

# Mathematische Methoden der Unsicherheitsquantifizierung

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Mathematik!  
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# Hilbert-Schmidt Operators

For normed linear spaces  $X$  and  $Y$ , we denote by  $\mathcal{L}(X, Y)$  the set of all bounded linear operators from  $X$  to  $Y$ .

## Definition C.1

Let  $X$  and  $Y$  be separable Hilbert spaces with norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  and let  $\{x_j\}_{j \in \mathbb{N}}$  denote a CONS of  $X$ . A linear operator  $L : X \rightarrow Y$  for which

$$\|L\|_{\text{HS}(X, Y)} := \left( \sum_{j=1}^{\infty} \|Lx_j\|_Y^2 \right)^{1/2} < \infty$$

is called a **Hilbert-Schmidt operator**. We shall write  $\|L\|_{\text{HS}}$  if  $X = Y$ .

## Proposition C.2

The mapping  $\|\cdot\|_{\text{HS}(X, Y)}$  is a norm, called the **Hilbert-Schmidt norm**, on the space of all Hilbert-Schmidt operators from  $X$  to  $Y$ , which we denote by  $\text{HS}(X, Y)$ .

In addition,  $(\text{HS}(X, Y), \|\cdot\|_{\text{HS}(X, Y)})$  is Banach space.

# Hilbert-Schmidt Operators

## Examples

### Example C.3

For  $X = Y = \mathbb{R}^n$  with the Euclidean norm  $\|\cdot\|$ , the Hilbert-Schmidt norm of a matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  coincides with the Frobenius-norm  $\|\mathbf{A}\|_F^2 = \sum_{i,j=1}^n a_{i,j}^2$ .

### Example C.4

Define  $L \in \mathcal{L}(L^2(0,1))$  by

$$(Lu)(x) = \int_0^x u(y) dy, \quad u \in L^2(0,1), \quad x \in (0,1).$$

For the CONS  $\{f_j(x) = \sqrt{2} \sin(j\pi x) : j \in \mathbb{N}\}$ , we have

$$(Lf_j)(x) = \frac{\sqrt{2}}{j\pi} (1 - \cos(j\pi x)).$$

$L$  is a Hilbert-Schmidt operator since  $\|Lf_j\|_{L^2(0,1)} \leq \frac{2\sqrt{2}}{j\pi}$ .

# Hilbert-Schmidt Operators

## Integral operators

### Lemma C.5

Let  $H$  be a separable Hilbert space. If  $L \in \text{HS}(H)$ , then  $\|L\|_{\mathcal{L}(H)} \leq \|L\|_{\text{HS}}$ . In particular, Hilbert-Schmidt operators are bounded.

### Definition C.6

For a domain  $D \subset \mathbb{R}^d$  and  $k \in L^2(D \times D)$ , the **integral operator with kernel function  $k$**  is defined as the linear operator

$$K : u \mapsto (Ku)(\mathbf{x}) := \int_D k(\mathbf{x}, \mathbf{y})u(\mathbf{y}) \, d\mathbf{y}, \quad \mathbf{x} \in D. \quad (\text{C.1})$$

### Theorem C.7

An integral operator with kernel function  $k \in L^2(D \times D)$  is a Hilbert-Schmidt operator on  $L^2(D)$ . Conversely, any Hilbert-Schmidt operator  $K$  on  $L^2(D)$  can be written in the form (C.1) with  $\|K\|_{\text{HS}} = \|k\|_{L^2(D \times D)}$ .

# Hilbert-Schmidt Operators

## Compact operators

### Definition C.8

A set  $B$  in a Banach space  $X$  is said to be **compact** if every sequence  $u_n \in B$  has a convergent subsequence  $u_{n_k}$  with limit  $u \in B$ .

### Definition C.9

A linear operator  $L : X \rightarrow Y$ , where  $X$  and  $Y$  are Banach spaces, is said to be **compact** if the image of any bounded set  $B \subset X$  has compact closure in  $Y$ , i.e., if  $\overline{L(B)}^{\|\cdot\|_Y}$  is a compact set in  $Y$  for all bounded  $B \subset X$ .

### Theorem C.10

For  $k \in L^2(D \times D)$  the associated integral operator  $K$  on  $L^2(D)$  with kernel function  $k$  is a compact operator.



# Hilbert-Schmidt Operators

Selfadjoint operators, eigenvalues

## Definition C.11

An operator  $L \in \mathcal{L}(H)$  on a Hilbert space  $H$  is said to be **selfadjoint** if

$$(Lu, v) = (u, Lv) \quad \forall u, v \in H.$$

## Proposition C.12

For a domain  $D \subset \mathbb{R}^d$ , if  $k \in L^2(D \times D)$  is symmetric, i.e.,  $k(\mathbf{x}, \mathbf{y}) = k(\mathbf{y}, \mathbf{x})$  for all  $\mathbf{x}, \mathbf{y} \in D$ , then the integral operator with kernel function  $k$  is selfadjoint with respect to the  $L^2(D)$  inner product.

