The definition of $p$ in (7.4a) thus yields

$$
p = -H^{-1}\chi. \quad (7.8)
$$

Consequently, (7.2a) implies

$$
p = \lambda (\sigma^D + \chi^D), \quad (7.9)
$$

which in particular implies that $\text{trace}(p) = 0$ a.e. in $\Omega$ such that $p \in Q$. In view of the definition of $p$ in (7.4a) and (7.8) we find $q = \left[\varepsilon(u) - p\right]^D - [Hp]^D$ and thus

$$
\int_{\Omega} q : q \, dx + \int_{\Omega} q : H \, p \, dx - \int_{\Omega} C(\varepsilon(u) - p) : q \, dx = 0 \quad \text{for all } q \in Q. \quad (7.10)
$$

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Solving the definition of \( p \) for \( \sigma \), inserting and adding the arising equation to (7.10) yields (6.10a). By the definition of \( q \) in (7.4a) and (7.2c) we immediately obtain \( |q| \leq \text{s.o.} \) a.e. in \( \Omega \). Moreover, from (7.9), the definition of \( q \), and the complementarity relation in (7.2c) it follows \( q : p = \lambda |\sigma^D + \chi^D|^2 = |\lambda (\sigma^D + \chi^D)| \text{s.o.} = \text{s.o.} \ |p| \), i.e. the remaining condition in (6.10b).

Step 2. The adjoint equation (6.10c):
Analogously to the derivation of (7.8) we test (7.2d) with \((\tau, -\tau)\) with arbitrary \( \tau \in S \), which gives

\[
r = -H^{-1}\psi. \tag{7.11}
\]

By solving this equation for \( \psi \), the definition of \( r \) and \( \pi \) in (7.4a) and (7.4b), respectively imply \( \pi = [\zeta + \psi]^D = [C(\varepsilon(w) - r) - Hr]^D \) so that

\[
\int_{\Omega} (Hr : q - C(\varepsilon(w) - r) : q + \pi : q) \, dx = 0 \quad \text{for all } q \in Q. \tag{7.12}
\]

Now, we can argue as in the case of the state equation. The definition of \( r \) gives \( \zeta = C(\varepsilon(w) - r) \). If we insert this into (7.2e) and add the arising equation to (7.12), then (6.10c) is obtained.

Step 3. The complementarity relations (6.10e)–(6.10h):
If ones inserts (7.11) together with the definition of \( q \) and \( \pi \) into (7.4a) and (7.4b), then similarly to (7.9)

\[
r = \lambda \pi + \theta q \tag{7.13}
\]

is obtained. Since \( \text{trace}(q) = \text{trace}(\pi) = 0 \) a.e. in \( \Omega \) by definition, this yields \( r \in Q \).
By (7.9) we have \( p(x) = 0 \) a.e. where \( \lambda(x) = 0 \) holds. If \( \lambda(x) \neq 0 \), then (7.2c) implies \( |\sigma^D(x) + \chi^D(x)| = \text{s.o.} \). Therefore, in any case, (7.9) gives \( |p| = \text{s.o.} \lambda \). Together with (7.13) and the definition of \( \theta \) in (7.4b) this yields (6.10f).

In view of (7.9) and the definition of \( \pi \) in (7.4b), (6.10e) follows directly from (7.2g).
Moreover, using the definitions of \( q \), \( \pi \), and \( \theta \) in (7.4a) and (7.4b), we obtain (6.10g) immediately from (7.2i). In addition, the complementarity relation in (7.2h) shows \( \theta(x) = 0 \) a.e. where \( |q(x)| = |\sigma^D(x) + \chi^D(x)| < \text{s.o.} \), which gives in turn (6.10h).
Since the gradient equations in (7.2f) and (6.10d) are the same, this ends the proof.

8 Numerical Experiments
We consider the numerical solution of a particular example of the regularized problem (4.1) for a sequence of increasing values of \( \gamma \). In fact, we apply some minor changes to the original problem setting in an effort to make the problem more realistic from a practical point of view. These modifications do not affect our theory, and the changes to the regularized optimality system (6.1) as well as the optimality system (6.10) for the original problem are going to be obvious.
8.1 First Example

In our first model problem, we restrict the discussion to boundary loads \( g \) as controls. These controls act only on a part \( \Gamma_C \) of the Neumann boundary \( \Gamma_N \), and \( g = 0 \) is fixed on \( \Gamma_N \setminus \Gamma_C \). We also slightly modify the first term in the objective so that only the first two components of the displacement field are observed, and the desired state \( u_d \) has only two components. Finally the observation takes place on part of the boundary \( \Gamma_O \subset \Gamma \), instead of inside the domain. We arrive at the following problem.

\[
\begin{align*}
\text{Minimize} \quad & J(u, g) := \frac{\beta}{2} \| u^{(1,2)} - u_d \|^2_{L^2(\Gamma_O; \mathbb{R}^2)} + \frac{\nu_2}{2} \| g \|^2_{L^2(\Gamma_C; \mathbb{R}^3)} \\
\text{s.t.} \quad & a(W, Y) + \int_{\Omega} h_\gamma(p) : q \, dx = \int_{\Gamma_C} g \cdot v \, ds \quad \text{for all } Y = (v, q) \in Z,
\end{align*}
\]  

(8.1)

with \( h_\gamma \) given in (4.2). For notational convenience, we denote the variables by \( u \) instead of \( u_\gamma \) etc.

