Mathematische Methoden der Unsicherheitsquantifizierung

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Hilbert-Schmidt Operators

For normed linear spaces X and Y, we denote by $\mathscr{L}(X, Y)$ the set of all bounded linear operators from X to Y.

Definition C.1

Let X and Y be separable Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ and let $\{x_j\}_{n\in\mathbb{N}}$ denote a CONS of X. A linear operator $L : X \to Y$ for which

$$\|L\|_{\mathrm{HS}(X,Y)} := \left(\sum_{j=1}^{\infty} \|Lx_j\|_Y^2 \right)^{1/2} < \infty$$

is called a Hilbert-Schmidt operator. We shall write $\|L\|_{\mathrm{HS}}$ if $X = Y$.

Proposition C.2

The mapping $\|\cdot\|_{\mathrm{HS}(X,Y)}$ is a norm, called the Hilbert-Schmidt norm, on the space of all Hilbert-Schmidt operators from X to Y, which we denote by $\mathrm{HS}(X, Y)$. In addition, $(\mathrm{HS}(X, Y), \|\cdot\|_{\mathrm{HS}(X,Y)})$ is Banach space.

Example C.3

For $X = Y = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$, the Hilbert-Schmidt norm of a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ coincides with the Frobenius-norm $\|\boldsymbol{A}\|_F^2 = \sum_{i,j=1}^n a_{i,j}^2$.

Example C.4

Define $L \in \mathscr{L}(L^2(0,1))$ by

$$(Lu)(x) = \int_0^x u(y)\,\mathrm{d}y, \qquad u \in L^2(0,1), \qquad x \in (0,1).$$

For the CONS $\{f_j(x) = \sqrt{2}\sin(j\pi x) : j \in \mathbb{N}\}$, we have

$$(Lf_j)(x) = \frac{\sqrt{2}}{j\pi}(1 - \cos(j\pi x)).$$

L is a Hilbert-Schmidt operator since $\|Lf_j\|_{L^2(0,1)} \leq \frac{2\sqrt{2}}{j\pi}$.

Lemma C.5

Let H be a separable Hilbert space. If $L \in \mathrm{HS}(H)$, then $\|L\|_{\mathscr{L}(H)} \leq \|L\|_{\mathrm{HS}}$. In particular, Hilbert-Schmidt operators are bounded.

Definition C.6

For a domain $D \subset \mathbb{R}^d$ and $k \in L^2(D \times D)$, the integral operator with kernel function k is defined as the linear operator

$$K : u \mapsto (Ku)(\boldsymbol{x}) := \int_D k(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}, \qquad \boldsymbol{x} \in D. \tag{C.1}$$

Theorem C.7

An integral operator with kernel function $k \in L^2(D \times D)$ is a Hilbert-Schmidt operator on $L^2(D)$. Conversely, any Hilbert-Schmidt operator K on $L^2(D)$ can be written in the form (C.1) with $\|K\|_{\mathrm{HS}} = \|k\|_{L^2(D\times D)}$.

Definition C.8

A set B in a Banach space X is said to be compact if every sequence $u_n \subset B$ has a convergent subsequence u_{n_k} with limit $u \in B$.

Definition C.9

A linear operator $L : X \to Y$, where X and Y are Banach spaces, is said to be compact if the image of any bounded set $B \subset X$ has compact closure in Y, i.e., if $\overline{L(B)}^{\|\cdot\|_Y}$ is a compact set in Y for all bounded $B \subset X$.

Theorem C.10

For $k \in L^2(D \times D)$ the associated integral operator K on $L^2(D)$ with kernel function k is a compact operator.

Definition C.11

An operator $L \in \mathscr{L}(H)$ on a Hilbert space H is said to be selfadjoint if

$$(Lu, v) = (u, Lv) \qquad \forall u, v \in H.$$

Proposition C.12

For a domain $D \subset \mathbb{R}^d$, if $k \in L^2(D \times D)$ is symmetric, i.e., $k(\boldsymbol{x}, \boldsymbol{y}) = k(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in D$, then the integral operator with kernel function k is selfadjoint with respect to the $L^2(D)$ inner product.

