Mathematische Methoden der Unsicherheitsquantifizierung

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Sommersemester 2014

TECHNISCHE UNIVERSITÄT
CHEMNITZ

Contents

Contents

Contents

Collocation methods are a long-established technique for solving integral or differential equations and are based on requiring the equation under consideration to hold at a finite number of collocation points sufficient to determine an approximate solution in an appropriate finite-dimensional function space.

They were introduced for solving PDEs with random inputs in [Xiu & Hesthaven, 2005] and [Babuška, Nobile & Tempone, 2007] and offer anumber of atractive features:

- Like MC, they reduce to a series of uncoupled deterministic subproblems for which legacy code can be used essentialy unmodified.
- Unlike MC, collocation can take advantage of smooth dependence of the solution on the random parameters to yield spectral convergence.
- Nonlinear problems pose no additional difficulty.

We consider the model problem on the bounded domain $D \subset \mathbb{R}^d$

$$-\nabla\cdot(a\nabla u) = f \quad \text{on } D, \qquad u|_{\partial D} = 0 \tag{4.1}$$

with random field data $\{a(\boldsymbol{x}), \boldsymbol{x} \in D\}$ and (possibly) $\{f(\boldsymbol{x}), \boldsymbol{x} \in D\}$.

We make the following assumptions:

Assumption 4.1

(a) $f \in L^2(\Omega; L^2(D))$.

(b) a is uniformly bounded from below, i.e., there exists a constant $a_{\min} > 0$ such that

$$a(\boldsymbol{x}) \geqslant a_{\min} \qquad \forall \boldsymbol{x} \in \overline{D}, \quad \mathbf{P}\text{-a.s.}$$

In addition to the space $\mathscr{V} := L^2(\Omega; H_0^1(D)) = L^2(\Omega; \mathbb{R}) \otimes H_0^1(D)$, we introduce the stochastic energy space

$$\mathscr{V}_a := \left\{ v \in \mathscr{V} : \|v\|_a := \mathbf{E}\left[(a\nabla v, \nabla v)_{L^2(D)} \right]^{1/2} < \infty \right\}.$$

Proposition 4.2

Under these assumptions $\mathscr{V}_a$ is continuously embedded in $\mathscr{V}$ and

$$\|v\|_{L^2(\Omega; H_0^1(D))} \leqslant \frac{1}{a_{\min}} \|v\|_{\mathscr{V}_a}.$$

With these definitions we give the following variational formulation of problem (4.1)

Find $u \in \mathscr{V}_a$ such that

$$\mathbf{E}\left[(a\nabla u, \nabla v)_{L^2(D)}\right] = \mathbf{E}\left[(f, v)_{L^2(D)}\right] \quad \forall v \in \mathscr{V}_a. \tag{4.2}$$

Lemma 4.3

Under Assumption 4.1, the variational problem (4.2) possesses a unique solution $u \in \mathscr{V}_a$ such that

$$\|u\|_{L^2(\Omega;H_0^1(D))} \leqslant \frac{C_D}{a_{\min}} \|f\|_{L^2(\Omega;L^2(D))},$$

where C_D denotes the Poincaré-Friedrichs constant of D.

If we assume the lower bound on the coefficient field a to hold only realization-wise, i.e.,

$$a(\boldsymbol{x}, \omega) \geqslant a_{\min}(\omega) > 0 \qquad \text{a.s. and a.e. on } D,$$

for a random variable $a_{\min}$, then Lemma 4.3 yields, for each $\omega \in \Omega$, a solution $u(\omega) \in H_0^1(D)$.

Lemma 4.4

Let $p, q \geqslant 0$ be conjugate exponents, i.e., $1/p + 1/q = 1$ and $k \in \mathbb{N}$. Then if $f \in L^{kp}(\Omega; L^2(D))$ and $1/a_{\min} \in L^{kq}(\Omega; \mathbb{R})$, we have $u \in L^k(\Omega; H_0^1(D))$.

