Stochastic Collocation for Elliptic PDEs with random data - the lognormal case

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Abstract We investigate the stochastic collocation method for parametric, elliptic partial differential equations (PDEs) with lognormally distributed random parameters in mixed formulation. Such problems arise, e.g., in uncertainty quantification studies for flow in porous media with random conductivity. We show the analytic dependence of the solution of the PDE w.r.t. the parameters and use this to show convergence of the sparse grid stochastic collocation method. This work fills some remaining theoretical gaps for the application of stochastic collocation in case of elliptic PDEs where the diffusion coefficient is not strictly bounded away from zero w.r.t. the parameters. We illustrate our results for a simple groundwater flow problem.

1 Introduction

The elliptic boundary value problem

$$-\nabla \cdot (a(\mathbf{x},\omega)\,\nabla p(\mathbf{x},\omega)) = f(\mathbf{x},\omega) \qquad \text{in } D, \qquad \mathbb{P}\text{-a.s.,} \tag{1a}$$

$$p(\mathbf{x},\omega) = g(\mathbf{x}) \qquad \text{on } \partial D, \qquad \mathbb{P}\text{-a.s.,} \tag{1b}$$

with random coefficient a and random source f, resp. its weak form, is of particular interest for studies on uncertainty quantification (UQ) methods. It is a rather simple mathematical model to study and, at the same time, of practical relevance, e.g., in groundwater flow modelling. There, the conductivity coeffcient a is typically

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uncertain and therefore modeled as a random field, in particular, a lognormal random field [11].

Recent methods for solving random PDEs such as stochastic collocation or Galerkin methods use a truncated Karhunen-Loève expansion of the random fields a and f in order to separate the deterministic and random parts of the problem (1) as well as reduce the randomness to a finite or countable number of random variables. This truncation leads to high-dimensional parametric problems, and approximation methods which are suited for higher dimensions, such as sparse grid collocation, have been successfully applied to this problem [1, 19, 18, 3]. In these works one often finds the assumption that the coefficient a is uniformly bounded away from zero, i.e., there exists a constant $c > 0$ such that $a(\mathbf{x}, \omega) \geq c$ $\mathbb{P}$-a.s. for all $\mathbf{x} \in D$. While this assumption simplifies the analysis, it fails to cover the important case where a has a (multivariate) lognormal distribution. For instance, in [1, 19, 18] the authors ensure uniform positivity by taking a to be the sum of a lognormal field and a positive constant $a_{\min}$. In [6] the analysis of full tensor-product collocation given in [1] is extended to the case of non-uniformly bounded coefficients a, but for deterministic sources f and homogeneous Dirichlet boundary conditions. Moreover, many works consider only the primal form (1) of the diffusion equation, but for many applications the numerical simulation of system (1) in mixed form

$$a^{-1}(\mathbf{x}, \omega)\,\mathbf{u}(\mathbf{x}, \omega) - \nabla p(\mathbf{x}, \omega) = 0 \quad \text{in } D, \tag{2a}$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}, \omega) = -f(\mathbf{x}, \omega) \quad \text{in } D, \tag{2b}$$

$$p(\mathbf{x}, \omega) = g(\mathbf{x}) \quad \text{on } \partial D, \tag{2c}$$

$\mathbb{P}$-almost surely, is more appropriate. This is the case, for instance, if the flux $\mathbf{u}$ is of particular interest, see [12] for numerical examples. In [4] a first study of stochastic Galerkin methods for mixed problems was given, but again the assumptions on a made there do not apply to lognormal or non-uniformly bounded random fields.

In this work, we fill the remaining gaps and present a convergence analysis of sparse grid collocation for (2) without assuming the existence of a deterministic $a_{\min} > 0$ such that $a(\mathbf{x}, \omega) \geq a_{\min}$. Therefore, we introduce in Section 2 the parametric variational problem under consideration and prove in Section 3 a regularity result for its solution. In Section 4 we then conduct the proof of convergence of sparse grid stochastic collocation in unbounded parameter domains for approximating smooth functions. Section 5 illustrates the theoretical results for a simple elliptic boundary value problem in mixed form and Section 6 closes with concluding remarks.

2 The Parametric Variational Problem

In this section we briefly recall how the elliptic boundary value problem (BVP) (1) with random diffusion coefficient $a(\mathbf{x}, \omega)$ is transformed into a BVP containing a high-dimensional parameter. We shall restrict our considerations to the mixed formulation (2).