Definition C.13

If $L \in \mathscr{L}(H)$, $\lambda \in \mathbb{C}$ is called an eigenvalue of L if there exists nonzero $\phi \in H$ such that $L\phi = \lambda\phi$. The element ϕ is called an eigenvector or eigenfunction of L.

Theorem C.14 (Spectral theorem for selfadjoint compact operators)

Let H be a separable Hilbert space and $K \subset \mathscr{L}(H)$ be selfadjoint and compact. Denote the eigenvalues of K by $\{\lambda_j\}_{j\in\mathbb{N}}$ ordered such that $|\lambda_{j+1}| \leq |\lambda_j|$ and denote the associated eigenfunctions by $\{\phi_j\}$. Then

(i) All eigenvalues are real and $\lambda_j \to 0$ as $j \to \infty$.

(ii) The sequence $\{\phi_j\}$ can be chosen as a CONS of the range $K(H)$ of K and,

(iii) for any $u \in H$,

$$Ku = \sum_{j=1}^{\infty} \lambda_k (u, \phi_j)\phi_j. \tag{C.2}$$

Definition C.15

A function $k : D \times D \to \mathbb{R}$ is nonnegative definite if for any set of points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in D$ and numbers $a_1, \dots, a_n \in \mathbb{R}$ there holds

$$\sum_{j,k=1}^{n} a_j a_k k(\boldsymbol{x}_j, \boldsymbol{x}_k) \geq 0.$$

A linear operator $L \in \mathscr{L}(H)$ on a Hilbert space H is nonnegative definite if

$$(u, Lu) \geq 0 \qquad \forall u \in H$$

and positive definite if

$$(u, Lu) > 0 \qquad \forall u \in H.$$

Lemma C.16

For a domain $D \subset \mathbb{R}^d$ and a nonnegative definite function $k \in C(D \times D)$, the integral operator K on $L^2(D)$ with kernel function k is nonnegative.

Lemma C.17 (Dini)

For a bounded domain D let $f_n \in C(\overline{D})$ be such that $f_n(\boldsymbol{x}) \le f_{n+1}(\boldsymbol{x})$ for $n \in \mathbb{N}$ and $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ as $n \to \infty$ for all $\boldsymbol{x} \in \overline{D}$. Then $\|f - f_n\|_\infty \to 0$ as $n \to \infty$.

Definition C.18

Let H be a separable Hilbert space. A nonnegative definite operator $L \in \mathscr{L}(H)$ is said to be of trace class if $\operatorname{trace}(L) < \infty$, where the trace of L is defined as

$$\operatorname{trace}(L) := \sum_{j=1}^{\infty} (L\psi_j, \psi_j)$$

for any CONS $\{\psi_j\}_{j \in \mathbb{N}}$ of H.

Theorem C.19 (Mercer)

For a bounded domain D, let $k \in C(\overline{D} \times \overline{D})$ be a symmetric and nonnegative definite function and let K be the integral operator with kernel function k. There exist eigenfunctions ϕ_j of K with eigenvalues $\lambda_j > 0$ such that $\phi_j \in C(\overline{D})$ and

$$k(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} \lambda_j \phi_j(\boldsymbol{x}) \phi_j(\boldsymbol{y}), \qquad \boldsymbol{x}, \boldsymbol{y} \in D,$$

where the series converges in $C(\overline{D} \times \overline{D})$. Furthermore,

$$\sup_{\boldsymbol{x},\boldsymbol{y} \in \overline{D}} \left| k(\boldsymbol{x}, \boldsymbol{y}) - \sum_{j=1}^{n} \lambda_j \phi_j(\boldsymbol{x}) \phi_j(\boldsymbol{y}) \right| \le \sup_{\boldsymbol{x} \in \overline{D}} \sum_{j=n+1}^{\infty} \lambda_j |\phi_j(\boldsymbol{x})|^2. \tag{C.3}$$

The operator K is of trace class and

$$\text{trace}(K) = \int_D k(\boldsymbol{x}, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$