Our computational domain is the scaled Fichera corner \( \Omega = (-L, L)^3 \setminus (0, L)^3 \) with \( L = 100 \,[\text{mm}] \). The control boundary \( \Gamma_C \) is the upper boundary at \( z = L \), and it coincides with the observation boundary \( \Gamma_O \). The Dirichlet boundary \( \Gamma_D \) is located at the opposite face, i.e., at \( z = -L \). Moreover, the remaining data in (8.1) is set to

- elasticity modulus \( E = 206 900 \, \text{N mm}^{-2} \)
- Poisson ratio \( \nu = 0.29 \)
- shear modulus \( \mu_L = \frac{E}{2(1+\nu)} \approx 80 194 \, \text{N mm}^{-2} \)
- dilation modulus \( \lambda_L = \frac{E \nu}{(1+\nu)(1-2\nu)} \approx 110 744 \, \text{N mm}^{-2} \)
- yield stress \( s.o. = 450 \sqrt{2/3} \, \text{N mm}^{-2} \approx 367.42 \, \text{N mm}^{-2} \)
- hardening parameter \( k_1 = 100 000 \, \text{N mm}^{-2} \)
- desired state \( u_d = (15, 0)^T \, \text{mm} \)
- coefficient \( \beta = 1 \, \text{mm}^{-4} \)
- coefficient \( \nu_2 = 1 \times 10^{-10} \, \text{mm}^2 \, \text{N}^{-2} \).

The boundary traction \( g \) has \( \text{N mm}^{-2} \) as its unit. The units of the coefficients \( \beta \) and \( \nu_2 \) are chosen in such a way that the objective \( J(u, g) \) becomes dimensionless. In our calculations, we choose a sequence of regularization parameters \( \gamma \) as shown in Table 8.1.

This numerical exercise was solved within the finite element framework FENICS (version 1.6) Logg et al. (2012a). We formulated the Lagrangian pertaining to (8.1) in the Unified Form Language UFL Alnæs (2012) and exploited the automatic differentiation capabilities of the form compiler FFC Logg et al. (2012b) in order to automatically generate the first-order optimality system, which is similar to (6.1). Minor changes to (6.1) are necessary due to the changes which led from (4.1) to (8.1).

The optimality system was discretized using vector-valued continuous \( P_1 \) elements for the primal and adjoint displacements \( u \) and \( w \) as well as the control \( g \), while symmetric
matrix-valued and trace free discontinuous $\mathbb{P}_0$ elements were used to discretize the primal and adjoint plastic strains $p$ and $r$. The quantities $q$ and $\pi$ were not introduced as extra variables, cf. (6.1b) and (6.1d). We then used Newton’s method (in the form of the solve method for a FENICs nonlinear variational problem, with the absolute stopping criterion set to $10^{-5}$ and relative stopping criterion disabled) with automatically generated second-order derivatives to solve the nonlinear optimality system. The overall algorithm for a fixed value of $\gamma$ is thus a basic sequential quadratic programming (SQP) approach. For each value of $\gamma$ in Table 8.1, we started the Newton iteration from the solution of the previous value. The first problem was started from an all-zeros initial guess for $u$, $p$, $g$, $w$ and $r$. It is noteworthy that, for this concrete model problem with the problem data as specified above, this simple procedure is sufficient to guarantee the fast convergence of Newton’s method (without globalization efforts).

For this study, sparse direct linear algebra was used to solve the arising linear systems. The uniform tetrahedral mesh has 7 000 nodes, approximately 35 000 cells and the total number of unknowns is roughly 415 000. To accelerate the solution, the built-in MPI-based parallel assembly and solution capabilities of FENICs were used to distribute the problem onto 24 cores. The wall clock time for one Newton step was approximately 100 seconds.

