Mathematische Methoden der Unsicherheitsquantifizierung

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Contents

Contents

Contents

Bessel Functions

Bessel functions arise, e.g., when constructing eigenfunction expansions for the Laplacian in cylindrical coordinates. They are solutions to Bessel's differential equation with parameter $\nu \in \mathbb{C}$

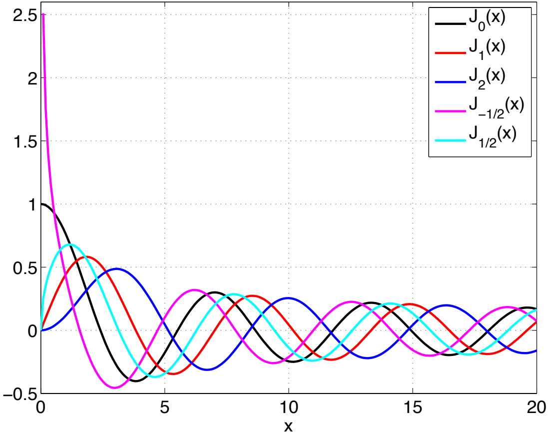
$$u''(z) + \frac{1}{z}u'(z) + \left(1 - \frac{\nu^2}{z_2}\right) u(z) = 0$$

for various boundary conditions.

We are interested in the Bessel functions of the first kind, denoted by $J_\nu(z)$ for real nonnegative argument $z = x \geqslant 0$. These are finite at $x = 0$ for ν nonnegative or integer and singular there for ν negative or non-integer.

For special values of $\boldsymbol{u}$ they possess simple expressions, e.g.

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x,$$



Bessel functions $J_\nu(x)$ may be evaluated in Matlab with the command `besselj(nu,x)`.

Functions $u : \mathbb{R}^d \to \mathbb{C}$ which are spherically symmetric w.r.t. the origin, i.e., for which

$$u(\boldsymbol{x}) = u(r), \qquad r = \|\boldsymbol{x}\|_2,$$

are sometimes called isotropic functions.

Theorem D.1

The Fourier transform $\hat{u}$ of an isotropic function $u : \mathbb{R}^d \to \mathbb{C}$ is isotropic. Using $u(r)$ and $\hat{u}(\lambda)$ with $\lambda = \|\boldsymbol{\lambda}\|_2$ to denote $u(\boldsymbol{x})$ and $\hat{u}(\boldsymbol{\lambda})$, there holds

$$\hat{u}(\lambda) = (2\pi)^{-d/2} \int_0^\infty J_\nu(\lambda r)(\lambda r)^{-\nu} u(r) r^{d-1} \, \mathrm{d}r, \qquad \nu = \frac{d}{2} - 1.$$

This special case of the Fourier transform is known as a Hankel transform.

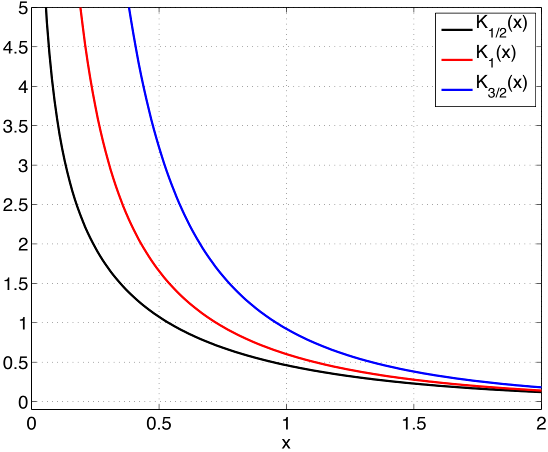
The modified Bessel functions (or hyperbolic Bessel functions) of the first and second kind are given by

$$I_\nu(x) = i^{-\nu} J_\nu(ix), \qquad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}.$$

The modified Bessel functions of the second kind K_ν arise in the definition of the Matérn class of isotropic covariance functions.

The special case $\nu = \frac{1}{2}$ has a simple expression:

$$K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}, \qquad x \geqslant 0. \tag{D.1}$$



Bessel functions $K_\nu(x)$ may be evaluated in Matlab with the command `besselk(nu,x)`.

In view of the following relations which hold for the modified Bessel functions (cf. [Lebedev, 1972])

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z), \qquad K_{-\nu}(z) = K_\nu(z)$$

along with (D.1), the modified Bessel functions of half-integral order $\nu + \frac{1}{2}$, $\nu \in \mathbb{N}_0$, may be expressed in terms of the exponential function, the square root, and a polynomial in $1/z$. In particular,

$$K_{3/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \frac{1}{z}\right),$$

$$K_{5/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \frac{3}{z} + \frac{3}{z^2}\right).$$

For the associated Matérn covariance kernel $c(r) = c(r; \sigma, \nu, \rho)$, we obtain

$$
\begin{aligned}
c(r; \sigma, 1/2, \rho) &= \sigma^2 \exp\left(\frac{-\sqrt{2}r}{\rho}\right), \\
c(r; \sigma, 3/2, \rho) &= \sigma^2 \exp\left(-\frac{\sqrt{6}r}{\rho}\right)\left(1 + \frac{\sqrt{6}r}{\rho}\right), \\
c(r; \sigma, 5/2, \rho) &= \sigma^2 \exp\left(-\frac{\sqrt{10}r}{\rho}\right)\left(1 + \frac{\sqrt{10}r}{\rho} + \frac{1}{3}\left(\frac{\sqrt{10}r}{\rho}\right)^2\right).
\end{aligned}
$$