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Mathematische Methoden der Unsicherheitsquantifizierung

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Exercise Sheet 6: Monte Carlo Finite Element Method

Exercise 1

Find an explicit form of the solution u of the following boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\,\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = f(x), \qquad u(0) = u_0,\ u(1) = u_1.$$

Which numerical methods can be used to calculate this solution?

Exercise 2

Consider a variational problem for a polygonal shaped domain $D \subset \mathbb{R}^n$ and find $u \in H^1(D)$ such that

$$\int_D a(x)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x = \int_D f(x)\,v(x)\,\mathrm{d}x \qquad \forall v \in H_0^1(D), \qquad u|_{\partial D} = g|_{\partial D},$$

with $a \in L^\infty(\bar{D})$, $\operatorname{essinf}_{x\in D} a(x) \geq c > 0$, $f \in L^2(D)$ and $g \in C^1(\widetilde{D})$ where $D \subset \widetilde{D}$.

(a) Show that the solution u of this variational problem depends in a linear way on f and g and that the corresponding mapping $L : (f,g) \mapsto u$, $L : L^2(D) \times C^1(\widetilde{D}) \to H^1(D)$ is bounded, i.e.

$$\|u\|^2_{H^1(D)} \leq C\left(\|f\|^2_{L^2(D)} + \|g\|^2_{C^1(\widetilde{D})}\right).$$

(b) Derive a variational problem for $\mathbf{E}[u]$ where u is a random variable with values in $H^1(D)$ satisfying $\mathbf{P}$ almost surely

$$\int_D a(x)\nabla u(\omega,x)\cdot\nabla v(x)\,\mathrm{d}x = \int_D f(\omega,x)\,v(x)\,\mathrm{d}x \qquad \forall v \in H_0^1(D), \qquad u(\omega)|_{\partial D} = g(\omega)|_{\partial D},$$

where a is given as above and f and g are random variables with values in $L^2(D)$ and $C^1(\widetilde{D})$ resp. We assume that all expectations exist.

Exercise 3

Wir consider the same variational problem as in Exercise 2 where now $D = [0,1]^2$ and

$$\begin{aligned} a(x,y) &\equiv 1,\\ f(x,y,\omega) &= \exp\left(\xi_1(\omega)\,\sin(\pi(x+y)) + \xi_2(\omega)\,\cos(\pi(x+y))\right),\\ g(x,y,\omega) &= (x - x^2)\,\sin\left(\frac{\pi}{3}\eta(\omega)(1+x+y)\right), \end{aligned}$$

with $\xi_1, \xi_2 \sim N(0,1)$, $\eta \sim U(0,1)$ independently.

(a) Compute by a Monte Carlo-FEM simulation the mean fields **E**$[f]$, **E**$[g]$ and **E**$[u]$. Use at least $M = 1000$ samples and a uniform triangulation with mesh width at most $h = 1/64$.

(b) Find analytical expression for the mean fields **E**$[f]$ and **E**$[g]$ and compute **E**$[u]$ diretly via an appropriate deterministic variational problem.

Compare your results with those from task (a).

(c) Let now $f(x, y) \equiv 1$ and $g(x, y) \equiv 0$ be deterministic and

$$a(x, y, \omega) = \exp\left(\xi(\omega)\, \sin(\pi x) \cos(\pi y)\right), \qquad \xi \sim N(0, 1).$$

Repeat the simulations and computations from task (a) and (b) for this data. What do you observe?

Exercise 4

Prove the following:

Lemma 1. *Let* $V \subseteq H_0^1(D)$, $D \subset \mathbb{R}^n$ *be a closed subspace and* $u_V : \Omega \to V$ *satisfies* **P** *almost surely*

$$\int_D a(x, \omega) \nabla u_V(x, \omega) \cdot \nabla v(x)\, \mathrm{d}x = \int_D f(x, \omega)\, v(x)\, \mathrm{d}x \qquad \forall v \in V, \qquad u_V(\omega)|_{\partial D} \equiv 0,$$

where **P** *a.s.*

$$0 < a_{\min}(\omega) \leq a(x, \omega) \leq a_{\max}(\omega), \qquad \textit{almost everywhere in } D,$$

and $f(\omega) \in L^2(D)$. *The two mappings* $a : \Omega \to L^\infty(D)$ *and* $f : \Omega \to L^2(D)$ *are assumed to be measurable. Then there holds:*

- *If* $f(\omega) \equiv f_0 \in L^2(D)$ **P** *a.s., then from* $1/a_{\min} \in L^p(\Omega; \mathbb{R})$ *for* $p \geq 1$, *follows* $u_V \in L^p(\Omega; H_0^1(D))$.
- *If* $1/a_{\min} \in L^q(\Omega; \mathbb{R})$ *and* $\|f(\omega)\|_{L^2(D)} \in L^r(\Omega; \mathbb{R})$ *with* $q, r \geq 1$ *s.t.* $1/q + 1/r = 1/p \leq 1$ *then there also follows* $u_V \in L^p(\Omega; H_0^1(D))$.

How can you weaken the assumptions if a and f are assumed to be independent? Do the problems from Exercise 1c and 3 meet these assumptions?

Exercise 5

The stationary groundwater level u in a domain $D = [0, 1]^2$ is given by the boundary value problem

$$-\nabla \cdot (a \nabla u) = f, \qquad u|_{\partial D} = g.$$

Here u behaves on the border ∂D like $g(x, y) = 1 + y$. Moreover, let the hydraulic conductivity be given by

$$a(x, y) = \exp\left(-1 + 0.5 \sin\left(\sqrt{3}\pi x\right) \cos\left(\sqrt{2}\pi y\right)\right)$$

and it is believed that there exists an outflow of groundwater in D whose exact position is unknown. Hence, set

$$f(x, y, \omega) = 1 - \frac{10}{(1 + 9(x - \eta_1(\omega))^2)(1 + 9(y - \eta_2(\omega))^2)},$$

with $\eta_1 \sim U(0.25, 0.75), \eta_2 \sim U(0, 0.5)$ being independent. The amount of water within the domain D is of interest. We therefore want to compute the expectation of

$$Q(\omega) := \int_D u(x, y, \omega) \mathrm{d}(x, y).$$

Use for numerical simulations linear triangles and a uniform triangulation as discretization.

(a) Derive a deterministic variational problem for the expectation of Q and solve this problem on a grid with $h = 1/128$.

(b) Use the theory of the finite element method to conject values for the rates α, β and γ of

$$\mathbf{E}[Q - Q_{h_\ell}] \propto h_\ell^\alpha, \qquad \mathbf{E}[Q_{h_{\ell+1}} - Q_{h_\ell}] \propto h_\ell^\beta, \qquad \mathrm{Cost}(Q_{h_\ell}) \propto h_\ell^{-\gamma}$$

where again $h_\ell = h_0/2^\ell$.

(c) Verfiy these rates numcerically.