Example 4.5

Lognormal Gaussian field

$$a(\boldsymbol{x}, \omega) = \exp\left(\sum_{m=1}^{M} g_m(\boldsymbol{x})\xi_m(\omega)\right), \qquad \xi_m \text{ i.i.d., } \xi_m \sim \mathrm{N}(0,1).$$

Assumption 4.6 (Finite-dimensional noise)

The coefficient and source term in (4.1) have the form

$$a(\boldsymbol{x},\omega) = a(\boldsymbol{x},\xi_1(\omega),\dots,\xi_M(\omega)), \quad f(\boldsymbol{x},\omega) = f(\boldsymbol{x},\boldsymbol{x},\xi_1(\omega),\dots,\xi_M(\omega))$$

with $M \in \mathbb{N}$ and real-valued random variables $\{\xi_m\}$ with mean zero and unit variance. We denote by $\Gamma_m = \xi_m(\Omega)$ the image of each ξ_m, $\Gamma := \prod_{m=1}^M \Gamma_m$ and assume that the random vector $\boldsymbol{\xi} = [\xi_1,\dots,\xi_M]$ has a joint pdf

$$\rho : \Gamma \to \mathbb{R}_0^+ \qquad \text{with} \qquad \rho \in L^\infty(\Gamma).$$

- An example of such a situation is a random field represented as a truncated KL expansion.
- Typically f and a are assumed independent, i.e., the first depends on a random vector $\boldsymbol{\xi}_a(\omega)$ and the second on $\boldsymbol{\xi}_f(\omega)$ with both random vectors independent.

The stochastic variational problem (4.2) may now be reformulated as a (deterministic) parametrized PDE with respect to the space

$$\mathscr{V}_{\rho,a} := L^2(\Gamma, \mathfrak{B}(\Gamma), \rho\, \mathrm{d}\boldsymbol{\xi}; H_0^1(D))$$

in place of $\mathscr{V}_a$:

Find $u \in \mathscr{V}_{a,\rho}$ such that

$$\int_\Gamma (a\nabla u, \nabla v)_{L^2(D)}\, \rho(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} = \int_\Gamma (f, v)_{L^2(D)}\, \rho(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \qquad \forall v \in \mathscr{V}_{\rho,a}. \tag{4.3}$$

The solution then also has the form $u = u(\boldsymbol{x}, \boldsymbol{\xi}) \in \mathscr{V}_{\rho,a}$ with $\boldsymbol{x} \in D$, $\boldsymbol{\xi} \in \Gamma$. It is convenient to view u as a mapping

$$u : \Gamma \to H_0^1(D).$$

Contents

To approximate a parameter-dependent object $u = u(\boldsymbol{\xi})$ with values in an abstract space V, fix a finite-dimensional subspace $V_N = \operatorname{span}\{u_1, \dots, u_N\} \subset V$ and set

$$u(\boldsymbol{\xi}) \approx u_N(\boldsymbol{\xi}) = \sum_{j=1}^{N} u_j \, \psi_j(\boldsymbol{\xi})$$

with coefficient functions $\psi_j : \Gamma \to \mathbb{R}$ determined by a fixed set of

collocation points $\quad \{\boldsymbol{\xi}_j\}_{j=1}^N \subset \Gamma.$

Simplest choice for ψ_j: Lagrange basis of multivariate (global) polynomials with respect to a system

$$\boldsymbol{\Xi} := \{\boldsymbol{\xi}_j\}_{j=1}^N \subset \Gamma$$

of unisolvent nodes.