2.1 Finite-Dimensional Noise Via Karhunen-Loève Expansion

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we denote by $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ the space of second-order real-valued random variables. We make the *finite-dimensional noise* assumption whereby the randomness in the coefficient $a(\mathbf{x}, \omega)$ and right hand side $f(\mathbf{x}, \omega)$ can be completely described by a finite set of M Gaussian random variables $\xi_1, \dots, \xi_M \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$.

Assumption 1 *There exist measurable functions $\tilde{a} : \mathbb{R}^M \to L^\infty(D)$ and $\tilde{f} : \mathbb{R}^M \to L^2(D)$ and M independent Gaussian random variables $\xi_1, \dots, \xi_M \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$, such that*

$$a(\mathbf{x}, \omega) = \tilde{a}(\mathbf{x}, \xi_1(\omega), \dots, \xi_M(\omega)) \quad \textit{and} \quad f(\mathbf{x}, \omega) = \tilde{f}(\mathbf{x}, \xi_1(\omega), \dots, \xi_M(\omega))$$

hold $\mathbb{P}$-almost surely almost everywhere in D.

We shall identify a with $\tilde{a}$ and f with $\tilde{f}$ in the following. Such finite-dimensional noise arises, e.g., when a random field is approximated by its truncated Karhunen-Loève expansion (KLE) [13].

Example 1 (KLE for lognormal random field). For a lognormal random field a, it is convenient to truncate the KLE of its logarithm $\log a$, yielding

$$a(\mathbf{x}, \omega) \approx a_M(\mathbf{x}, \omega) := \exp\left(\psi_0(\mathbf{x}) + \sum_{m=1}^{M} \sqrt{\lambda_m}\, \psi_m(\mathbf{x}) \xi_m(\omega) \right), \tag{3}$$

where $\psi_0(\mathbf{x}) := \mathbb{E}\left[\log a(\mathbf{x}, \cdot)\right]$ and $\{(\lambda_m, \psi_m)\}_{m \geq 0}$ denotes the sequence of eigenpairs of the covariance operator C associated with $\log a$,

$$(C\psi)(\mathbf{x}) = \int_D c(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \quad \text{where } c(\mathbf{x}, \mathbf{y}) = \operatorname{Cov}\left(\log a(\mathbf{x}, \cdot), \log a(\mathbf{y}, \cdot)\right), \tag{4}$$

and where the $\{\xi_m\}_{m \geq 0}$ are i.i.d. standard normally distributed random variables. For a discussion on approximating a directly by a (generalized) truncated polynomial chaos expansion see [10]. For an analysis of the effect of truncating the KLE see [6]. We neglect any truncation error in the following and identify a_M with a resp. $\tilde{a}$.

2.2 The Parametric Elliptic Problem in Mixed Variational Form

We set $\boldsymbol{\xi} := (\xi_1, \dots, \xi_M)$ and denote by $\rho(\boldsymbol{\xi}) = \prod_{m=1}^{M} \frac{\exp(-\xi_m^2/2)}{\sqrt{2\pi}}$ the joint probability density function (pdf) of the i.i.d standard normally distributed $\xi_1, \dots, \xi_M$. We rewrite the random mixed elliptic problem (2) as the parametric mixed elliptic problem

$$a^{-1}(\mathbf{x},\boldsymbol{\xi})\,\mathbf{u}(\mathbf{x},\boldsymbol{\xi}) - \nabla p(\mathbf{x},\boldsymbol{\xi}) = 0 \qquad \text{in } D, \tag{5a}$$

$$\nabla\cdot\mathbf{u}(\mathbf{x},\boldsymbol{\xi}) = -f(\mathbf{x},\boldsymbol{\xi}) \qquad \text{in } D, \tag{5b}$$

$$p(\mathbf{x},\boldsymbol{\xi}) = g(\mathbf{x}) \qquad \text{on } \partial D, \tag{5c}$$

where the equations are taken to hold $\rho\,\mathrm{d}\boldsymbol{\xi}$-almost everywhere.