Figure 8.1 shows the displacement $u$ obtained at the solution with regularization parameter $\gamma = 10^4$. The desired state $u_d$ is achieved rather closely. Figure 8.1 also shows the control $g$, acting on the upper surface. The boundary stresses are in the range $|g| \in [0, 2,918]$ N mm$^{-2}$. Moreover, we show in Figure 8.2 the Frobenius norm of the deviator of the combined stress $\sigma + \chi = C(\varepsilon(u) - p) - \mathbb{H} p$, compare (7.3a). We point out that $|\sigma^D + \chi^D|$ nearly reaches the upper bound of $s.o. = 367.4235$ N mm$^{-2}$. Figure 8.2 also shows the Frobenius norm of the plastic strain $|p|$. As expected, nonzero plastic strains are concentrated in the areas where the stresses are at the yield stress limit. It is also as expected that these large stresses occur in areas around the edges leading to the re-entrant corner. All visualizations were done in PARAVIEW.

Figure 8.1: Displacement magnitude $|u|$, and control $g$ and its magnitude $|g|$ at the solution.

Table 8.1 gives some insight into the behavior of the Huber-type regularization. We see that below $\gamma = 10^3$, the plastic behavior is effectively suppressed. Between $\gamma = 10^3$
Figure 8.2: Sum of stress deviators $|\sigma^D + \chi^D|$ and plastic strain $|p|$ at the solution.

and $\gamma = 10^4$, we observe the sudden onset of a pronounced elastoplastic behavior, which is accompanied by a moderate increase in the number of Newton iterations due to increasing nonlinear effects. This observation was found to be independent of the size of the discretization.

Table 8.1: Dependence of number of Newton steps (starting from the solution at the previous value of $\gamma$ as initial guess), value of the deviator sum $\|\sigma^D + \chi^D\|_{L^\infty(\Omega; Q)}$, and distance to the dual feasibility constraint $s.o. - \|\sigma^D + \chi^D\|_{L^\infty(\Omega; Q)} \geq 0$ on the choice of the regularization parameter $\gamma$ at fixed discretization level for the first example.

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<th>deviator</th>
<th>distance</th>
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<td>$3.6109 \cdot 10^2$</td>
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<td>2</td>
<td>367.4235</td>
<td>$4.7829 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$1.0000 \cdot 10^4$</td>
<td>3</td>
<td>367.4235</td>
<td>$4.7417 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

As Table 8.1 demonstrates, the regularization approach is capable to efficiently solve the first example. The number of Newton iterations remain small, even for larger values of $\gamma$, and the algorithm can cope with a substantial increase of $\gamma$ without further globalization efforts. The behavior of the algorithm however strongly depends on the nature of the specific problem under consideration. When the nonsmooth character of (1.1) is more pronounced, then the performance of our smoothing method decreases significantly, as the next example shows.
8.2 Second Example

The second example corresponds to problem \texttt{mpcdist2} from the problem collection \textsc{OPTPDE}. This problem is a two-dimensional version of (1.3) again with only minor modifications, which do not affect the theory. In this problem, the yield stress \(\sigma_0\) and the control cost parameter \(\alpha\) are free parameters, and the exact solution is known and depends on these parameters. In our experiment, we set \(\sigma_0 = 1\) and \(\alpha = 10^6\). In the collection \textsc{OPTPDE}, this problem is formulated in its dual, i.e., stress-based form, cf. (7.2). An essential feature of this example is that the biactive set (where \(\lambda\) and \(\mid \sigma^D + \chi^D \mid^2 - \tilde{\sigma}_0^2\) vanish simultaneously) at the solution has a positive measure. The control-to-state mapping is therefore not Gâteaux-differentiable in the optimal solution, and this nonsmoothness indeed affects our algorithm, as we will see in the following.

The triangular mesh in this example has approximately 73 000 nodes, 141 000 cells and the total number of unknowns is roughly 1 000 000. We applied the same discretization and algorithmic framework as in our first example. This means in particular that we use an SQP method without globalization for the solution of the regularized problems with fixed \(\gamma\) and that we apply a “warm start”, i.e., we took the solution from the previous \(\gamma\)-iteration as initial value for the SQP method with updated \(\gamma\). This time however, this procedure does not show such a satisfactory convergence behavior as it did in the first example. First of all it is not possible to rapidly increase the \(\gamma\)-values without a breakdown of the SQP convergence so that 41 \(\gamma\)-updates are needed to increase \(\gamma\) from 3.5 to 35.0. Moreover, even with this moderate enlargement of the \(\gamma\)-values, the number of SQP-iterations increase substantially, as Table 8.2 shows. Notice that the relevant range of \(\gamma\) is different here from the first example due to the order of magnitude for the artificial material parameters. A further increase in \(\gamma\) would require globalization of the Newton solver. Nevertheless, we can see from Table 8.2 that both the feasibility constraint and the displacement field (for which the exact, unregularized solution is known) have converged to reasonable accuracy.

