Mathematatische Methoden der Unsicherheitsquantifizierung

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Mathematik!
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Elliptic Boundary Value Problem

We consider the elliptic boundary value problem of finding the solution of the partial differential equation with Dirichlet boundary condition

$$-\nabla\cdot(a\nabla u) = f \qquad \text{on } D \subset \mathbb{R}^2, \tag{B.1a}$$

$$u = g \qquad \text{on } \partial D, \tag{B.1b}$$

given a convex bounded domain D with sufficiently smooth boundary ∂D, a coefficient function $a : D \to \mathbb{R}^+$, a source term $f : D \to \mathbb{R}$ and boundary data in the form of a function $g : \partial D \to \mathbb{R}$.

The differential operator in (B.1a) is short for

$$\nabla\cdot(a\nabla u) = \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(a(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_j}\right)$$

Equation (B.1a) is a model for diffusion phenomena occurring in, e.g., heat conduction, electrostatics, potential flow and elasticity. Generalizations of (B.1) involve the addition of lower-order terms, other boundary conditions, a matrix-valued coefficient function and dependence of a on u.

If $f \in C(\overline{D})$ and $a \in C^1(\overline{D})$, then a function $u \in C^2(D) \cap C^1(\overline{D})$ which satisfies (B.1) is called a classical solution or a strong solution of the boundary value problem.

There are (theoretical and practical) reasons for generalizing the classical solution concept. The key to this generalization lies in reformulating (B.1) as a variational problem. Multiplying both sides of (B.1a) by an arbitrary function $\phi \in C_0^\infty(D)$, in this context known as a test function, and integrating by parts, we observe that any (classical) solution of (B.1) also satisfies the equation

$$a(u,\phi) = \ell(\phi) \qquad \forall \phi \in C_0^\infty(D), \tag{B.2}$$

with the symmetric bilinear form $a(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ given by

$$a(u,\phi) = \int_D a(\boldsymbol{x}) \nabla u(x) \cdot \nabla \phi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \qquad \ell(\phi) = \int_D f(\boldsymbol{x}) \phi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}. \tag{B.3}$$

For (B.2) to make sense, it is sufficient that the integrals and derivatives are well-defined.

This is the case if u and ϕ are taken to lie in the Sobolev space

$$H^1(D) := \{v \in L^2(D) : \nabla v \in L^2(D)^2\},$$

which is a Hilbert space with respect to the inner product

$$(u, v)_{H^1(D)} = \int_D (\nabla u \cdot \nabla v + uv)\, \mathrm{d}\boldsymbol{x} = (\nabla u, \nabla v) + (u, v),$$

where we use $(\cdot, \cdot)$ to denote the inner product in $L^2(D)$. The associated norm on $H^1(D)$ is

$$\|u\|^2_{H^1(D)} = \int_D \left(|\nabla u|^2 + u^2\right) \mathrm{d}\boldsymbol{x}.$$

The gradients are in terms of weak derivatives in the sense

$$\left(\frac{\partial u}{\partial x_j}, \phi\right) = -\left(u, \frac{\partial \phi}{\partial x_j}\right) \qquad \forall \phi \in C_0^\infty(D).$$

Stating the boundary condition (B.1b) requires a well-defined notion of evaluating a function from $H^1(D)$ on the lower-dimensional manifold ∂D.

- Functions in $H^1(D)$ satisfying the BC with homogeneous boundary data $g \equiv 0$ are easily defined as lying in the subspace

$$H_0^1(D) := \overline{C_0^\infty(D)}^{\|\cdot\|_{H_1(D)}} \subset H^1(D).$$

- For inhomogeneous boundary data we define the space

$$W := H_g^1(D) := \{v \in H^1(D) : u_{|\partial D} = g\}.$$

  The evaluation on the boundary is understood in the following sense: for a sufficiently smooth boundary there exists a bounded trace operator $\gamma : H^1(D) \to L^2(\partial D)$ such that for all $u \in C^1(\overline{D})$ there holds $\gamma u = u_{|\partial D}$. Since $C^1(\overline{D})$ is dense in $H^1(D)$, we have $\gamma u = \lim_{n\to\infty} u_{|\partial D}$ for any approximating sequence $\{u_n\} \subset C^1(\overline{D})$ converging to u in $H^1(D)$.

Definition B.1

The trace space of $H^1(D)$ for a sufficiently smooth domain D is defined as

$$H^{1/2}(\partial D) := \gamma(H^1(D)) = \{\gamma u : u \in H^1(D)\}.$$

$H^{1/2}(\partial D)$ is a Hilbert space with norm

$$\|g\|_{H^{1/2}(\partial D)} := \inf\{\|u\|_{H^1(D)} : \gamma u = g, u \in H^1(D)\}.$$

Sine in general $H^{1/2}(\partial D) \subsetneq L^2(\partial D)$, boundary data g in (B.1b) must be chosen from $H^{1/2}(\partial D)$.

Lemma B.2

There exists $K_\gamma > 0$ such that, for all $g \in H^{1/2}(\partial D)$, we can find $u_g \in H^1(D)$ with $\gamma u_g = g$ and

$$\|u_g\|_{H^1(D)} \leq K_\gamma \|g\|_{H^{1/2}(\partial D)}$$

We denote the spaces of trial and test functions by

$$W := H^1_g(D), \quad \text{and} \quad V := H^1_0(D).$$

Assumption B.3

The coefficient function $a = a(\boldsymbol{x})$ in (B.1a) satisfies

$$0 < a_{\min} \le a(\boldsymbol{x}) \le a_{\max} < \infty \qquad \text{for almost all } \boldsymbol{x} \in D$$

for positive constants $a_{\min}$ and $a_{\max}$. In particular, $a \in L^\infty(D)$ and a is uniformly bounded away from zero.

By Assumption B.3, the bilinear form $a(\cdot,\cdot)$ is bounded on $H^1(D)$, i.e.,

$$|a(u,v)| \le C\|u\|_{H^1(D)}\|v\|_{H^1(D)}, \qquad \forall u, v \in H^1(D)$$

with a constant $C \le \|a\|_{L^\infty(D)}$.