To state the weak mixed form of (5), we assume $g \in H^{1/2}(\partial D)$ and introduce the space

$$H(\mathrm{div};D) = \left\{\mathbf{v}\in L^2(D) : \nabla\cdot\mathbf{v}\in L^2(D)\right\} \tag{6}$$

with norm $\|\mathbf{v}\|^2_{H(\mathrm{div};D)} = \|\mathbf{v}\|^2_{L^2(D)} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(D)}$ as well as the bilinear and linear forms

$$A_{\boldsymbol{\xi}}(\mathbf{u},\mathbf{v}) = \int_D a^{-1}(\mathbf{x},\boldsymbol{\xi})\mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{7}$$

$$B(\mathbf{v},q) = -\int_D q(\mathbf{x})\ \nabla\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{8}$$

$$h_{\boldsymbol{\xi}}(q) = -\int_D f(\mathbf{x},\boldsymbol{\xi})\,q(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{9}$$

$$\ell(\mathbf{v}) = -\int_{\partial D} g(\mathbf{x})\,\mathbf{v}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{10}$$

for $\mathbf{u},\mathbf{v}\in H(\mathrm{div};D)$ and $q\in L^2(D)$, where in the last line $\mathbf{n}$ denotes the unit outward normal vector along the boundary ∂D and the integral is understood as a linear functional on $H^{1/2}(\partial D)$, see [9, Appendix B.3]. The weak form of (5) then reads

$$A_{\boldsymbol{\xi}}(\mathbf{u}(\cdot,\boldsymbol{\xi}),\mathbf{v}) + B(\mathbf{v},p(\cdot,\boldsymbol{\xi})) = \ell(\mathbf{v}) \qquad \forall\mathbf{v}\in H(\mathrm{div};D), \tag{11a}$$

$$B(\mathbf{u}(\cdot,\boldsymbol{\xi}),q) = h_{\boldsymbol{\xi}}(q) \qquad \forall q\in L^2(D), \tag{11b}$$

$\rho\,\mathrm{d}\boldsymbol{\xi}$-almost everywhere. Hence, setting $\mathcal{S} := L^2(\mathbb{R}^M,\mathcal{B}(\mathbb{R}^M),\rho\,\mathrm{d}\boldsymbol{\xi})$, where $\mathcal{B}$ denotes the σ-algebra of Borel sets, and $\mathcal{V} := L^2(D)\times H(\mathrm{div};D)$, we are thus seeking a solution $(p,\mathbf{u})\in\mathcal{S}\otimes\mathcal{V}$ which satisfies (11) $\rho\,\mathrm{d}\boldsymbol{\xi}$-a.e. That such a solution exists and is unique will be shown in Section 3.

Remark 1. Note that, due to Assumption 1 and the continuous (hence measurable) dependence of the solution $(p,\mathbf{u})$ of a variational problem such as (11) on the coefficient a and the source term f, we can deduce by means of the Doob-Dynkin lemma [17, Lemma 1.13, p. 7] that the solution of (11) may be identified with the weak solution of (2) by way of

$$(p(\omega),\mathbf{u}(\omega)) = (p(\boldsymbol{\xi}),\mathbf{u}(\boldsymbol{\xi}))\,, \qquad \boldsymbol{\xi} = \boldsymbol{\xi}(\omega),$$

$\mathbb{P}$-almost surely as functions in $L^2(D)$ and $H(\mathrm{div};D)$, respectively.

3 Analytic Dependence on the Parameter

Subsequently, we denote by $(\cdot,\cdot)$ the inner product in $L^2(D)$, where for vector-valued functions we set $(\mathbf{u},\mathbf{v}) := \int_D \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}$.

In this section we prove existence and analytic dependence of the solution $(p(\boldsymbol{\xi}),\mathbf{u}(\boldsymbol{\xi}))$ of the mixed problem (11) on the parameter $\boldsymbol{\xi}$. In particular, we will prove analyticity of $(p(\cdot),\mathbf{u}(\cdot))$ in a subdomain of $\mathbb{C}^M$. To this end, we consider problem (11) with the parameter vector $\boldsymbol{\xi}$ extended to *complex* values $\boldsymbol{\zeta} = \boldsymbol{\xi} + i\boldsymbol{\eta} \in \mathbb{C}^M$, $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^M$, along with suitable extensions of the functions $a(\mathbf{x},\cdot)$ and $f(\mathbf{x},\cdot)$. To ensure well-posedness of (11) for this complex extension

$$A_{\boldsymbol{\zeta}}(\mathbf{u},\mathbf{v}) = \int_D a^{-1}(\mathbf{x},\boldsymbol{\zeta})\nabla\mathbf{u}(\mathbf{x})\cdot\overline{\nabla\mathbf{v}(\mathbf{x})}\,\mathrm{d}\mathbf{x},$$

