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Mathematische Methoden der Unsicherheitsquantifizierung

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Exercise Sheet 8: Numerical Computation of Karhunen-Loève Expansions

Exercise 1

Let $D \subset \mathbb{R}^n$ be a bounded domain and $a(x, \omega) : D \times \Omega \to \mathbb{R}$ a second order random field with mean value 0 and continuous covariance function $c(x, y) : D^2 \to \mathbb{R}_+$. The Karhunen-Loève expansion (KLE) of a can be calculated numerically using the Galerkin method. The Galerkin method tests the eigenvalue equation of the covariance operator C of a with elements $v \in V_N$ of a finite dimensional Galerkin space V_N.

$$\langle C\phi, v\rangle = \lambda \langle \phi, v\rangle \qquad \forall v \in V_N, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of the corresponding space of functions $L^2(D)$.

(a) If we use $\phi \in V_N$ as eigenfunctions then (1) leads to a generalized linear eigenvalue problem with basis coefficients $\boldsymbol{u}$ of the eigenfunction ϕ with respect to a basis $\{v_1, \dots, v_N\}$ of V_N:

$$A\boldsymbol{u} = \lambda M\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^N. \tag{2}$$

What are the components of A and M?

(b) We want to implement the Galerkin method for calculating the KLE of a random field a on $D = [0, 1]$ in MATLAB. This implementation should be done by a function

$$[\Lambda, \Phi] = \texttt{ComputeKL1D}(\texttt{c}, \texttt{N})$$

where

- $\texttt{N} + 1$ is the dimension of the Galerkin space and `c` is a function handle to the covariance function c, which allows vector valued returns,
- Λ is the column vector of `N` **decreasing** eigenvalues and Φ is a matrix where every column is a basis coefficient vector of **normalized** eigenfunctions in the Galerkin space.

We use the space of piecewise linear functions as Galerkin space V_{N+1} for a discretization of $D = [0, 1]$ in N intervals $[x_i, x_{i+1}]$, $x_i = i/N$ with $i = 0, \dots, N$.

A tensor version of the trapezoid rule with points $\{x_i\}_{i=0}^N$ is used as a quadrature formula in $D^2 = [0, 1]^2$:

$$\begin{aligned}\int_0^1 \int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{N^2}\Bigg(&\frac{1}{4}f(0,0) + \frac{1}{2}\sum_{i=1}^{N-1} f(x_i, 0) + \frac{1}{4}f(1,0) + \frac{1}{2}\sum_{i=1}^{N-1} f(1, x_i)\\ &+ \frac{1}{4}f(1,1) + \frac{1}{2}\sum_{i=1}^{N-1} f(x_i, 1) + \frac{1}{4}f(0,1) + \frac{1}{2}\sum_{i=1}^{N-1} f(0, x_i)\\ &+ \sum_{i,j=1}^{N-1} f(x_i, x_j)\Bigg).\end{aligned}$$

Implement the function `ComputeKL1D` in MATLAB and test your implementation with $c(x, y) = \min(x, y) - xy$ (see the last exercise sheet for this covariance function).

Advice 1: This quadrature formula allows a fast assemblation of A in (2) without any loops.

Advice 2: The elements of the mass matrix M in (2) can be calculated exactly in advance.

Advice 3: You can use the MATLAB function `eig` for the calculation of eigenvalues and eigenvectors.

Exercise 2

Consider the following three covariance functions $c : [0,1]^2 \to \mathbb{R}_+$ of Matérn class:

$$
\begin{aligned}
c_{1,R}(x,y) &= \exp\left(-\frac{|x-y|}{R}\right),\\
c_{2,R}(x,y) &= \left(1+\frac{\sqrt{3}\,|x-y|}{R}\right)\exp\left(-\frac{\sqrt{3}\,|x-y|}{R}\right),\\
c_{3,R}(x,y) &= \exp\left(-\frac{|x-y|^2}{R}\right).
\end{aligned}
$$

The correlation length R is given by $R = 1$.

(a) Generate realizations of a Gaussian field G with mean value 0 and covariance functions $c_{1,R}$, $c_{2,R}$ and $c_{3,R}$ for $R = 1$.
For these realizations calculate G at every discretization point $x_i = i/N$, $i = 0, \ldots, 1000$.

(b) Compute the KLE of the three covariance functions with `ComputeKL1D(c,N)` from Exercise 1 with $N = 1000$ and compare the behavior of the eigenvalues in a log-log plot.

(c) We now repeat (a) and (b) for $R = 0.1$ and $R = 0.01$. What do you observe?