Given a univariate nodal sequence of distinct nodes

$$\chi_k = \{\xi_1^{(k)}, \dots, \xi_{n_k}^{(k)}\}, \qquad k \in \mathbb{N},$$

we denote by $\{\ell_j^{(k)}\}_{j=1}^{n_k}$ with $\ell_j^{(k)} \in \mathscr{P}_{n_k-1}$ the associated Lagrange basis, i.e., the uniquely determined polynomials of degree $n_k - 1$ satisfying

$$\ell_j^{(k)}(\xi_i^{(k)}) = \delta_{i,j}, \qquad j = 1, \dots, n_k.$$

We introduce the univariate interpolation operator

$$I_k : f \mapsto I_k f = \sum_{j=1}^{n_k} f(\xi_j^{(k)})\, \ell_j^{(k)} \in \mathscr{P}_{n_k-1}$$

- We will later analyze tensor-product interpolation in the variable ξ and its approximation properties, which can be derived from the constituent univariate interpolations.
- For univariate interpolation, good nodal sequences are, e.g., zeros of orthogonal polynomials, Clenshaw-Curtiss nodes (extremal values of the Chebyshev polynomials) and Leja points.
- We will restrict ourselves to zeros of orthogonal polynomials. Since these are, at the same time, the nodes of high-order quadrature schemes, this will simplify the computation of integrals involving the collocation approximation, e.g., to compute moments of the solution of (4.1).

- If we assume Γ is the M-fold Cartesian product of the same (bounded or unbounded) real interval. In this case we may choose the same nodal sequence in all coordinates, and set

$$\Xi_k := \chi_k \times \cdots \times \chi_k = \{\boldsymbol{\xi}_{\boldsymbol{\alpha}} = (\xi^{(k)}_{\alpha_1}, \dots, \xi^{(k)}_{\alpha_M}) : 1 \leqslant \alpha_m \leqslant n_k\}.$$

  Note that $N := |\Xi_k| = n_k^M$.
- The tensor-product interpolation operator is then defined as

$$\mathscr{I}_k := I_k \otimes \cdots \otimes I_k : u \mapsto \sum_{|\boldsymbol{\alpha}|_\infty \leqslant n_k} u(\boldsymbol{\xi}_{\boldsymbol{\alpha}})\, \ell^{(k)}_{\alpha_1} \cdot \ldots \cdot \ell^{(k)}_{\alpha_M},$$

  where $|\boldsymbol{\alpha}|_\infty = \max_{m=1}^M |\alpha_m|$.
- The range of the tensor-product interpolation operator $\mathscr{I}_k$ is the space $\mathscr{Q}_{n_k-1,M}$ of multivariate polynomials of degree $n_k - 1$ defined as

$$\mathscr{Q}_{p,M} = \left\{ \prod_{m=1}^{M} p_m(\xi_m) : p_m \in \mathscr{P}_p \right\}.$$

Stochastic Collocation

Semi-discrete problem

- The semi-discrete problem is obtained by replacing $V = H_0^1(D)$ with a finite-dimensional subspace, say, a finite-element space $V^h \subset H_0^1(D)$.
- If we require the discrete variational problem to hold pointwise in Γ, we obtain the problem

Find $u^h : \Gamma \to V^h$ such that

$$(a(\boldsymbol{\xi})\nabla u(\boldsymbol{\xi}), \nabla v)_{L^2(D)} = (f(\boldsymbol{\xi}), v)_{L^2(D)} \qquad \forall v \in V^h \text{ and } \forall \boldsymbol{\xi} \in \Gamma. \tag{4.4}$$

The fully discrete problem is obtained by approximating the semidiscrete solution $u^h : \Gamma \to V^h$ by

$$u^h(\boldsymbol{x}, \boldsymbol{\xi}) \approx u^{h,p}(\boldsymbol{x}, \boldsymbol{\xi}) := (\mathscr{I}_p u^h)(\boldsymbol{x}, \boldsymbol{\xi}),$$

where $\mathscr{I}_p$ is the tensor-product interpolant constructed from univariate Lagrange interpolants of degree p, i.e., based on $p+1$ disctinct nodes in each variable.