8.3 Discussion

It is interesting that, in both test cases, the Huber-type regularization of the primal formulation of elastoplasticity has the effect that the combined stresses \(\sigma + \chi\) in the dual formulation actually stay below the yield threshold. When one starts with the dual formulation (7.1), it is natural to employ a penalty-type regularization, which leads to a convergence of the combined stresses from above. We point out that the penalized dual formulation can be interpreted as an elastoviscoplastic model. A similar interpretation seems to be lacking in the primal formulation and deserves further investigation. With regard to the numerical performance, the Huber-type regularization showed a different behavior in the two examples. A possible explanation for this observation consists in the nonsmooth nature of the second example, whereas the first example seems to be mainly unaffected by the nonsmoothness of the control-to-state map. A more efficient
Table 8.2: Dependence of number of Newton steps (starting from the solution at the previous value of $\gamma$ as initial guess), value of the deviator sum $\|\sigma^D + \chi^D\|_{L^\infty(\Omega;Q)}$, distance to the dual feasibility constraint s. o. $-\|\sigma^D + \chi^D\|_{L^\infty(\Omega;Q)} \geq 0$ and relative displacement error $\|u - u_{\text{exact}}\|_{L^\infty(\Omega;\mathbb{R}^2)}/\|u_{\text{exact}}\|_{L^\infty(\Omega;\mathbb{R}^2)}$ on the choice of the regularization parameter $\gamma$ at fixed discretization level for the second example.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Newton</th>
<th>deviator distance</th>
<th>rel. disp. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.5000 \cdot 10^0$</td>
<td>1</td>
<td>$0.7527 \cdot 10^{-1}$</td>
<td>$1.3777 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$3.7074 \cdot 10^0$</td>
<td>1</td>
<td>$0.7643 \cdot 10^{-1}$</td>
<td>$1.3188 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$7.8355 \cdot 10^0$</td>
<td>1</td>
<td>$0.8891 \cdot 10^{-1}$</td>
<td>$7.0413 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$8.2998 \cdot 10^0$</td>
<td>2</td>
<td>$0.8968 \cdot 10^{-1}$</td>
<td>$6.6810 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$1.8581 \cdot 10^1$</td>
<td>2</td>
<td>$0.9761 \cdot 10^{-2}$</td>
<td>$3.0164 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$1.9682 \cdot 10^1$</td>
<td>3</td>
<td>$0.9800 \cdot 10^{-2}$</td>
<td>$2.8369 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.0848 \cdot 10^1$</td>
<td>3</td>
<td>$0.9837 \cdot 10^{-2}$</td>
<td>$2.6662 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.2084 \cdot 10^1$</td>
<td>3</td>
<td>$0.9872 \cdot 10^{-2}$</td>
<td>$2.5040 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.3392 \cdot 10^1$</td>
<td>3</td>
<td>$0.9906 \cdot 10^{-3}$</td>
<td>$2.3499 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.4778 \cdot 10^1$</td>
<td>4</td>
<td>$0.9937 \cdot 10^{-3}$</td>
<td>$2.2036 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.6246 \cdot 10^1$</td>
<td>4</td>
<td>$0.9966 \cdot 10^{-3}$</td>
<td>$2.0646 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.7801 \cdot 10^1$</td>
<td>5</td>
<td>$0.9993 \cdot 10^{-4}$</td>
<td>$1.9327 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$2.9449 \cdot 10^1$</td>
<td>6</td>
<td>$1.0000 \cdot 10^{-8}$</td>
<td>$1.8076 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$3.1194 \cdot 10^1$</td>
<td>6</td>
<td>$1.0000 \cdot 10^{-8}$</td>
<td>$1.6888 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$3.3042 \cdot 10^1$</td>
<td>6</td>
<td>$1.0000 \cdot 10^{-8}$</td>
<td>$1.5763 \cdot 10^{-2}$</td>
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<tr>
<td>$3.5000 \cdot 10^1$</td>
<td>7</td>
<td>$1.0000 \cdot 10^{-8}$</td>
<td>$1.4696 \cdot 10^{-2}$</td>
</tr>
</tbody>
</table>

and robust algorithm calls for several modifications of our straightforward implementation. First of all, the SQP method has to be globalized by means of a suitable merit function, and preconditioned iterative solvers for the Newton systems have to be employed. Moreover, regularization and discretization parameters have to be coupled adaptively, and more attention should be paid to the update of the regularization parameter. At this point an update rule based on a model function could be helpful. Such a strategy has successfully been applied to optimal control problems governed by the obstacle problem; cf. e.g. Kunisch and Wachsmuth (2011). Another issue concerns the choice of the initial value for the SQP method. Instead of just taking the value of the previous $\gamma$-iteration, it may be possible to apply an extrapolation strategy in the spirit of Hintermüller and Yousept (2010). The final issue concerns a suitable stopping criterion for the SQP method and the $\gamma$ update, since the relation between residual and error norms is $\gamma$-dependent. All these questions however are anything else than trivial to answer and give rise to future research.
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REFERENCES


