

Mathematische Methoden der Unsicherheitsquantifizierung

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- 5 Probability Theory
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Hilbert-Schmidt Operators

For normed linear spaces X and Y , we denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y .

Definition C.1

Let X and Y be separable Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ and let $\{x_j\}_{j \in \mathbb{N}}$ denote a CONS of X . A linear operator $L : X \rightarrow Y$ for which

$$\|L\|_{\text{HS}(X, Y)} := \left(\sum_{j=1}^{\infty} \|Lx_j\|_Y^2 \right)^{1/2} < \infty$$

is called a **Hilbert-Schmidt operator**. We shall write $\|L\|_{\text{HS}}$ if $X = Y$.

Proposition C.2

The mapping $\|\cdot\|_{\text{HS}(X, Y)}$ is a norm, called the **Hilbert-Schmidt norm**, on the space of all Hilbert-Schmidt operators from X to Y , which we denote by $\text{HS}(X, Y)$. In addition, $(\text{HS}(X, Y), \|\cdot\|_{\text{HS}(X, Y)})$ is Banach space.

Hilbert-Schmidt Operators

Examples

Example C.3

For $X = Y = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$, the Hilbert-Schmidt norm of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ coincides with the Frobenius-norm $\|\mathbf{A}\|_F^2 = \sum_{i,j=1}^n a_{i,j}^2$.

Example C.4

Define $L \in \mathcal{L}(L^2(0,1))$ by

$$(Lu)(x) = \int_0^x u(y) \, dy, \quad u \in L^2(0,1), \quad x \in (0,1).$$

For the CONS $\{f_j(x) = \sqrt{2} \sin(j\pi x) : j \in \mathbb{N}\}$, we have

$$(Lf_j)(x) = \frac{\sqrt{2}}{j\pi} (1 - \cos(j\pi x)).$$

L is a Hilbert-Schmidt operator since $\|Lf_j\|_{L^2(0,1)} \leq \frac{2\sqrt{2}}{j\pi}$.

Hilbert-Schmidt Operators

Integral operators

Lemma C.5

Let H be a separable Hilbert space. If $L \in \text{HS}(H)$, then $\|L\|_{\mathcal{L}(H)} \leq \|L\|_{\text{HS}}$. In particular, Hilbert-Schmidt operators are bounded.

Definition C.6

For a domain $D \subset \mathbb{R}^d$ and $k \in L^2(D \times D)$, the **integral operator with kernel function k** is defined as the linear operator

$$K : u \mapsto (Ku)(\mathbf{x}) := \int_D k(\mathbf{x}, \mathbf{y})u(\mathbf{y}) \, d\mathbf{y}, \quad \mathbf{x} \in D. \quad (\text{C.1})$$

Theorem C.7

An integral operator with kernel function $k \in L^2(D \times D)$ is a Hilbert-Schmidt operator on $L^2(D)$. Conversely, any Hilbert-Schmidt operator K on $L^2(D)$ can be written in the form (C.1) with $\|K\|_{\text{HS}} = \|k\|_{L^2(D \times D)}$.

Hilbert-Schmidt Operators

Compact operators

Definition C.8

A set B in a Banach space X is said to be **compact** if every sequence $u_n \subset B$ has a convergent subsequence u_{n_k} with limit $u \in B$.

Definition C.9

A linear operator $L : X \rightarrow Y$, where X and Y are Banach spaces, is said to be **compact** if the image of any bounded set $B \subset X$ has compact closure in Y , i.e., if $\overline{L(B)}^{\|\cdot\|_Y}$ is a compact set in Y for all bounded $B \subset X$.

Theorem C.10

For $k \in L^2(D \times D)$ the associated integral operator K on $L^2(D)$ with kernel function k is a compact operator.

Hilbert-Schmidt Operators

Selfadjoint operators, eigenvalues

Definition C.11

An operator $L \in \mathcal{L}(H)$ on a Hilbert space H is said to be **selfadjoint** if

$$(Lu, v) = (u, Lv) \quad \forall u, v \in H.$$

Proposition C.12

For a domain $D \subset \mathbb{R}^d$, if $k \in L^2(D \times D)$ is symmetric, i.e., $k(\mathbf{x}, \mathbf{y}) = k(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in D$, then the integral operator with kernel function k is selfadjoint with respect to the $L^2(D)$ inner product.