This entails solving a (deterministic) version of (4.1) for each of the tensor-product interpolation nodes:

Find $u(\boldsymbol{\xi}_{\boldsymbol{\alpha}}) \in V^h$ for all $\boldsymbol{\xi}_{\boldsymbol{\alpha}} \in \boldsymbol{\Xi}$ such that

$$(a(\boldsymbol{\xi}_{\boldsymbol{\alpha}}) \nabla u(\boldsymbol{\xi}_{\boldsymbol{\alpha}}), \nabla v)_{L^2(D)} = (f(\boldsymbol{\xi}_{\boldsymbol{\alpha}}), v)_{L^2(D)} \qquad \forall v \in V^h. \tag{4.5}$$

Stochastic Collocation

Auxiliary density

- We have not made the assumption that the random variables $\{\xi_m\}_{m=1}^M$ are independent. Expansions containing non-independent random variables arise naturally when other expansion functions than the covariance eigenfunctions are employed.
- Both analysis and computation, however, are considerably simplified when independence holds. To this end we introduce an auxiliary density function $\hat{\rho} : \Gamma \to \mathbb{R}_0^+$ with the properties

$$\hat{\rho}(\boldsymbol{\xi}) = \prod_{m=1}^{M} \hat{\rho}_m(\xi_m) \ \forall \boldsymbol{\xi} \in \Gamma \quad \text{and} \quad \left\| \frac{\rho}{\hat{\rho}} \right\|_{L^\infty(\Gamma)} < \infty. \tag{4.6}$$

- Since the density separates, it can be viewed as the joint pdf of M independent random variables.
- We choose as interpolation nodes the tensor product of univariate nodal sets consisting of the zeros of the orthogonal polynomials associated with the weight function $\hat{\rho}_m(\xi_m)$ in each of the M coordinates $\xi_1, \dots, \xi_M$.

Contents

Our analysis requires assumptions on f and the densities $\hat{\rho}$ and ρ:

- f is a continuous function of $\boldsymbol{\xi}$ which, in case of unbounded parameter domain Γ, grows at most exponentially at infinity.
- ρ and $\hat{\rho}$ behave at infinity like a Gaussian density.

To make these assumptions explicit we introduce a weight function

$$\sigma(\boldsymbol{\xi}) := \prod_{m=1}^{M} \sigma_m(\xi_m) \leqslant 1, \quad \sigma_m(\xi_m) = \begin{cases} 1 & \text{if } \Gamma_m \text{ bounded,} \\ e^{-\alpha_m |\xi_m|} & \text{otherwise,} \end{cases} \tag{4.7}$$

as well as the function space

$$C_\sigma(\Gamma; W) := \left\{ v : \Gamma \to W,\ v \text{ continuous in } \boldsymbol{\xi},\ \max_{\boldsymbol{\xi} \in \Gamma} \|\sigma(\boldsymbol{\xi}) v(\boldsymbol{\xi})\|_W < \infty \right\}$$

where W is a Banach space of functions defined on D.

Assumption 4.7 (Growth at infinity)

In what follows we assume that

(a) $f \in C_\sigma(\Gamma; L^2(D))$ and

(b) the joint probability density ρ satisfies

$$\rho(\boldsymbol{\xi}) \leqslant C_\rho \, e^{-\sum_{m=1}^{M}(\delta_m \xi_m)^2} \quad \forall \boldsymbol{\xi} \in \Gamma \tag{4.8}$$

for some $C_\rho > 0$ and $\delta_m > 0$ if Γ_m is unbounded and $\delta_m = 0$ otherwise.