Definition B.4

A weak solution of (B.1) is a function $u \in W$ such that

$$a(u, v) = \ell(v) \qquad \forall v \in V, \tag{B.4}$$

with $a(\cdot, \cdot)$ and $\ell(\cdot)$ as defined in (B.3).

Definition B.5

A bilinear form $a : H \times H \to \mathbb{R}$ on a Hilbert space H is said to be coercive if there exists a constant $\alpha > 0$ such that

$$a(u, u) \geq \alpha \|u\|_H^2 \qquad \forall u \in H.$$

Lemma B.6 (Lax & Milgram)

Let H be a real Hilbert space with norm $\|\cdot\|$ and let ℓ be a bounded linear functional on H. Let $a : H \times H \to \mathbb{R}$ be a bilinear form that is bounded and coercive. Then there exists a unique $u_\ell \in H$ such that $a(u_\ell, v) = \ell(v)$ for all $v \in H$.

For functions in $H^1(D)$ we introduce the H^1 semi-norm

$$|u|_{H^1(D)} := \left(\int_D |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} \right)^{1/2}.$$

as well as the energy norm associated with the coefficient function a as

$$|u|_a := a(u,u)^{1/2} = \left(\int_D a \nabla u \cdot \nabla u \,\mathrm{d}\boldsymbol{x} \right)^{1/2}.$$

Theorem B.7 (Poincaré-Friedrichs inequality)

For a bounded domain D there exists a constant $C = C_D > 0$ such that

$$\|u\|_{L^2(D)} \le C_D |u|_{H^1(D)} \qquad \forall u \in H_0^1(D).$$

Lemma B.8

Under Assumption B.3 the bilinear form $a : H^1(D) \times H^1_0(D) \to \mathbb{R}$ is bounded and the energy norm is equivalent to the H^1 semi-norm on $H^1(D)$.

Theorem B.9

Let Assumption B.3 hold, $f \in L^2(D)$ and $g \in H^{1/2}(\partial D)$. Then (B.1) has a unique weak solution $u \in W = H^1_g(D)$.

Theorem B.10

Under the conditions of Theorem B.9 the weak solution $u \in W$ satisfies

$$|u|_{H^1(D)} \leq K \left(\|f\|_{L^2(D)} + \|g\|_{H^{1/2}(\partial D)} \right)$$

where $K = \max\{C_D/a_{\min}, K_\gamma(1 + a_{\max}/a_{\min})\}$.

Replacing a und f in (B.1) by approximations $\tilde{a}$ and $\tilde{f}$, leads to the perturbed problem of finding $\tilde{u} \in W$ such that

$$\tilde{a}(\tilde{u}, v) = \tilde{\ell}(v) \qquad \forall v \in V \tag{B.5}$$

with $\tilde{a} : W \times V \to \mathbb{R}$ and $\tilde{\ell} : V \to \mathbb{R}$ defined by

$$\tilde{a}(u, v) = \int_D \tilde{a}(\boldsymbol{x}) \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \qquad \tilde{\ell}(\phi) = \int_D \tilde{f}(\boldsymbol{x}) v(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{B.6}$$

Theorem B.11

Let Assumption B.3 hold for a as well as for $\tilde{a}$ with constants $\tilde{a}_{\min}$, $\tilde{a}_{\max}$. If, furthermore, $\tilde{f} \in L^2(D)$ and $g \in H^{1/2}(\partial D)$, then problem (B.5) has a unique weak solution $\tilde{u} \in W = H^1_g(D)$.

Theorem B.12

Under the conditions of Theorems B.9 and B.11, if $u, \tilde{u} \in W$ denote the solutions of (B.4) and (B.5), respectively, then

$$|u - \tilde{u}|_{H^1(D)} \leq C_D \tilde{a}_{\min}^{-1} \|f - \tilde{f}\|_{L^2(D)} + \tilde{a}_{\min}^{-1} \|a - \tilde{a}\|_{L^\infty(D)} |u|_{H^1(D)}$$

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Given: linear variational problem of finding $u \in V$, V a Hilbert space with norm $\|\cdot\|$, such that

$$a(u, v) = \ell(v) \qquad \forall v \in V \tag{B.7}$$

with a bilinear form $a(\cdot, \cdot)$ and linear form $\ell(\cdot)$ on V which satisfy the assumptions of the Lax-Milgram lemma.

Galerkin method for finding approximate solutions of (B.7) proceeds by restricting the problem to a finite-dimensional subspace $V_n \subset V$: denote by $u_n \in V_n$ the solution of

$$a(u_n, v) = \ell(v) \qquad \forall v \in V_n. \tag{B.8}$$

Note: The Galerkin approximation u_n of u with respect to the space V_n is uniquely determined since the conditions of the Lax-Milgram lemma are satisfied for Problem (B.8) by inclusion.

The simple structure of a linear variational problem allows its reduction to a problem of best approximation.

Lemma B.13 (Céa)

If the assumptions of the Lax-Milgram lemma apply to Problem (B.7) with solution $u \in V$, then the Galerkin approximation u_n, i.e., the solution of (B.8), satisfies

$$\|u - u_n\| \le \frac{C}{\alpha} \inf_{v \in V_n} \|u - v\|. \tag{B.9}$$

- If the bilinear form $a(\cdot,\cdot)$ is, in addition, symmetric (Hermitian) then, because of coercivity, it defines an inner product on V.
- Galerkin orthogonality then implies u_n is the a-orthogonal projection of u onto V_n and therefore the best approximation to u from V_n with respect to the associated (energy) norm.
- In the energy norm (B.9) is therefore satisfied with $C = \alpha = 1$.
- Coercivity and boundedness also imply that the energy norm is equivalent with $\|\cdot\|$, i.e.,

$$\sqrt{\alpha}\|v\| \le |v|_a \le \sqrt{C}\|v\| \qquad \forall v \in V,$$

which leads to the improved estimate over (B.9)

$$\|u - u_n\| \le \sqrt{\frac{C}{\alpha}} \inf_{v \in V_n} \|u - v\|.$$

We have seen that, for the elliptic BVP (B.1), we have the equivalences

$$\|\cdot\|_{H^1(D)} \asymp |\cdot|_{H^1(D)} \asymp |\cdot|_a.$$

Corollary B.14

Under Assumption B.3, the Galerkin approximation u_n to the solution of the elliptic boundary value problem (B.1), with respect to any subspace V_n of $V = H_0^1(D)$, satisfies

$$|u - u_n|_a = \inf_{v \in V_n} |u - v|_a,$$

$$|u - u_n|_{H^1(D)} \leq \sqrt{\frac{a_{\min}}{a_{\max}}}\, |u - v|_{H^1(D)} \qquad \forall v \in V_n.$$

