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Exercise Sheet 7: Random fields and Karhunen-Loève Expansion

For this exercise we denote by $(\Omega, \mathcal{A}, \mathbf{P})$ a probability space, by $D \subset \mathbb{R}^n$ a bounded domain and by H a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Exercise 1

Let $a(x, \omega) : D \times \Omega \to D$ be a second-order random field with

$$m(x) := \mathbf{E}[a(x)], \qquad c(x, y) := \operatorname{Cov}(a(x), a(y)).$$

Furthermore, we assume that $a(\cdot, \omega) \in L^2(D)$ **P**-a.s. and $a(\cdot, \omega) : \Omega \to L^2(D)$ is a measurable function.

(a) Which conditions are required for c and m such that $a \in L^2(\Omega; L^2(D))$, i.e. under which conditions is a a quadratic integrable random variable with values in $L^2(D)$?

(b) In general, for a random variable $X \in L^2(\Omega; H)$ with values in a Hilbert space H, the *covariance* $\operatorname{Cov}(X)$ is a *operator* $C : H \to H$ fulfilling

$$\langle C\, u, v \rangle = \mathbf{E}\left[\langle X - \mathbf{E}X, u \rangle \langle X - \mathbf{E}X, v \rangle \right] \qquad \forall u, v \in H.$$

Show that the integral operator

$$\left(C\phi \right)(x) := \int_D c(x, y) \phi(y) \, \mathrm{d}y$$

equals the covariance operator of a random variable $a \in L^2(\Omega; L^2(D))$ with values in a Hilbert space.

(c) We assume that realizations of a are **P**-a.s. continuously differentiable with

$$\operatorname{Cov}(\nabla a(x), \nabla a(y)) = \nabla_x \nabla_y c(x, y) \qquad \text{and} \qquad \operatorname{Cov}(\nabla a(x), a(y)) = \nabla_x c(x, y),$$

and in particular there holds $a \in L^2(\Omega; H^1(D))$. Which form does a covariance operator of the $H^1(D)$-valued random variable $a(\cdot, \omega) : \Omega \to H^1(D)$ take?

Exercise 2

(a) Let $\{e_n\}_{n \in \mathbb{N}}$ be a random, complete orthonormal system of H. Under which conditions is the random Fourier series

$$X(\omega) := \sum_{n \in \mathbb{N}} \xi_n(\omega) \, e_n,$$

where $\{\xi_n\}_{n \in \mathbb{N}}$ are real-valued and independent random variables, again a random variable $X \in L^2(\Omega; H)$?

(b) Let $a(x, \omega) : D \times \Omega \to D$ be a Gaussian random field with $a \in L^2(\Omega; L^2(D))$. Furthermore, $\{\phi_n(x)\}_{n \in \mathbb{N}}$ is a complete orthonormal system of $L^2(D)$. Then there follows

$$a(x, \omega) = \sum_{n \in \mathbb{N}} \xi_n(\omega) \, \phi_n(x)$$

with respect to $L^2(\Omega; L^2(D))$. Find the joint cumulative distribution function of $(\xi_n)_{n=1}^N$ for any $N \in \mathbb{N}$!

Exercise 3

We study the covariance function of a *Brownian bridge* B which is a Gaussian process with mean 0 on the domain $D = [0, 1]$ and

$$c(x, y) := \min(x, y) - xy.$$

The covariance operator is again denoted by C, such that $C\phi(x) := \int_0^1 c(x, y)\phi(y)\,\mathrm{d}y$.

(a) Derive a boundary value problem for the eigenfunctions of C by differentiating the eigenvalue equation twice.

(b) Solve the boundary value problem and calculate eigenvalues and eigenfunctions of C. Finally, derive the Karhunen-Loève expansion (KLE) of a Brownian bridge.

(c) Plot realizations of the truncated KLE B_N of B for $N = 10, 100, 1000$.

(d) Estimate the error by truncating the KLE, i.e. calculate $\mathbf{E}[\|B - B_N\|^2_{L^2(D)}]$. How many terms N are necessary, such that the truncation error in terms of the $L^2(\Omega; L^2(D))$ norm is smaller than 10%?

(e) Task (a) showed that C equals the inverse Laplace operator, $C = \Delta^{-1} = \left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^{-1}$ (on an appropriate subspace of $L^2(D)$). It is now possible to define $\Delta^{-\alpha}$, $\alpha > 0$, with respect to the eigensystem $(\lambda_n, \phi_n)_{n\in\mathbb{N}}$ of Δ^{-1}:

$$\Delta^{-\alpha}u(x) := \sum_{n=1}^{\infty} \lambda_n^{\alpha}\langle u, \phi_n\rangle \phi_n(x), \qquad \langle u, \phi_n\rangle = \int_0^1 u(x)\phi_n(x)\,\mathrm{d}x.$$

For which $\alpha > 0$ is $\Delta^{-\alpha} : L^2(D) \to L^2(D)$ again an operator of the trace class and hence a covariance operator?

We study for such α Gaussian processes on $D = [0, 1]$ with mean 0 and covariance operator $\Delta^{-\alpha}$. Calculate realizations of the corresponding KLE with $M = 10, 100, 1000$ for each $\alpha = 0.75, 1, 1.5, 2$ in MATLAB. Plot theses approximations of the covariance function in MATLAB.