of $A_{\boldsymbol{\xi}}$, and $h_{\boldsymbol{\zeta}}(q) = -\int_D f(\mathbf{x},\boldsymbol{\zeta})\,\overline{q(\mathbf{x})}\,\mathrm{d}\mathbf{x}$, we restrict the complex parameter $\boldsymbol{\zeta}$ to the domain

$$\Sigma := \left\{\boldsymbol{\zeta}\in\mathbb{C}^M : a_{\min}(\boldsymbol{\zeta}) > 0 \quad \text{and} \quad a_{\max}(\boldsymbol{\zeta}) < +\infty\right\},$$

where

$$a_{\max}(\boldsymbol{\zeta}) := \operatorname{ess\,sup}_{\mathbf{x}\in D}\operatorname{Re} a(\mathbf{x},\boldsymbol{\zeta}), \quad a_{\min}(\boldsymbol{\zeta}) := \operatorname{ess\,inf}_{\mathbf{x}\in D}\operatorname{Re} a(\mathbf{x},\boldsymbol{\zeta}).$$

For a general Banach space W, we denote by $L^q_\rho(\mathbb{R}^M;W)$ the Bochner space $L^q(\mathbb{R}^M,\mathcal{B}(\mathbb{R}^M),\rho d\boldsymbol{\xi};W)$ of W-valued functions of $\boldsymbol{\xi}$ and make the following assumptions for proving the existence of a solution to (11) for *real-valued* parameters $\boldsymbol{\xi}\in\mathbb{R}^M$:

Assumption 2 *The data a, f and g defining problem* (11) *satisfy*

(1) $g \in H^{1/2}(\partial D)$*,*
(2) $a \in L^q_\rho(\mathbb{R}^M;L^\infty(D))$ *for all* $q\in[1,\infty)$*,*
(3) $a_{\min}(\boldsymbol{\xi}) > 0$ *for all* $\boldsymbol{\xi}\in\mathbb{R}^M$ *and* $1/a_{\min}\in L^q_\rho(\mathbb{R}^M;\mathbb{R}_+)$ *for all* $q\in[1,\infty)$*,*
(4) $f \in L^{q^*}_\rho(\mathbb{R}^M;L^2(D))$ *for some* $q^* > 2$*.*

Note that, under Assumption 2, we have $\mathbb{R}^M \subset \Sigma$. We can now state

Lemma 1 (cf. [4, Lemma 2.3]). *Let Assumption 2 be satisfied. Then there exists a unique solution* $(p,\mathbf{u})\in\mathcal{S}\otimes\mathcal{V}$ *of* (11)*.*

Lemma 1 will be proven together with the following existence and continuity result for the solution to (11) for *complex* parameters $\boldsymbol{\zeta}\in\mathbb{C}^M$. In order to state this result, we introduce the spaces

$$C_\sigma(\Sigma;W) := \{v:\Sigma\to W \text{ continuous, strongly measurable and } \|v\|_{C_\sigma} = \max_{\boldsymbol{\zeta}\in\Sigma}\sigma(\operatorname{Re}\boldsymbol{\zeta})\|v(\boldsymbol{\zeta})\|_W < \infty\},$$

where $\sigma:\mathbb{R}^M\to\mathbb{R}_+$ is an arbitrary nonnegative weight function and W a Banach space.

Assumption 3 *For* $\sigma : \mathbb{R}^M \to \mathbb{R}_+$ *there holds*

(5.) $f \in C_\sigma(\Sigma; L^2(D))$ *and* $a \in C_\sigma(\Sigma; L^\infty(D))$,
(6.) $a_{\max} \in C_\sigma(\Sigma;\mathbb{R})$ *and* $1/a_{\min} \in C_\sigma(\Sigma;\mathbb{R})$.

Lemma 2. *Let Assumptions 2 and 3 be satisfied. Then for each* $\boldsymbol{\zeta} \in \Sigma$ *there exists a unique* $(p(\boldsymbol{\zeta}), \mathbf{u}(\boldsymbol{\zeta}))$ *which solves* (11) *with* $(p,\mathbf{u}) \in C_{\sigma^4}(\Sigma;\mathcal{V})$.