Given a basis $\{v_1, \dots, v_n\}$ of V_n and the solution $u_n = \sum_{j=1}^n \xi_j v_j$, then the Galerkin variational equation (B.8) is equivalent with

$$\sum_{j=1}^n \xi_j \, a(v_j, v_i) = \ell(v_i), \qquad i = 1, \dots, n,$$

which, when rewritten as a linear system of equation, becomes the Galerkin system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{B.10}$$

with Galerkin matrix $[\boldsymbol{A}]_{i,j} = a(v_j, v_i)$, unknown vector $[\boldsymbol{x}]_i = \xi_i$ and right-hand side vector $[\boldsymbol{b}]_i = \ell(v_i)$.

- If $a(\cdot, \cdot)$ is symmetric, then so is $\boldsymbol{A}$.
- If $a(\cdot, \cdot)$ is coercive, then $\boldsymbol{A}$ is (uniformly) positive definite.

- Different Galerkin methods result from different choices of subspaces.
- Wavelets.
- Trigonometric functions, global polynomials (spectral methods).
- Radial basis functions.
- The finite element method employs finite dimensional subspaces of the variational spaces (trial and test spaces) consisting of piecewise polynomials with respect to a partition of D.
- We shall assume in the following that D is a polygon (polyhedron), but the finite element method can also be applied to domains with curved boundaries.

Assumptions on the partition of the domain D, denoted by $\mathscr{T}_h$ with elements K:

(Z$_1$) $\overline{D} = \cup_{K \in \mathscr{T}_h} K$.

(Z$_2$) Each $K \in \mathscr{T}_h$ is a closed set with nonempty interor $\mathring{K}$.

(Z$_3$) For two distinct $K_1, K_2 \in \mathscr{T}_h$ there holds $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$.

(Z$_4$) Each $K \in \mathscr{T}_h$ has a Lipschitz-continuous boundary ∂K.

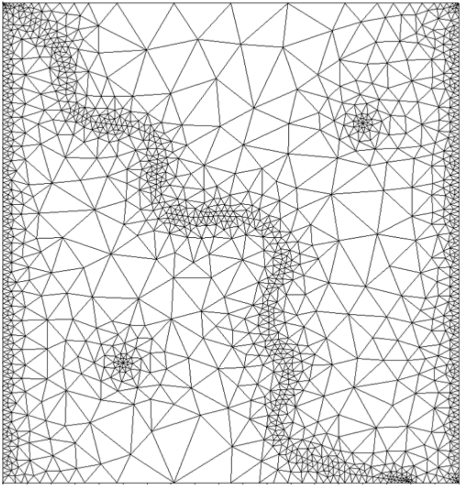
The partition is usually assigned a discretization parameter $h > 0$ given by

$$h := \max_{K \in \mathscr{T}^h} \operatorname{diam} K,$$

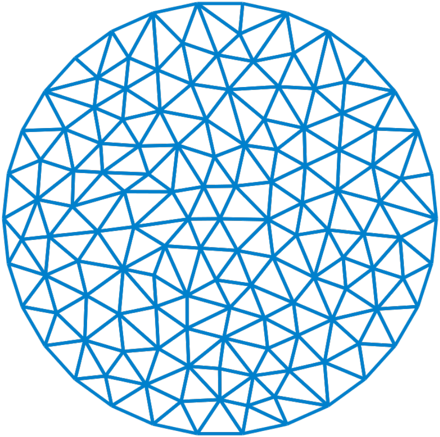
which is a measure of how fine the partition is.

Finite Element Approximation

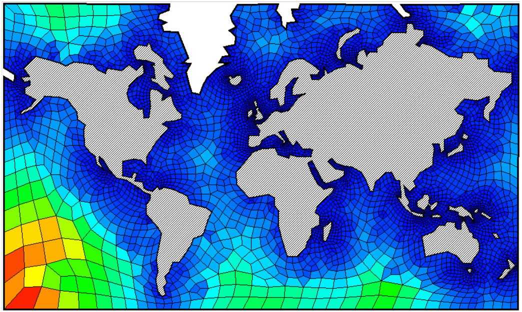
Triangulations



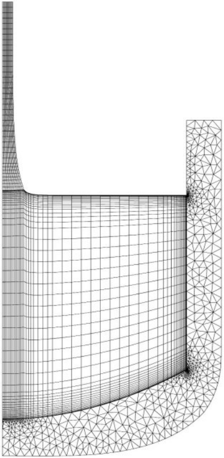
Triangular mesh on a square domain.



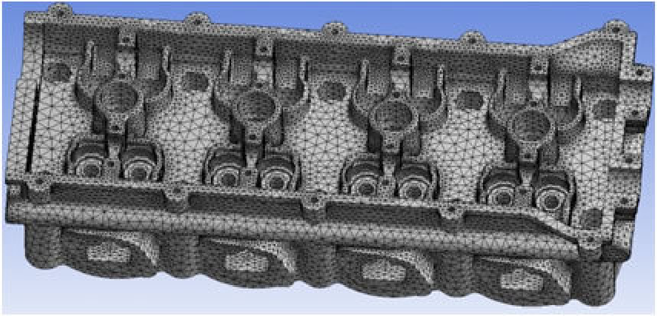
Triangular mesh on a polygonal approximation of a circle.



Quadrilateral mesh on a rectangular (exterior) domain.



Mesh consisting of triangles and quadrilaterals.



Tetrahedral mesh of complex 3D geometry (engine block).

A conforming Galerkin approximation is one which employs finite-dimensional spaces V_n such that $V_n \subset V$.