- We can now choose any suitable auxiliary density $\hat{\rho}(\boldsymbol{\xi}) = \prod_{m=1}^{M} \hat{\rho}_m(\xi_m)$ that satisfies, for each $m = 1, \dots, M$,

$$C_{\min}^{(m)} e^{-(\delta_m \xi_m)^2} \leqslant \hat{\rho}_m(\xi_m) \leqslant C_{\max}^{(m)} e^{-(\delta_m \xi_m)^2}, \quad \forall \xi_m \in \Gamma_m, \tag{4.9}$$

  for positive constants $C_{\min}^{(m)}$, $C_{\max}^{(m)}$ independent of ξ_m.
- This choice satisfies the requirement (4.6) with

$$\left\| \frac{\rho}{\hat{\rho}} \right\|_{L^\infty(\Gamma)} \leqslant \frac{C_\rho}{C_{\min}}, \qquad C_{\min} := \prod_{m=1}^{M} C_{\min}^{(m)}.$$

- Under the above assumptions we have the inclusions

$$C_\sigma(\Gamma; W) \subset L^2_{\hat{\rho}}(\Gamma; W) \subset L^2_\rho(\Gamma; W)$$

  with continuous imbeddings. $(L^2_\rho(\Gamma; W) := L^2(\Gamma, \mathfrak{B}(\Gamma), \rho(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}; W))$

Lemma 4.8

If $f \in C_\sigma(\Gamma; L^2(D))$ and $a \in C_{\mathsf{loc}}(\Gamma; L^\infty(D))$, uniformly bounded away from zero, then the solution to problem (4.3) satisfies $u \in C_\sigma(\Gamma; H_0^1(D))$.

We next show that, if a and f possess partial derivatives of all orders with respect to $\boldsymbol{\xi}$ with mild growth, then the solution u is analytic as a function of each individual parameter ξ_m. This requires a one-dimensional analysis, for which we introduce the following notation:

$$\Gamma_m^* := \bigtimes_{\substack{j=1,\\ j\neq m}}^{M} \Gamma_j \quad \text{with generic elements denoted } \boldsymbol{\xi}_m^*, \quad m = 1, \dots, M.$$

Similarly, we set

$$\hat{\rho}_m^* := \prod_{\substack{j=1,\\ j\neq m}}^{M} \hat{\rho}_j \quad \text{and} \quad \sigma_m^* := \prod_{\substack{j=1,\\ j\neq m}}^{M} \sigma_j \ .$$

Lemma 4.9

Under the assumption that, for every $\boldsymbol{\xi} = (\xi_m, \boldsymbol{\xi}_m^*) \in \Gamma$, there exists $\gamma_m < \infty$ such that

$$\left\| \frac{\partial_{\xi_m}^k a(\boldsymbol{\xi})}{a(\boldsymbol{\xi})} \right\|_{L^\infty(D)} \leqslant \gamma_m^k \, k! \qquad \text{and} \qquad \frac{\|\partial_{\xi_m}^k f(\boldsymbol{\xi})\|_{L^2(D)}}{1 + \|f(\boldsymbol{\xi})\|_{L^2(D)}} \leqslant \gamma_m^k \, k!, \tag{4.10}$$

the solution $u(\xi_m, \boldsymbol{\xi}_m^*)$ as a function of ξ_m. $u : \Gamma_m \to C_{\sigma_m^*}(\Gamma_m^*; H_0^1(D))$ admits an analytic continuation $u(\zeta, \boldsymbol{\xi}_m^*)$, $\zeta \in \mathbb{C}$, to the region of the complex plane

$$\Sigma(\Gamma_m; \tau_m) := \{\zeta \in \mathbb{C} : \operatorname{dist}(\zeta, \Gamma_m) \leqslant \tau_m\} \tag{4.11}$$

with $0 < \tau_m < 1/(2\gamma_m)$. Moreover, for all $\zeta \in \Sigma(\Gamma_m; \tau_n)$ there holds

$$\|\sigma_m(\operatorname{Re}\zeta) u(\zeta)\|_{C_{\sigma_m}^*(\Gamma_m^*; H_0^1(D))} \leqslant \frac{C_D}{a_{\min}} \frac{e^{\alpha_m \tau_m}}{1 - 2\tau_m\gamma_m} (1 + 2\|f\|_{C_\sigma(\Gamma; H_0^1(D))}). \tag{4.12}$$