Proof (of Lemma 1 and Lemma 2). We first observe that, for $\mathbf{u},\mathbf{v} \in H(\mathrm{div};D)$ and $q \in L^2(D)$, we obtain

$$\begin{aligned} |A_{\boldsymbol{\zeta}}(\mathbf{u},\mathbf{v})| &= \left|\left(a^{-1}(\boldsymbol{\zeta})\mathbf{u}, \mathbf{v}\right)\right| \le \frac{1}{a_{\min}(\boldsymbol{\zeta})} \|\mathbf{u}\|_{H(\mathrm{div};D)} \|\mathbf{v}\|_{H(\mathrm{div};D)}, \\ |B(\mathbf{v},q)| &= |(q, \nabla\cdot \mathbf{v})| \le \|q\|_{L^2(D)} \|\mathbf{v}\|_{H(\mathrm{div};D)}, \\ |\ell(\mathbf{v})| &\le \|\mathbf{v}\|_{H(\mathrm{div};D)} \|g\|_{H^{1/2}(\partial D)}, \\ |h_{\boldsymbol{\zeta}}(q)| &= |(f(\boldsymbol{\zeta}),q)| \le \|f(\boldsymbol{\zeta})\|_{L^2(D)} \|q\|_{L^2(D)}. \end{aligned}$$

Moreover, $A_{\boldsymbol{\zeta}}$ is coercive on

$$\begin{aligned} V &= \{\mathbf{v} \in H(\mathrm{div};D) : B(\mathbf{v},q) = -(q, \nabla\cdot\mathbf{v}) = 0 \quad \forall q \in L^2(D)\} \\ &= \{\mathbf{v} \in H(\mathrm{div};D) : \|\nabla\cdot\mathbf{v}\|_{L^2(D)} = 0\}, \end{aligned}$$

since for $\mathbf{v} \in V$ there holds

$$\mathrm{Re} A_{\boldsymbol{\zeta}}(\mathbf{v},\mathbf{v}) = \mathrm{Re}\left(a^{-1}(\boldsymbol{\zeta})\mathbf{v}, \mathbf{v}\right) \ge \operatorname{ess\,inf}_{\mathbf{x}\in D} \mathrm{Re}\left(a^{-1}(\mathbf{x},\boldsymbol{\zeta})\right) \|\mathbf{v}\|^2_{L^2(D)} \ge \frac{\|\mathbf{v}\|^2_{H(\mathrm{div};D)}}{a_{\max}(\boldsymbol{\zeta})}.$$

According to [5, p. 136], for any $q \in L^2(D)$ there exists $\mathbf{v}_q \in V$ such that

$$\|\nabla\cdot\mathbf{v}_q - q\|_{L^2(D)} = 0 \quad \text{and} \quad \|\mathbf{v}_q\|_{H(\mathrm{div};D)} \le C_D \|q\|_{L^2(D)},$$

with a constant C_D depending only on the domain D. Thus, the inf-sup-condition follows since, for any $q \in L^2(D)$,

$$\sup_{\mathbf{v}\in H(\mathrm{div};D)} \frac{B(\mathbf{v},q)}{\|\mathbf{v}\|_{H(\mathrm{div};D)}} \ge \frac{(q, \nabla\cdot\mathbf{v}_q)}{\|\mathbf{v}_q\|_{H(\mathrm{div};D)}} = \frac{\|q\|^2_{L^2(D)}}{\|\mathbf{v}_q\|_{H(\mathrm{div};D)}} \ge \frac{\|q\|_{L^2(D)}}{C_D}.$$

Therefore, by applying [5, Theorem II.1.1], resp. its generalization to variational problems in complex Hilbert spaces (hereby applying the complex version of the Lax-Milgram-lemma), we obtain for each $\boldsymbol{\zeta} \in \mathbb{C}^M$ a unique solution $(p(\boldsymbol{\zeta}), \mathbf{u}(\boldsymbol{\zeta}))$ to the associated deterministic variational problem. Moreover, there holds

$$\begin{aligned} \|\mathbf{u}(\boldsymbol{\zeta})\|_{H(\mathrm{div};D)} &\le \|g\|_{H^{1/2}(\partial D)} a_{\max}(\boldsymbol{\zeta}) + 2C_D \frac{a_{\max}(\boldsymbol{\zeta})}{a_{\min}(\boldsymbol{\zeta})} \|f(\boldsymbol{\zeta})\|_{L^2(D)}, \\ \|p(\boldsymbol{\zeta})\|_{L^2(D)} &\le 2C_D \|g\|_{H^{1/2}(\partial D)} \frac{a_{\max}(\boldsymbol{\zeta})}{a_{\min}(\boldsymbol{\zeta})} + 2C_D \frac{a_{\max}(\boldsymbol{\zeta})}{a^2_{\min}(\boldsymbol{\zeta})} \|f(\boldsymbol{\zeta})\|_{L^2(D)}. \end{aligned}$$

Further, we observe that $p : \mathbb{R}^M \to L^2(D)$ and $\mathbf{u} : \mathbb{R}^M \to H(\mathrm{div};D)$ are measurable, since they are continuous functions of a, f and g.