Let V^h denote a space of piecewise continuous functions $v : \overline{D} \to \mathbb{R}$ with respect to an admissible triangulation $\mathscr{T}_h$ of D, i.e., such that each restriction $v|_K$ to any $K \in \mathscr{T}_h$ is continuous on K.

Theorem B.15

With the notation defined above, there holds $V^h \subset H^1(D)$ if, and only if,

$$V^h \subset C(\overline{D}) \qquad \text{and} \qquad \{v|_K : v \in V^h\} \subset H^1(K).$$

In this case $\{v \in V^h : v = 0 \text{ on } \partial D\} \subset H^1_0(D)$.

According to [Ciarlet, 1978], a finite element is a triple (K, P_K, Ψ_K) such that

(1) K is a nonempty set

(2) P_K is a finite-dimensional space of functions defined on K and

(3) Ψ_K is a set of linearly independent linear functionals ψ on P_K with the property that, for any $p \in P_K$,

$$\psi(p) = 0 \ \ \forall \psi \in \Psi_K \qquad \Rightarrow \qquad p = 0.$$

We shall consider a single finite element, the so-called linear triangle, where

(1) $K \in \mathbb{R}^2$ is a triangle with (non-collinear) vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$,

(2) P_K is the space of all affine functions on K and

(3) Ψ_K consists of the three functionals

$$\Psi_K = \{\psi_j : P_K \to \mathbb{R}, \psi_j(p) = p(\boldsymbol{x}_j), j = 1, 2, 3\}.$$

- To construct a (global) finite element space V^h based on linear triangle elements consider a triangulation $\mathscr{T}^h$ of D consisting of (closed) triangles K which satisfy properties (**Z1**)–(**Z4**).
- The functions in V^h will also lie in $H^1(D)$ if they are continuous on $\overline{D}$, which, for piecewise linear (polynomial) functions, is equivalent with their being continuous across triangle boundaries.
- We thus obtain the space

$$V^h := \{v \in C(\overline{D}) : v|_K \in \mathscr{P}_1 \ \forall K \in \mathscr{T}^h\},$$

  where $\mathscr{P}_k$ denotes the space of (multivariate) polynomials of (complete) degree k.
- A subspace V_0^h of V^h is given by

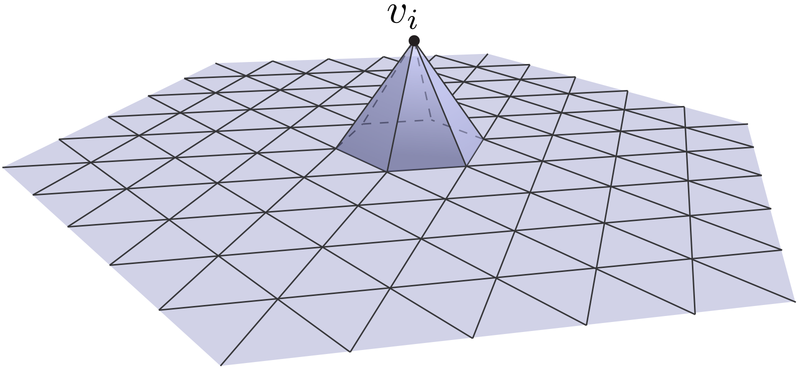
$$V_0^h := \{v \in V^h : v|_{\partial D} = 0\} \subset H_0^1(D).$$

- A continuous piecewise linear function in V^h is completely determined by its values at all triangle vertices.
- Such a (finite) set of parameters which uniquely determine a finite element function is called a set of degrees of freedom (DOF).
- In V_0^h these are the values at all nodes which do not lie on ∂D; denote their number by n.
- A particularly convenient basis $\{\phi_1, \dots, \phi_n\}$ of V_0^h is the so-called nodal basis characterized by

$$\phi_j(\boldsymbol{x}_i) = \delta_{i,j} \qquad i, j = 1, \dots, n.$$

- If $\mathscr{N}^h = \{x_1, \dots, x_n\}$ denotes the set of vertices $x_j \notin \partial D$, then

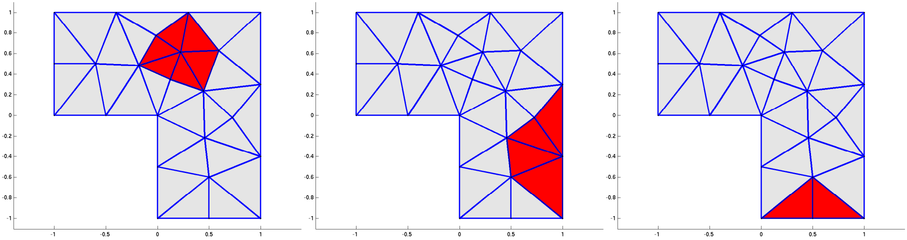
$$\operatorname{supp} \phi_j = \bigcup_{\substack{K \in \mathscr{T}^h \\ \boldsymbol{x}_j \in K}} K.$$



A nodal basis function with its support.

Finite Element Approximation

Nodal basis for linear triangles



Triangulation of an L-shaped domain with the supports of several basis functions.

Implications for Galerkin system (B.10):

$$[\boldsymbol{b}]_i = \ell(\phi_i) = \int_D f\phi_i \,\mathrm{d}\boldsymbol{x} = \int_{\mathrm{supp}\,\phi_i} f\phi_i \,\mathrm{d}\boldsymbol{x},$$

$$\begin{aligned}[\boldsymbol{A}]_{i,j} = a(\phi_j, \phi_i) &= \int_D a(\boldsymbol{x})\nabla\phi_i(\boldsymbol{x}) \cdot \nabla\phi_j(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathrm{supp}\,\phi_i \cap \mathrm{supp}\,\phi_j} a(\boldsymbol{x})\nabla\phi_i(\boldsymbol{x}) \cdot \nabla\phi_j(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

In particular: Galerkin matrix $\boldsymbol{A}$ is sparse.