If the diffusion coefficient is expanded in a finite linear KL series

$$a(\boldsymbol{x},\omega) = \overline{a}(\boldsymbol{x}) + \sum_{m=1}^{M} \sqrt{\lambda_m}\, a_m(\boldsymbol{x})\, \xi_m(\omega),$$

assuming $a(\boldsymbol{x},\omega) \geqslant a_{\min}$ **P**-a.s. and a.e. in D, then we have the bounds

$$\left\| \frac{\partial_{\xi_m}^k a}{a} \right\|_{L^\infty(\Gamma\times D)} \leqslant \begin{cases} \dfrac{\sqrt{\lambda_m}\|a_m\|_{L^\infty(D)}}{a_{\min}}, & k = 1, \\ 0, & k > 1. \end{cases}$$

and we may choose

$$\gamma_m = \frac{\sqrt{\lambda_m}\|a_m\|_{L^\infty(D)}}{a_{\min}}$$

in (4.10).

If the diffusion coefficient is expanded in a finite exponential KL series

$$a(\boldsymbol{x},\omega) = a_{\min} + \exp\left(\overline{a}(\boldsymbol{x}) + \sum_{m=1}^{M} \sqrt{\lambda_m}\, a_m(\boldsymbol{x})\, \xi_m(\omega)\right)$$

we have

$$\left\| \frac{\partial_{\xi_m}^k a}{a} \right\|_{L^\infty(\Gamma\times D)} \leqslant \left(\sqrt{\lambda_m}\|a_m\|_{L^\infty(D)}\right)^k$$

and we can set

$$\gamma_m = \sqrt{\lambda_m}\|a_m\|_{L^\infty(D)}$$

in (4.10).

If the source term f has the form

$$f(\boldsymbol{x},\omega) = \overline{f}(\boldsymbol{x}) + \sum_{m=1}^{M} f_m(\boldsymbol{x})\,\xi_m(\omega)$$

with Gaussian RV ξ_m (not necessarily independent) and the functions f_m are square integrable, then f belongs to $C_\sigma(\Gamma; L^2(D))$ with weight σ as defined in (4.7) for any choice of exponential coefficients $\alpha_m > 0$.

Moreover

$$\frac{\|\partial^k_{\xi_m} f(\boldsymbol{\xi})\|_{L^2(D)}}{1 + \|f(\boldsymbol{\xi})\|_{L^2(D)}} \leqslant \begin{cases} \|f_m\|_{L^2(D)}, & k = 1, \\ 0, & k > 1, \end{cases}$$

and we can take $\gamma = \|f_m\|_{L^2(D)}$ in (4.10).

Thus such a source term satisfies the assumptions of Lemma 4.9.

Note also that, in case a is deterministic, the solution u is linear in the ξ_m, and hence clearly analytic.

Contents

We collect some classical results on interpolation theory and consider univariate functions f defined on a bounded or unbounded interval $\Gamma \subset \mathbb{R}$ with values in a Hilbert space V.

- As before, assume ρ is a positive weight function on Γ which satisfies

$$\rho(\xi) \leqslant C_M \, e^{-(\delta\xi)^2} \qquad \text{for some } C_M > 0$$

  and $\delta > 0$ for unbounded Γ and $\delta = 0$ otherwise.
- We let $\{\vartheta_j\}_{j=1}^{p+1}$ denote the zeros of the orthogonal polynomial of degree $p+1$ associated with the weight function ρ.
- Let σ be an additional positive weight function such that

$$\sigma(\xi) \geqslant C_m \, e^{-(\delta\xi)^2/4} \qquad \text{for some } C_m > 0.$$

- Observe that the condition on σ is satisfied both by a Gaussian weight $\sigma(\xi) = e^{-(\mu\xi)^2}$ with $\mu \leqslant \delta/2$ and by an exponential weight $\sigma(\xi) = e^{-\alpha|\xi|}$ for any $\alpha \geqslant 0$.