## Definition C.13

If  $L \in \mathcal{L}(H)$ ,  $\lambda \in \mathbb{C}$  is called an **eigenvalue** of  $L$  if there exists nonzero  $\phi \in H$  such that  $L\phi = \lambda\phi$ . The element  $\phi$  is called an eigenvector or eigenfunction of  $L$ .

# Hilbert-Schmidt Operators

## Spectral theorem

### Theorem C.14 (Spectral theorem for selfadjoint compact operators)

Let  $H$  be a separable Hilbert space and  $K \in \mathcal{L}(H)$  be selfadjoint and compact. Denote the eigenvalues of  $K$  by  $\{\lambda_j\}_{j \in \mathbb{N}}$  ordered such that  $|\lambda_{j+1}| \leq |\lambda_j|$  and denote the associated eigenfunctions by  $\{\phi_j\}$ . Then

- (i) All eigenvalues are real and  $\lambda_j \rightarrow 0$  as  $j \rightarrow \infty$ .
- (ii) The sequence  $\{\phi_j\}$  can be chosen as a CONS of the range  $K(H)$  of  $K$  and,
- (iii) for any  $u \in H$ ,

$$Ku = \sum_{j=1}^{\infty} \lambda_k(u, \phi_j) \phi_j. \quad (\text{C.2})$$

## Definition C.15

A function  $k : D \times D \rightarrow \mathbb{R}$  is **nonnegative definite** if for any set of points  $\mathbf{x}_1, \dots, \mathbf{x}_n \in D$  and numbers  $a_1, \dots, a_n \in \mathbb{R}$  there holds

$$\sum_{j,k=1}^n a_j a_k k(\mathbf{x}_j, \mathbf{x}_k) \geq 0.$$

A linear operator  $L \in \mathcal{L}(H)$  on a Hilbert space  $H$  is **nonnegative definite** if

$$(u, Lu) \geq 0 \quad \forall u \in H$$

and **positive definite** if

$$(u, Lu) > 0 \quad \forall u \in H.$$

# Hilbert-Schmidt Operators

Nonnegative functions, operators, trace class operators

## Lemma C.16

For a domain  $D \subset \mathbb{R}^d$  and a nonnegative definite function  $k \in C(D \times D)$ , the integral operator  $K$  on  $L^2(D)$  with kernel function  $k$  is nonnegative.

## Lemma C.17 (Dini)

For a bounded domain  $D$  let  $f_n \in C(\overline{D})$  be such that  $f_n(\mathbf{x}) \leq f_{n+1}(\mathbf{x})$  for  $n \in \mathbb{N}$  and  $f_n(\mathbf{x}) \rightarrow f(\mathbf{x})$  as  $n \rightarrow \infty$  for all  $\mathbf{x} \in \overline{D}$ . Then  $\|f - f_n\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ .

## Definition C.18

Let  $H$  be a separable Hilbert space. A nonnegative definite operator  $L \in \mathcal{L}(H)$  is said to be of **trace class** if  $\text{trace}(L) < \infty$ , where the **trace** of  $L$  is defined as

$$\text{trace}(L) := \sum_{j=1}^{\infty} (L\psi_j, \psi_j)$$

for any CONS  $\{\psi_j\}_{j \in \mathbb{N}}$  of  $H$ .

### Theorem C.19 (Mercer)

For a bounded domain  $D$ , let  $k \in C(\overline{D} \times \overline{D})$  be a symmetric and nonnegative definite function and let  $K$  be the integral operator with kernel function  $k$ . There exist eigenfunctions  $\phi_j$  of  $K$  with eigenvalues  $\lambda_j > 0$  such that  $\phi_j \in C(\overline{D})$  and

$$k(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} \lambda_j \phi_j(\mathbf{x}) \phi_j(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in D,$$

where the series converges in  $C(\overline{D} \times \overline{D})$ . Furthermore,

$$\sup_{\mathbf{x}, \mathbf{y} \in \overline{D}} \left| k(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^n \lambda_j \phi_j(\mathbf{x}) \phi_j(\mathbf{y}) \right| \leq \sup_{\mathbf{x} \in \overline{D}} \sum_{j=n+1}^{\infty} \lambda_j |\phi_j(\mathbf{x})|^2. \quad (\text{C.3})$$

The operator  $K$  is of trace class and

$$\text{trace}(K) = \int_D k(\mathbf{x}, \mathbf{x}) \, d\mathbf{x}.$$