Common procedure in assembling the Galerkin system:

(1) Ignore boundary condition initially, i.e., consider all of V^h with nodal basis

$$\{\phi_1, \phi_2, \ldots, \phi_n, \phi_{n+1}, \ldots, \phi_{\tilde{n}}\},$$

$\tilde{n} - n$ the number of vertices on the boundary ∂D.
Yields matrix $\tilde{\boldsymbol{A}} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, vector $\tilde{\boldsymbol{b}} \in \mathbb{R}^{\tilde{n}}$.

(2) Then eliminate the DOF associated with boundary vertices.
Yields matrix $\boldsymbol{A}$, vector $\boldsymbol{b}$.

Note:

- Initial approach for step (1): compute $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}$, entry by entry, i.e., basis function by basis function
- But: shape and connectivity of supports typically very different.
- Simpler: compute $\boldsymbol{A}$, $\boldsymbol{b}$ element by element.

$K \in \mathscr{T}^h$: then for $i, j = 1, 2 \dots, \tilde{n}$:

$$a(\phi_j, \phi_i) = \int_D a \nabla \phi_j \cdot \nabla \phi_i \,\mathrm{d}\boldsymbol{x} = \sum_{K \in \mathscr{T}^h} \int_K a \nabla \phi_j \cdot \nabla \phi_i \,\mathrm{d}\boldsymbol{x} =: \sum_{K \in \mathscr{T}^h} a_K(\phi_j, \phi_i),$$

$$\ell(\phi_i) = \int_D f \phi_i \,\mathrm{d}\boldsymbol{x} = \sum_{K \in \mathscr{T}^h} \int_K f \phi_i \,\mathrm{d}\boldsymbol{x} =: \sum_{K \in \mathscr{T}^h} \ell_K(\phi_i).$$

Setting

$$\begin{aligned} [\tilde{\boldsymbol{A}}_K]_{i,j} &:= a_K(\phi_j, \phi_i) & \qquad & i, j = 1, 2, \dots, \tilde{n}, \\ [\tilde{\boldsymbol{b}}_K]_i &:= \ell_K(\phi_i, & & i = 1, 2, \dots, \tilde{n}, \end{aligned}$$

we obtain

$$\tilde{\boldsymbol{A}} = \sum_{K \in \mathscr{T}^h} \tilde{\boldsymbol{A}}_K, \qquad \tilde{\boldsymbol{b}} = \sum_{K \in \mathscr{T}^h} \tilde{\boldsymbol{b}}_K.$$

Since each element belongs to the support of exactly three basis functions, only (at most) nine entries of $\tilde{\boldsymbol{A}}_K$ and three entries of $\tilde{\boldsymbol{b}}_K$ are nonzero.
Which entries these are can be determined by maintaining an element table:

$[ET(i,j)]_{i=1,2,3;j=1,\dots,n_K}$:

Element	K_1	K_2	$\dots$	K_{n_K}
first vertex	$i_1^{(1)}$	$i_1^{(2)}$	$\dots$	$i_1^{(n_K)}$
second vertex	$i_2^{(1)}$	$i_2^{(2)}$	$\dots$	$i_2^{(n_K)}$
third vertex	$i_3^{(1)}$	$i_3^{(2)}$	$\dots$	$i_3^{(n_K)}$

Here n_K denotes the number of triangles in $\mathscr{T}^h$.

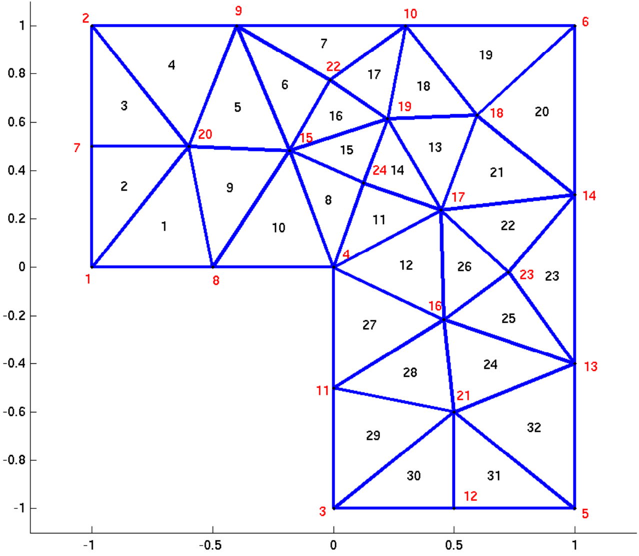
Besides the global vertex numbering

$$x_1, x_2, \dots, x_{\tilde{n}},$$

the element table introduces a second, local vertex numbering

$$x_1^{(K)},\; x_2^{(K)},\; x_3^{(K)}$$

of the vertices (DOFs) associated with K.



Global numbering of vertices (red) and elements (black) in a triangulation of an L-shaped domain.

With this notation the nonzero submatrix $\boldsymbol{A}_K$ of $\tilde{\boldsymbol{A}}_K$ and nonzero subvector $\boldsymbol{b}_K$ of $\tilde{\boldsymbol{b}}_K$ are given by

$$\boldsymbol{A}_K := \begin{bmatrix} a_K(\phi_1^{(K)}, \phi_1^{(K)}) & a_K(\phi_2^{(K)}, \phi_1^{(K)}) & a_K(\phi_3^{(K)}, \phi_1^{(K)}) \\ a_K(\phi_1^{(K)}, \phi_2^{(K)}) & a_K(\phi_2^{(K)}, \phi_2^{(K)}) & a_K(\phi_3^{(K)}, \phi_2^{(K)}) \\ a_K(\phi_1^{(K)}, \phi_3^{(K)}) & a_K(\phi_2^{(K)}, \phi_3^{(K)}) & a_K(\phi_3^{(K)}, \phi_3^{(K)}) \end{bmatrix}, \qquad \boldsymbol{b}_K := \begin{bmatrix} \ell_K(\phi_1^{(K)}) \\ \ell_K(\phi_2^{(K)}) \\ \ell_K(\phi_3^{(K)}) \end{bmatrix}.$$

If K has number k in the enumeration of the elements, then the association of the local numbering $\{\phi_i^{(K)}\}_{i=1,2,3}$ of the three basis functions whose support contains K with the global numbering $\{\phi_j\}_{j=1}^{\tilde{n}}$ of all basis functions is given by

$$\phi_i^{(K)} = \phi_j, \qquad j = ET(i,k), \quad i = 1,2,3.$$

$\boldsymbol{A}_K$ and $\boldsymbol{b}_K$ are sometimes called the element stiffness matrix and element load vector.

