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11. Juli 2016

Mathematische Methoden der Unsicherheitsquantifizierung

Sommersemester 2016

Exercise Sheet 1: Review Probability Theory

Exercise 1

Denote by $(\Omega, \mathfrak{A}, \mathbf{P})$ a probability space and by $(E, \mathfrak{E})$ a measurable space. Let $X : (\Omega, \mathfrak{A}) \to (E, \mathfrak{E})$ be a random variable. Show that $\sigma(X) := \{X^{-1}(B), B \in \mathfrak{E}\} \subseteq \mathfrak{A}$ defines a sub σ-algebra of $\mathfrak{A}$ and $\mathbf{P}_X$ with $\mathbf{P}_X(B) := \mathbf{P}(X^{-1}(B))$ for all $B \in \mathfrak{E}$ defines a probability measure on $(E, \mathfrak{E})$.

Exercise 2

What is the probability density function of the normal distribution and why does a normally distributed random variable take almost surely irrational values?

Exercise 3

Let Y be a *log-normally* distributed random variable, i.e. $\log(Y) \sim N(\mu, \sigma^2)$.

(a) Derive the probability density function of Y.

(b) Calculate the expected value and variance of Y.
Tip: In order to do so calculate $\mathbf{E}[\exp(X)]$ and $\mathrm{Var}[\exp(X)]$ directly for $X \sim N(\mu, \sigma^2)$.

(c) For which $c \in \mathbb{R}$ does $\mathbf{E}[\exp(c^2 X^2)]$ with $X \sim N(\mu, \sigma^2)$ exist?

Exercise 4

Let $X, Y \in L^2(\Omega, \mathfrak{A}, \mathbf{P}; \mathbb{R})$. Show that

$$\mathrm{Var}[X+Y] \leq 2\,\mathrm{Var}[X] + 2\,\mathrm{Var}[Y]$$

and

$$\begin{aligned} X, Y \text{ are uncorrelated } &\iff \mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y] \\ &\iff \mathrm{Var}[X+Y] = \mathrm{Var}(X) + \mathrm{Var}(Y). \end{aligned}$$

Exercise 5

What is the probability density function of the multivariate normal distribution? Give an example for two normally distributed random variables X, Y, such that $(X, Y)^\top$ is not a normally distributed random vector.

Tip: Use

$$Y := \begin{cases} -X, & \text{if } |X| > 1, \\ X, & \text{if } |X| \leq 1. \end{cases}$$

Exercise 6

Derive a flow chart of the following different convergence concepts and add arrows for every implication. If the implication only holds under certain conditions add a number and explain these conditions:

- Convergence in $L^p(\Omega)$, $p \in [1, \infty)$,
- Convergence **P**-almost surely,
- Convergence in probability,
- Convergence in distribution.

Exercise 7

For which of the above mentioned convergence concepts follows the convergence of the expected value? Which conditions need to be satisfied?

Exercise 8

Let $X, X_n \in L^2(\Omega, \mathfrak{A}, \mathbf{P}; \mathbb{R})$, $n \in \mathbb{N}$. Show that

(a) From $X_n \xrightarrow{L^2} X$ follows $X_n \xrightarrow{\mathbf{P}} X$ (for $n \to \infty$).

(b) If there holds $X_n \xrightarrow{\mathbf{P}\text{-a.s.}} X$ and **P**-almost surely $|X_n| \leq Y$ with $Y \in L^2(\Omega, \mathfrak{A}, \mathbf{P}; \mathbb{R})$, then there follows $X_n \xrightarrow{L^2} X$.

Exercise 9

Let $X \sim N(0,1)$ and $X_n \sim N(\mu_n, \sigma_n^2)$ with $\mu_n \to 0$ and $\sigma_n \to 1$. Show that $X_n \xrightarrow{\mathcal{D}} X$ holds and in particular

$$\lim_{n\to\infty} \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbf{P}(X_n \in A) - \mathbf{P}(X \in A)| = 0$$

or

$$\lim_{n\to\infty} \mathbf{E}[f(X_n)] = \mathbf{E}[f(X)] \qquad \forall f : \mathbb{R} \to \mathbb{R} \text{ bounded and measurable.}$$

Tip: Make use of Riesz' Theorem.