TU Chemnitz – Professur Numerische Mathematik
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Mathematische Methoden der Unsicherheitsquantifizierung

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Exercise Sheet 5: Multilevel Monte Carlo

We study the following system of differential equations from the lecture

$$\boldsymbol{u}' = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} u_1(1-u_2) \\ u_2(u_1-1) \end{bmatrix} =: f(\boldsymbol{u}), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0,$$

where

$$\boldsymbol{u}_0 = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix} + 0.2 \cdot \boldsymbol{\xi}, \qquad \boldsymbol{\xi} \sim \mathrm{U}[-1,1]^2.$$

We are now interested in the expected value of $Q = u_1(T)$. As numerical solution for this initial value problem we use the Euler method

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + h \cdot f(\boldsymbol{u}_n), \qquad n = 0, 1, \ldots$$

with a stepsize parameter $h > 0$. The discretization error is given by

$$|\boldsymbol{u}(nh) - \boldsymbol{u}_n| \le h \frac{M}{2L}(\mathrm{e}^{LT} - 1), \qquad M = \max_{t \in [0,T]} \|\boldsymbol{u}''(t)\|,$$

where L is the Lipschitz constant of the right hand side f of the ODE with respect to $\boldsymbol{u}$.

Exercise 1

Develop a MATLAB script with input values $\boldsymbol{u}_0$, T and h and returning a matrix where each column is an iterate $\boldsymbol{u}_n$, $n = 0, \ldots, T/h$ of the Euler approximation applied to the above initial value problem.

Exercise 2

(a) Given the discretisation error of the Euler approximation derive the rates α, β and γ of

$$\mathbf{E}[Q - Q_{h_\ell}] \propto h_\ell^{\alpha}, \qquad \mathrm{Var}(Q_{h_\ell} - Q_{h_{\ell-1}}) \propto h_\ell^{\beta}, \qquad \mathcal{C}(h_\ell) \propto h_\ell^{-\gamma},$$

where $h_\ell = 2^{-\ell} h_0$ and $\mathcal{C}(h_\ell)$ are the costs of calculating u_h.

(b) Estimate the values α, β and γ numerically. Use at least $M = 10^3$ samples for the Monte Carlo simulation for the expected values and $h_0 = 2^{-3}$ as coarsest discretization as well as $l = 0, \ldots, 7$. Take

$$\mathbb{E}[Q] = 1.493005355.$$

as the exact value. Furthermore, check the usual condition of $\mathrm{Var}(Q_h) = \mathrm{const.}$

Exercise 3

Run various standard Monte Carlo simulations with $h_\ell = 2^{-\ell} h_0$, $l = 0, \dots, 7$ and $h_0 = 2^{-3}$. Fix the cost of every single Monte Carlo run (for example $3s$). Estimate the mean square error and plot confidence intervals. What do you observe? Explain these observations and discuss the relation to finding an optimal h_ℓ for fixed costs.

Exercise 4

How would you estimate asymptotic confidence intervals in case of the MLMC method?

Exercise 5

Develop a 2-level Monte Carlo algorithm using the estimated values α, β and γ and run this algorithm several times with finest discretisation $h_L = 2^{-5}, 2^{-6}, \dots, 2^{-10}$. Every MLMC run should again have fixed costs / runtime of $\mathcal{C} = 3s$ for example. Estimate the asymptotic confidence intervals and plot them. Compare your results with the results obtained from the standard Monte Carlo simulation in Exercise 3. What do you observe and try to explain it?