We summarize phase (1) of the finite element assembly process in the following algorithm[5]

Algorithm 2: Phase (1) of finite element assembly.

Initialize $\tilde{\boldsymbol{A}} := \boldsymbol{O}$, $\tilde{\boldsymbol{b}} := \boldsymbol{0}$.
foreach $K \in \mathscr{T}_h$ **do**
    Compute $\boldsymbol{A}_K$ and $\boldsymbol{b}_K$
    $k \leftarrow$ [index of element K]
    $i_1 \leftarrow ET(1,k)$, $i_2 \leftarrow ET(2,k)$, $i_3 \leftarrow ET(3,k)$
    $\tilde{\boldsymbol{A}}([i_1\, i_2\, i_3],[i_1\, i_2\, i_3]) \leftarrow \tilde{\boldsymbol{A}}([i_1\, i_2\, i_3],[i_1\, i_2\, i_3]) + \boldsymbol{A}_K$
    $\tilde{\boldsymbol{b}}([i_1\, i_2\, i_3]) \leftarrow \tilde{\boldsymbol{b}}([i_1\, i_2\, i_3]) + \boldsymbol{b}_K$

[5] We use the following Matlab-inspired notation:

$$\boldsymbol{A}([i_1\, i_2\, i_3],[i_1\, i_2\, i_3]) = \begin{bmatrix} a_{i_1,i_1} & a_{i_1,i_2} & a_{i_1,i_3} \\ a_{i_2,i_1} & a_{i_2,i_2} & a_{i_2,i_3} \\ a_{i_3,i_1} & a_{i_3,i_2} & a_{i_3,i_3} \end{bmatrix}, \quad \boldsymbol{b}([i_1\, i_2\, i_3]) = \begin{bmatrix} b_{i_1} \\ b_{i_2} \\ b_{i_3} \end{bmatrix}.$$

Both the numerical integration as well as the error analysis benefit from a change of variables to a reference element $\hat{K} \subset \mathbb{R}^2$. Each element $K \in \mathscr{T}^h$ then has a parametrization $K = F_K(\hat{K})$, where

$$F_K : \hat{K} \to K, \qquad \hat{K} \ni \boldsymbol{\xi} \mapsto \boldsymbol{x} \in K, \quad \boldsymbol{x} = F_K(\boldsymbol{\xi}) = B_K \boldsymbol{\xi} + \boldsymbol{b}_K.$$

Most common for triangular elements: unit simplex

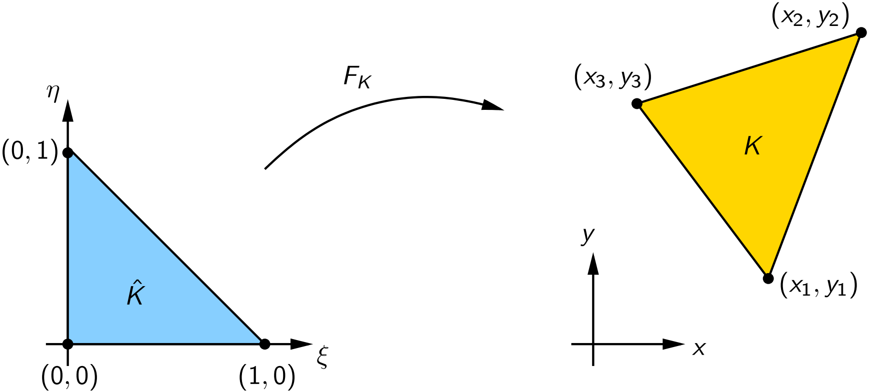
$$\hat{K} = \{(\xi, \eta) \in \mathbb{R}^2 : 0 \le \xi \le 1, 0 \le \eta \le 1 - \xi\}.$$

For each triangle $K \in \mathscr{T}^h$ the affine mapping F_K is determined by prescribing, e.g.,

$$\begin{aligned} (1,0) &\mapsto (x_1, y_1), \\ (0,1) &\mapsto (x_2, y_2), \\ (0,0) &\mapsto (x_3, y_3), \quad \text{i.e.} \end{aligned}$$

Finite Element Approximation

Reference element



$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}}_{B_K} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \underbrace{\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}}_{\boldsymbol{b}_K}$$

Local (nodal) basis on $\hat{K}$: (dual basis of DOF)

$$\hat{\phi}_1(\xi,\eta) = \xi, \quad \hat{\phi}_2(\xi,\eta) = \eta, \quad \hat{\phi}_3(\xi,\eta) = 1 - \xi - \eta, \qquad (\xi,\eta) \in \hat{K}.$$

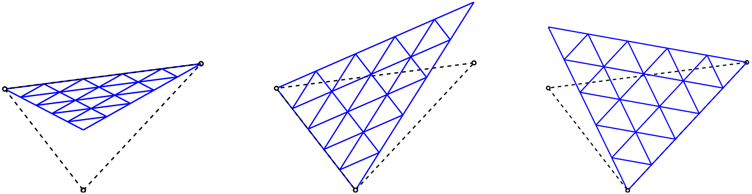
The correspondence

$$\hat{\phi} \mapsto \phi := \hat{\phi} \circ F_K^{-1}, \quad \text{d.h.} \quad \phi(\boldsymbol{x}) := \hat{\phi}(\boldsymbol{\xi}(\boldsymbol{x})) = \hat{\phi}(F_K^{-1}(\boldsymbol{x}))$$

assigns to $\hat{\phi}$ on $\hat{K}$ a unique function ϕ on K.