Definition C.13

If $L \in \mathcal{L}(H)$, $\lambda \in \mathbb{C}$ is called an **eigenvalue** of L if there exists nonzero $\phi \in H$ such that $L\phi = \lambda\phi$. The element ϕ is called an eigenvector or eigenfunction of L .

Hilbert-Schmidt Operators

Spectral theorem

Theorem C.14 (Spectral theorem for selfadjoint compact operators)

Let H be a separable Hilbert space and $K \in \mathcal{L}(H)$ be selfadjoint and compact. Denote the eigenvalues of K by $\{\lambda_j\}_{j \in \mathbb{N}}$ ordered such that $|\lambda_{j+1}| \leq |\lambda_j|$ and denote the associated eigenfunctions by $\{\phi_j\}$. Then

- (i) All eigenvalues are real and $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$.
- (ii) The sequence $\{\phi_j\}$ can be chosen as a CONS of the range $K(H)$ of K and,
- (iii) for any $u \in H$,

$$Ku = \sum_{j=1}^{\infty} \lambda_k(u, \phi_j) \phi_j. \quad (\text{C.2})$$

Definition C.15

A function $k : D \times D \rightarrow \mathbb{R}$ is **nonnegative definite** if for any set of points $\mathbf{x}_1, \dots, \mathbf{x}_n \in D$ and numbers $a_1, \dots, a_n \in \mathbb{R}$ there holds

$$\sum_{j,k=1}^n a_j a_k k(\mathbf{x}_j, \mathbf{x}_k) \geq 0.$$

A linear operator $L \in \mathcal{L}(H)$ on a Hilbert space H is **nonnegative definite** if

$$(u, Lu) \geq 0 \quad \forall u \in H$$

and **positive definite** if

$$(u, Lu) > 0 \quad \forall u \in H.$$

Hilbert-Schmidt Operators

Nonnegative functions, operators, trace class operators

Lemma C.16

For a domain $D \subset \mathbb{R}^d$ and a nonnegative definite function $k \in C(D \times D)$, the integral operator K on $L^2(D)$ with kernel function k is nonnegative.

Lemma C.17 (Dini)

For a bounded domain D let $f_n \in C(\overline{D})$ be such that $f_n(\mathbf{x}) \leq f_{n+1}(\mathbf{x})$ for $n \in \mathbb{N}$ and $f_n(\mathbf{x}) \rightarrow f(\mathbf{x})$ as $n \rightarrow \infty$ for all $\mathbf{x} \in \overline{D}$. Then $\|f - f_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

Definition C.18

Let H be a separable Hilbert space. A nonnegative definite operator $L \in \mathcal{L}(H)$ is said to be of **trace class** if $\text{trace}(L) < \infty$, where the **trace** of L is defined as

$$\text{trace}(L) := \sum_{j=1}^{\infty} (L\psi_j, \psi_j)$$

for any CONS $\{\psi_j\}_{j \in \mathbb{N}}$ of H .

Theorem C.19 (Mercer)

For a bounded domain D , let $k \in C(\overline{D} \times \overline{D})$ be a symmetric and nonnegative definite function and let K be the integral operator with kernel function k . There exist eigenfunctions ϕ_j of K with eigenvalues $\lambda_j > 0$ such that $\phi_j \in C(\overline{D})$ and

$$k(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} \lambda_j \phi_j(\mathbf{x}) \phi_j(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in D,$$

where the series converges in $C(\overline{D} \times \overline{D})$. Furthermore,

$$\sup_{\mathbf{x}, \mathbf{y} \in \overline{D}} \left| k(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^n \lambda_j \phi_j(\mathbf{x}) \phi_j(\mathbf{y}) \right| \leq \sup_{\mathbf{x} \in \overline{D}} \sum_{j=n+1}^{\infty} \lambda_j |\phi_j(\mathbf{x})|^2. \quad (\text{C.3})$$

The operator K is of trace class and

$$\text{trace}(K) = \int_D k(\mathbf{x}, \mathbf{x}) \, d\mathbf{x}.$$