By applying the Hölder inequality for the exponents $r = q^*/2$ and $q > 0$ (such that $1/r + 1/q = 1$) and by taking into account the above estimate and the assumptions, we easily obtain that $p \in \mathcal{S} \otimes L^2(D)$ and $\mathbf{u} \in \mathcal{S} \otimes H(\mathrm{div};D)$, which yields $(p,\mathbf{u}) \in \mathcal{S} \otimes \mathcal{V}$. Uniqueness follows immediately. The continuity of $(p(\boldsymbol{\zeta}),\mathbf{u}(\boldsymbol{\zeta}))$ w.r.t. $\boldsymbol{\zeta} \in \Sigma$ follows from our assumptions on the continuity of a, f w.r.t. $\boldsymbol{\zeta}$. Finally, $p \in C_{\sigma^4}(\Sigma; L^2(D))$ and $\mathbf{u} \in C_{\sigma^4}(\Sigma; H(\mathrm{div};D))$ follow again directly from the estimates above and the assumptions. This completes the proof. $\square$

In an analogous way to [8, Lemma 2.2] we can show the analyticity of the parameter-to-solution map $\boldsymbol{\zeta} \mapsto (p(\boldsymbol{\zeta}),\mathbf{u}(\boldsymbol{\zeta}))$.

Lemma 3. *Let Assumptions 2 and 3 be satisfied and let the functions $a^{-1} : \Sigma \to L^\infty(D)$ and $f : \Sigma \to L^2(D)$ be analytic. Then also the mapping $\boldsymbol{\zeta} \mapsto (p(\boldsymbol{\zeta}),\mathbf{u}(\boldsymbol{\zeta}))$ is analytic in Σ.*

Proof. We prove the statement by showing the existence of each partial complex derivative $\partial_m(p(\boldsymbol{\zeta}),\mathbf{u}(\boldsymbol{\zeta}))$, $m = 1,\dots,M$. A deep theorem by Hartogs [14] then yields analyticity as a function of all M complex variables. Therefore, we fix $m \in \{1,\dots,M\}$, denote by e_m the m-th coordinate in $\mathbb{R}^M$ and set for $z \in \mathbb{C}\backslash\{0\}$

$$(q_z,\mathbf{v}_z)(\boldsymbol{\zeta}) := \frac{(p,\mathbf{u})(\boldsymbol{\zeta}+z\mathbf{e}_m) - (p,\mathbf{u})(\boldsymbol{\zeta})}{z}.$$

Note, that Σ is an open set due to the continuity of $a_{\max}$ and $a_{\min}$. Therefore, for each $\boldsymbol{\zeta} \in \Sigma$, there exists $\varepsilon_{\boldsymbol{\zeta}}$ such that, for $|z| \le \varepsilon_{\boldsymbol{\zeta}}$, the solution $(p,\mathbf{u})(\boldsymbol{\zeta}+z\mathbf{e}_m)$ and thus also the quotient above are well defined.

To simplify the presentation, we rewrite the variational problem (11) as a coupled linear system in the corresponding dual spaces, denoting again by $A_{\boldsymbol{\zeta}} : H(\mathrm{div};D) \to H(\mathrm{div};D)^*$ the linear mapping $[A_{\boldsymbol{\zeta}}\mathbf{u}](\mathbf{v}) := (a^{-1}(\boldsymbol{\zeta})\mathbf{u},\mathbf{v})$, by $B : L^2(D) \to H(\mathrm{div};D)^*$ the linear map $[Bp](\mathbf{v}) := (p, \nabla\cdot\mathbf{v})$ and by $B^\top : H(\mathrm{div};D) \to L^2(D)^*$ the map $[B^\top\mathbf{u}](q) := (q, \nabla\cdot\mathbf{u})$. Moreover, by ℓ and $h_{\boldsymbol{\zeta}}$ we denote the linear functionals corresponding to the right hand side of (11). Thus, the variational problem (11) reads

$$\begin{pmatrix} A_{\boldsymbol{\zeta}}\mathbf{u} + Bp \\ B^\top\mathbf{u} \end{pmatrix} = \begin{pmatrix} \ell \\ h_{\boldsymbol{\zeta}} \end{pmatrix}. \tag{12}$$