Local basis functions on K:

$$\phi_j = \hat{\phi}_j \circ F_K^{-1} : K \to \mathbb{R}, \qquad j = 1, 2, 3.$$



The chain rule[6] applied to $\phi(\boldsymbol{x}) = \hat{\phi}(\boldsymbol{\xi}(\boldsymbol{x}))$ gives

$$\nabla\phi = \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \begin{bmatrix} \hat{\phi}_\xi \xi_x + \hat{\phi}_\eta \eta_x \\ \hat{\phi}_\xi \xi_y + \hat{\phi}_\eta \eta_y \end{bmatrix} = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \begin{bmatrix} \hat{\phi}_\xi \\ \hat{\phi}_\eta \end{bmatrix} = (DF_K^{-1})^\top \hat{\nabla}\hat{\phi}.$$

Since

$$\begin{aligned} \boldsymbol{x} &= F_K(\boldsymbol{\xi}) = B_K\boldsymbol{\xi} + \boldsymbol{b}_K, && \text{i.e. } DF_K \equiv B_K, \\ \boldsymbol{\xi} &= F_K^{-1}(\boldsymbol{x}) = B_K^{-1}(\boldsymbol{x} - \boldsymbol{b}_K), && \text{i.e. } DF_K^{-1} \equiv B_K^{-1} \end{aligned}$$

we obtain

$$\nabla\phi = B_K^{-\top}\hat{\nabla}\hat{\phi}.$$

[6] $\hat{\nabla}$ indicates differentiation with respect to the variables ξ and η.

This finally gives the element integrals ($\phi_i = \phi_i^{(K)}$, $i = 1, 2, 3$)

$$
\begin{aligned}
a_K(\phi_j, \phi_i) &= \int_K a(\boldsymbol{x})\, \nabla\phi_j(\boldsymbol{x}) \cdot \nabla\phi_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&= \int_{\hat{K}} a(\boldsymbol{x}(\boldsymbol{\xi}))\, \left(B_K^{-\top}\hat{\nabla}\hat{\phi}_j(\boldsymbol{\xi})\right) \cdot \left(B_K^{-\top}\hat{\nabla}\hat{\phi}_i(\boldsymbol{\xi})\right) |\det B_K|\, \mathrm{d}\boldsymbol{\xi}.
\end{aligned}
\tag{B.11}
$$

The determinant is given by (note K is a triangle)

$$
|\det B_K| = 2|K|,
$$

$$
B_K^{-\top} = \frac{1}{2|K|} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \end{bmatrix},
$$

$$
\begin{bmatrix} \hat{\nabla}\hat{\phi}_1 & \hat{\nabla}\hat{\phi}_2 & \hat{\nabla}\hat{\phi}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.
$$

To impose the Dirichlet boundary condition we require that the Galerkin approximation $u^h \in V^h$ satisfy

$$u^h(\boldsymbol{x}_j) = g(\boldsymbol{x}_j) \quad \text{at all boundary vertices } \{\boldsymbol{x}_j\}_{j=n+1}^{\tilde{n}}. \tag{B.12}$$

- We partition the coefficient vector $\boldsymbol{u} \in \mathbb{R}^{\tilde{n}}$ into a first block $\boldsymbol{u}_I \in \mathbb{R}^n$ containing the coefficients associated with the interior vertices $\{\boldsymbol{x}_j\}_{j=1}^{n}$ and a second block $\boldsymbol{u}_B \in \mathbb{R}^{\tilde{n}-n}$ containing the constrained coefficients associated with boundary vertices.
- For the assembled matrix $\tilde{\boldsymbol{A}}$ and vector $\tilde{\boldsymbol{b}}$ this induces the partitionings

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \tilde{\boldsymbol{A}}_{II} & \tilde{\boldsymbol{A}}_{IB} \\ \tilde{\boldsymbol{A}}_{BI} & \tilde{\boldsymbol{A}}_{BB} \end{bmatrix}, \qquad \tilde{\boldsymbol{b}} = \begin{bmatrix} \tilde{\boldsymbol{b}}_I \\ \tilde{\boldsymbol{b}}_B \end{bmatrix}.$$

- The constraint (B.12) now reads $\boldsymbol{u}_B = \boldsymbol{g}$, where $\boldsymbol{g} \in \mathbb{R}^{\tilde{n}-n}$ contains the boundary data $\{g(\boldsymbol{x}_j)\}_{j=n+1}^{\tilde{n}}$.

This constraint is characterized by there being no coupling of the boundary DOF to either interior DOF or among themselves, resulting in the modified linear system of equations

$$\begin{bmatrix} \tilde{\boldsymbol{A}}_{II} & \tilde{\boldsymbol{A}}_{IB} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_B \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_I \\ \boldsymbol{g} \end{bmatrix},$$

which gives the reduced system

$$\boldsymbol{A}\boldsymbol{u}_I = \boldsymbol{b}, \qquad \boldsymbol{A} = \tilde{\boldsymbol{A}}_{II}, \quad \boldsymbol{b} = \boldsymbol{b}_I - \tilde{\boldsymbol{A}}_{IB}\boldsymbol{g}$$

for the interior DOF.

Note that this procedure is a discrete variant of the reformulation of the BVP with inhomogeneous Dirichlet boundary conditions to an equivalent one with homogeneous Dirichlet boundary conditions.

Contents

Finite Element Convergence

...in a nutshell

- Céa's lemma characterizes the Galerkin error as one of best appproximation from the FE subspace V^h.
- An upper bound for this error is the distance of the true solution from its interpolant from the FE subspace. This is the uniquely determined function from V^h which possesses the same global DOF as the exact solution.
- The asymptotic behavior of the interpolant is then analyzed on a sequence of meshes $\{\mathscr{T}_{h_n}\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} h_n = 0$.
- For the interpolation error to become small, the mesh sequence has to be shape-regular: if ρ_K denotes the radius of the inscribed circle in K and $h_K = \operatorname{diam} K$, then a sequence of meshes is shape-regular provided the ratio

$$\frac{\rho_K}{h_K}, \qquad K \in \mathscr{T}_h$$

  is bounded below uniformly for all $\{\mathscr{T}_{h_n}\}$.
- A priori convergence bounds are obtained by relating the smoothness of the exact solution to the convergence rate h^α of the interpolation error as $h \to 0$.

Finite Element Convergence

Extra regularity

Interpolation estimates for a solution u which is only in $H^1(D)$ do not yield a useful rate h^α with an $\alpha > 0$. For this reason one usually tries to show that the solution possesses more regularity.