Lemma 4.10

Let $\Gamma \subset \mathbb{R}$ be an interval (bounded or unbounded) and let $\rho : \Gamma \to \mathbb{R}^+$ denote a weight function such that all integer moments are finite, i.e., $\int_\Gamma \xi^n \rho(\xi)\,\mathrm{d}\xi < \infty,\ n \in \mathbb{N}_0$. Then for each $p \in \mathbb{N}$ there exist polynomials $\{q_j\}_{j=1}^{p+1}$ of degree p such that for all $1 \leqslant j, k \leqslant p+1$ there holds

$$\begin{aligned} (q_j, q_k)_\rho &:= \int_\Gamma q_m(\xi) q_n(\xi)\,\rho(\xi)\,\mathrm{d}\xi = \delta_{j,k}, \\ \text{and} \quad (q_j, q_k)_{\tilde{\rho}} &= \vartheta_j \delta_{j,k,n}, \end{aligned} \tag{4.13}$$

where $\tilde{\rho}(\xi) := \xi\rho(\xi)$. Moreover, the q_j are, up to a constant factor, the Lagrange basis polynomials $\{\ell_j\}_{j=1}^{p+1}$ constructed with the $p+1$ (distinct) zeros of the orthogonal polynomial of degree $p+1$ associated with the weight function ρ.

The ϑ_j are the nodes of the associated $(p+1)$-point Gauss quadrature rule with weights given by

$$\omega_j = \int_\Gamma \ell_j(\xi)\rho(\xi)\,\mathrm{d}\xi = \int_\Gamma \ell_j(\xi)^2 \rho(\xi)\,\mathrm{d}\xi, \quad j = 1, \dots, p+1.$$

By $I_p : C(\Gamma) \to \mathscr{P}_p$ we denote the Lagrange interpolation operator

$$(I_p f)(\xi) = \sum_{j=1}^{p+1} f(\vartheta_j)\ell_j(\xi), \qquad \xi \in \Gamma.$$

Lemma 4.11

The operator $I_p : C_\sigma(\Gamma, V) \to L^2_\rho(\Gamma; V)$ is continuous.

Lemma 4.12

For every function $v \in L^2_\rho(\Gamma; V)$ the interpolation error satisfies

$$\|v - I_p v\|_{L^2_\rho(\Gamma;V)} \leqslant C \inf_{w \in \mathscr{P}_p \otimes V} \|v - w\|_{C_\sigma(\Gamma;V)}$$

with a constant C independent of p.

Lemma 4.13

Given a function $v \in C(\Gamma; V)$ which admits an analytic extension to the region

$$\Sigma(\Gamma;\tau) := \{z \in \mathbb{C} : \operatorname{dist}(z,\Gamma) \leqslant \tau\}$$

of the complex plane for some $\tau > 0$, then there holds

$$\min_{w \in \mathscr{P}_p \otimes V} \|v - w\|_{C(\Gamma;V)} \leqslant \frac{2}{\rho - 1} e^{-p \log \rho} \max_{z \in \Sigma(\Gamma;\tau)} \|v(z)\|_V,$$

where

$$\rho := \frac{2\tau}{|\Gamma|} + \sqrt{1 + \frac{4\tau^2}{|\Gamma|^2}} \geqslant 1.$$

A proof can be found in [Babuška et al., 2007] Lemma 4.4 and general results on best approximation of analytic functions by polynomials in [DeVore & Lorentz, 1993] Chapter 7, Section 8.

In case of unbounded Γ we recall a theorem of [Hille, 1940] on the convergence of Hermite series and the decay of the associated expansion coefficients.

