

Mathematische Methoden der Unsicherheitsquantifizierung

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Mathematik!
TU Chemnitz

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 - 7.1 Bessel functions

Bessel Functions

Bessel functions arise, e.g., when constructing eigenfunction expansions for the Laplacian in cylindrical coordinates. They are solutions to **Bessel's differential equation** with parameter $\nu \in \mathbb{C}$

$$u''(z) + \frac{1}{z}u'(z) + \left(1 - \frac{\nu^2}{z^2}\right)u(z) = 0$$

for various boundary conditions.

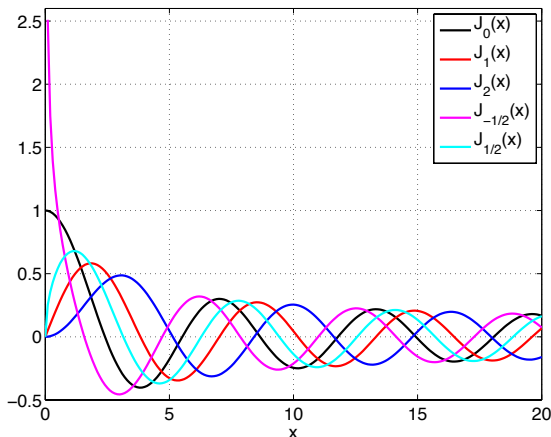
We are interested in the **Bessel functions of the first kind**, denoted by $J_\nu(z)$ for real nonnegative argument $z = x \geq 0$. These are finite at $x = 0$ for ν nonnegative or integer and singular there for ν negative or non-integer.

For special values of ν they possess simple expressions, e.g.

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x,$$

Bessel Functions

Bessel functions of the first kind



Bessel functions $J_\nu(x)$ may be evaluated in Matlab with the command `besselj(nu, x)`.

Bessel Functions

Fourier transforms spherically symmetric functions

Functions $u : \mathbb{R}^d \rightarrow \mathbb{C}$ which are spherically symmetric w.r.t. the origin, i.e., for which

$$u(\mathbf{x}) = u(r), \quad r = \|\mathbf{x}\|_2,$$

are sometimes called **isotropic functions**.

Theorem D.1

The Fourier transform \hat{u} of an isotropic function $u : \mathbb{R}^d \rightarrow \mathbb{C}$ is isotropic. Using $u(r)$ and $\hat{u}(\lambda)$ with $\lambda = \|\boldsymbol{\lambda}\|_2$ to denote $u(\mathbf{x})$ and $\hat{u}(\boldsymbol{\lambda})$, there holds

$$\hat{u}(\lambda) = (2\pi)^{-d/2} \int_0^\infty J_\nu(\lambda r) (\lambda r)^{-\nu} u(r) r^{d-1} dr, \quad \nu = \frac{d}{2} - 1.$$

This special case of the Fourier transform is known as a **Hankel transform**.

Bessel Functions

Modified Bessel functions

The **modified Bessel functions** (or hyperbolic Bessel functions) of the first and second kind are given by

$$I_\nu(x) = i^{-\nu} J_\nu(ix), \quad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

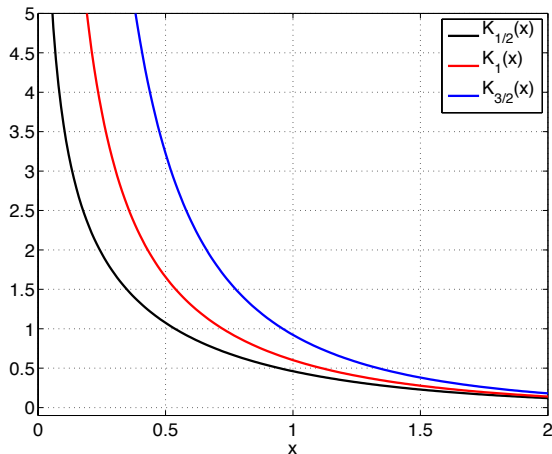
The modified Bessel functions of the second kind K_ν arise in the definition of the Matérn class of isotropic covariance functions.

The special case $\nu = \frac{1}{2}$ has a simple expression:

$$K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}, \quad x \geq 0. \quad (\text{D.1})$$

Bessel Functions

Modified Bessel functions



Bessel functions $K_\nu(x)$ may be evaluated in Matlab with the command `besselk(nu, x)`.

Bessel Functions

Modified Bessel functions, half integral order

In view of the following relations which hold for the modified Bessel functions (cf. [\[Lebedev, 1972\]](#))

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}K_{\nu}(z), \quad K_{-\nu}(z) = K_{\nu}(z)$$

along with (D.1), the modified Bessel functions of half-integral order $\nu + \frac{1}{2}$, $\nu \in \mathbb{N}_0$, may be expressed in terms of the exponential function, the square root, and a polynomial in $1/z$. In particular,

$$K_{3/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \frac{1}{z}\right),$$
$$K_{5/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \frac{3}{z} + \frac{3}{z^2}\right).$$

Bessel Functions

Modified Bessel functions, half integral order, Matérn kernels

For the associated Matérn covariance kernel $c(r) = c(r; \sigma, \nu, \rho)$, we obtain

$$c(r; \sigma, 1/2, \rho) = \sigma^2 \exp\left(\frac{-\sqrt{2}r}{\rho}\right),$$

$$c(r; \sigma, 3/2, \rho) = \sigma^2 \exp\left(-\frac{\sqrt{6}r}{\rho}\right) \left(1 + \frac{\sqrt{6}r}{\rho}\right),$$

$$c(r; \sigma, 5/2, \rho) = \sigma^2 \exp\left(-\frac{\sqrt{10}r}{\rho}\right) \left(1 + \frac{\sqrt{10}r}{\rho} + \frac{1}{3} \left(\frac{\sqrt{10}r}{\rho}\right)^2\right).$$