Definition B.16

For $r \in \mathbb{N}$ and $D \subset \mathbb{R}^d$ bounded, we denote by $H^r(D)$ the Sobolev space

$$H^r(D) := \{v \in L^2(D) : D^{\boldsymbol{\alpha}} u \in L^2(D) \text{ for all } \boldsymbol{\alpha} \in \mathbb{N}_0^d, |\boldsymbol{\alpha}| \le r\}$$

$H^r(D)$ is a Hilbert space with the inner product

$$(u, v)_{H^r(D)} = \sum_{|\boldsymbol{\alpha}| \le r} \int_D (D^{\boldsymbol{\alpha}} u)(D^{\boldsymbol{\alpha}} v) \,\mathrm{d}\boldsymbol{x}.$$

Finite Element Convergence

Extra regularity, fractional index

For any $r \in \mathbb{R} \setminus \mathbb{N}_0$ we set $r = k + s$, $k \in \mathbb{N}_0$, $s \in (0,1)$ and denote for a bounded domain $D \subset \mathbb{R}^d$ by $|\cdot|_{H^r(D)}$ and $\|\cdot\|_{H^r(D)}$ the Sobolev-Slobodetskii semi-norm and norm defined for $v \in H^k(D)$ by

$$
|v|_{H^r(D)} = \left(\int_{D \times D} \sum_{|\boldsymbol{\alpha}|=k} \frac{[D^{\boldsymbol{\alpha}} v(\boldsymbol{x}) - D^{\boldsymbol{\alpha}} v(\boldsymbol{y})]^2}{|\boldsymbol{x} - \boldsymbol{y}|^{d+2s}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right)^{1/2} \quad \text{and}
$$

$$
\|v\|_{H^r(D)} = \left(\|v\|_{H^k(D)}^2 + |v|_{H^r(D)}^2 \right)^{1/2}.
$$

The Sobolev space $H^r(D)$ is then defined as the space of functions $v \in H^k(D)$ such that $|v|_{H^r(D)}^2$ is finite.

- Let V^h denote the space of piecewise linear functions subject to a shape-regular, admissible triangulation $\mathscr{T}_h$ of D.
- Denote by $I_h : C(\overline{D}) \to V^h$ the (global) interpolation operator assigning to each continuous function v the interpolant $v_h \in V^h$ determined by the condition that v_h agrees with v at all vertices of $\mathscr{T}_h$.
- Then the error of best approximation of $u \in C(\overline{D})$ is bounded by the interpolation error

$$\inf_{v \in V^h} |u - v|_{H^1(D)} \leq |u - I_h u|_{H^1(D)}.$$

- If the solution u of (B.4) has additional regularity $u \in H^2(D)$, then the Sobolev imbedding theorem assures that u agrees a.e. with a function in $C(\overline{D})$, so that pointwise evaluation of u and thus the interpolant is well-defined.
- In this case a scaling argument can be used to show

$$|u - I_h u|_{H^1(D)} \leq K\, h\, |u|_{H^2(D)}$$

with a constant K independent of h and u.

Assumption B.17 (H^2 regularity)

There exists a constant $K_2 > 0$ such that, for every $f \in L^2(D)$, the solution of (B.4) belongs to $H^2(D)$ and satisfies
$$|u|_{H^2(D)} \le K_2 \|f\|_{L^2(D)}.$$

Theorem B.18

Under Assumptions B.3 and B.17, the solution u of (B.4) with $f \in L^2(D)$ and the piecewise linear finite element approximation u_h on a sequence of shape-regular meshes satisfy

$$|u - u_h|_a \le K\sqrt{a_{\max}}|u|_{H^2(D)}\, h \le KK_2\sqrt{a_{\max}}\|f\|_{L^2(D)}\, h \tag{B.13}$$

with a constant K independent of h.

Corollary B.19

Under the assumptions of Theorem B.18 there holds

$$|u - u_h|_{H^1(D)} \le K\sqrt{\frac{a_{\max}}{a_{\min}}}|u|_{H^2(D)}\, h \le KK_2\sqrt{\frac{a_{\max}}{a_{\min}}}\|f\|_{L^2(D)}\, h.$$

When the coefficient function a and the source term f are replaced by approximations $\tilde{a} \approx a$ and $\tilde{f} \approx f$, then with the modified bilinear and linear forms defined as in (B.6), we may consider the discrete problem

$$\tilde{a}(\tilde{u}_h, v) = \tilde{\ell}(v) \quad \forall v \in V^h. \tag{B.14}$$

In analogy to Theorem B.11 we obtain

Theorem B.20

Under Assumption B.3 let $\tilde{f} \in L^2(D)$ and $g \in H^{1/2}(\partial D)$. Then (B.14) has a unique solution $\tilde{u}_h \in V^h$.

By the triangle inequality, we have

$$|u - \tilde{u}_h|_{H^1(D)} \leq |u - \tilde{u}|_{H^1(D)} + |\tilde{u} - \tilde{u}_h|_{H^1(D)}.$$

By an obvious extension of Corollary B.14, we obtain the bound

$$|\tilde{u} - \tilde{u}_h|_{H^1(D)} \leq \sqrt{\frac{\tilde{a}_{\max}}{\tilde{a}_{\min}}} \inf_{v \in V^h} |\tilde{u} - v|_{H^1(D)}.$$

Alternatively, if we approximate the data at the discrete level only, we may consider the following splitting as more natural:

$$|u - \tilde{u}_h|_{H^1(D)} \le |u - u_h|_{H^1(D)} + |u_h - \tilde{u}_h|_{H^1(D)}.$$

The second term arises, e.g., if we approximate the Galerkin approximation u_h by approximating the bilinear and linear forms using, e.g., piecewise constant approximations of the coefficient a and source term f.

Straightforward modification of the proof of Theorem B.12 yields

$$|u - \tilde{u}_h|_{H^1(D)} \le C_D \tilde{a}_{\min}^{-1} \|f - \tilde{f}\|_{L^2(D)} + \tilde{a}_{\min}^{-1} \|a - \tilde{a}\|_{L^\infty(D)} |u_h|_{H^1(D)}.$$