- Let $H_n \in \mathscr{P}_n$ denote the (univariate) Hermite polynomial of degree n

$$H_n(\xi) = \frac{(-1)^n}{\sqrt{\pi^{1/2} 2^n n!}} \, e^{\xi^2} \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} e^{-\xi^2}, \qquad n \in \mathbb{N}_0,$$

  and by $h_n(\xi) = e^{-\xi^2/2} H_n(\xi)$ the associated Hermite function.
- The Hermite polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function $e^{-\xi^2}$ and form a complete orthonormal system of $L^2(\mathbb{R})$ with respect to the associated inner product.
- The Hermite polynomials and functions as defined above are normalized in such a way that

$$\int_{\mathbb{R}} h_k(\xi) h_\ell(\xi) \, \mathrm{d}\xi = \int_{\mathbb{R}} H_k(\xi) H_\ell(\xi) e^{-\xi^2} \, \mathrm{d}\xi = \delta_{k,\ell}, \qquad k, \ell \in \mathbb{N}_0.$$

Lemma 4.14 (Hille, 1940)

Let the function f be analytic in the strip $\{|\operatorname{Im} z| \leqslant \tau\}$. A necessary and sufficient condition for the Fourier-Hermite series

$$\sum_{k=0}^{\infty} f_k \, h_k(z), \qquad f_k := \int_{\mathbb{R}} f(\xi) \, h_k(\xi) \, \mathrm{d}\xi, \tag{4.14}$$

to converge to $f(z)$ in $\Sigma(\mathbb{R}; \tau)$ is that for every $\beta \in [0, \tau)$ there exist a finite positive $C(\beta)$ such that

$$|f(x+iy)| \leqslant C(\beta) e^{-|x|\sqrt{\beta^2 - y^2}}, \qquad x \in \mathbb{R}, |y| \leqslant \beta. \tag{4.15}$$

Moreover, the Fourier coefficients satisfy

$$|f_k| \leqslant C e^{-\tau\sqrt{2k+1}}. \tag{4.16}$$

Lemma 4.15

Assume that $v \in C_\sigma(\mathbb{R}; V)$ admits an analytic extension to the strip

$$\Sigma(\mathbb{R};\tau) = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{R}) \leqslant \tau\} \qquad \text{for some} \qquad \tau > 0$$

and that

$$\sigma(x)\|v(z)\|_V \leqslant C_v(\tau) \qquad \forall z = x + iy \in \Sigma(\mathbb{R};\tau).$$

Then for any $\delta > 0$ there exists a constant C independent of p and a function $\Theta(p) = O(p)$ such that

$$\min_{w \in \mathscr{P}_p \otimes V} \max_{\xi \in \mathbb{R}} \left| \|v(\xi) - w(\xi)\|_V \, e^{-(\delta\xi)^2/4} \right| \leqslant C\,\Theta(p)\, e^{-\tau\delta\sqrt{p}}.$$

Theorem 4.16

Under the assumptions of Lemmas 4.8 and 4.9 there exist positive constants $\{r_m\}_{m=1}^M$ and C independent of h and p such that

$$\|u - u^{h,p}\|_{L^2_\rho(\Gamma,V)} \leqslant \frac{1}{\sqrt{a_{\min}}} \inf_{v \in L^2_\rho(\Gamma,V^h)} \|u - v\|_{\mathscr{V}_a} + C \sum_{m=1}^M \beta_m(p_m) \exp(-r_m p_m^{\theta_m}) \qquad (4.17)$$

where, if Γ_m is bounded,

$$\theta_m = \beta_m = 1, \qquad r_m = \log\left[\frac{2\tau_m}{|\Gamma_m|}\left(1 + \sqrt{1 + \frac{|\Gamma_m|^2}{4\tau_m^2}}\right)\right]$$

and, if Γ_m is unbounded,

$$\theta_m = 1/2, \qquad \beta_m = O(\sqrt{p_m}), \qquad r_m = \tau_m \delta_m.$$

τ_m is smaller than the distance between Γ_m and the nearest singularity of u as defined in Lemma 4.9 and δ_m is as defined in (4.8).