Hence, by denoting $\boldsymbol{\zeta}_z = \boldsymbol{\zeta} + z\mathbf{e}_m$ we have

$$\begin{pmatrix} A_{\boldsymbol{\zeta}}\mathbf{v}_z + Bq_z \\ B^\top\mathbf{v}_z \end{pmatrix} - \begin{pmatrix} \frac{A_{\boldsymbol{\zeta}} - A_{\boldsymbol{\zeta}_z}}{z}\mathbf{u}(\boldsymbol{\zeta}_z) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{h_{\boldsymbol{\zeta}_z} - h_{\boldsymbol{\zeta}}}{z} \end{pmatrix},$$

i.e., the pair $(q_z,\mathbf{v}_z)$ solves the linear system (12) for the right hand side

$$L_z := \frac{1}{z}\begin{pmatrix} -(A_{\boldsymbol{\zeta}_z} - A_{\boldsymbol{\zeta}})\mathbf{u}(\boldsymbol{\zeta}_z) \\ h_{\boldsymbol{\zeta}_z} - h_{\boldsymbol{\zeta}} \end{pmatrix}.$$

We now show that

$$L_z \to L_0 := \begin{pmatrix} -\partial_m A_{\boldsymbol{\zeta}} \mathbf{u}(\boldsymbol{\zeta}) \\ \partial_m h_{\boldsymbol{\zeta}} \end{pmatrix} \quad \text{as } z \to 0,$$

where $[\partial_m A_{\boldsymbol{\zeta}} \mathbf{u}](\mathbf{v}) := (\partial_m a^{-1}(\boldsymbol{\zeta})\mathbf{u}, \mathbf{v})$ and $\partial_m h_{\boldsymbol{\zeta}}(q) := (\partial_m f(\boldsymbol{\zeta}), q)$. Note first that there holds

$$\lim_{h\to 0} \left\| \frac{h_{\boldsymbol{\zeta}+z\mathbf{e}_m} - h_{\boldsymbol{\zeta}}}{z} - \partial_m h_{\boldsymbol{\zeta}} \right\|_{L^2(D)^*} = 0,$$

which can be easily seen by applying the Cauchy-Schwarz inequality and the assumption about the analyticity of f. Moreover, we have

$$\begin{aligned} \left\| \frac{A_{\boldsymbol{\zeta}_z} - A_{\boldsymbol{\zeta}}}{z} \mathbf{u}(\boldsymbol{\zeta}_z) - \partial_m A_{\boldsymbol{\zeta}} \mathbf{u}(\boldsymbol{\zeta}) \right\|_{H(\mathrm{div};D)^*} &\le \left\| \frac{A_{\boldsymbol{\zeta}_z} - A_{\boldsymbol{\zeta}}}{z} \right\| \|\mathbf{u}(\boldsymbol{\zeta}_z) - \mathbf{u}(\boldsymbol{\zeta})\|_{H(\mathrm{div};D)} \\ &+ \left\| \frac{A_{\boldsymbol{\zeta}_z} - A_{\boldsymbol{\zeta}}}{z} - \partial_m A_{\boldsymbol{\zeta}} \right\| \|\mathbf{u}(\boldsymbol{\zeta})\|_{H(\mathrm{div};D)}. \end{aligned}$$

There holds

$$\lim_{z\to 0} \|\mathbf{u}(\boldsymbol{\zeta} + z\mathbf{e}_m) - \mathbf{u}(\boldsymbol{\zeta})\| = 0,$$

since $\mathbf{u}(\boldsymbol{\zeta})$ depends continuously on $\boldsymbol{\zeta}$ as shown before. Furthermore, there holds

$$\frac{\left| \int_D \frac{a^{-1}(\boldsymbol{\zeta}_z) - a^{-1}(\boldsymbol{\zeta})}{z} \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \right|}{\|\mathbf{u}\|_{H(\mathrm{div};D)} \|\mathbf{v}\|_{H(\mathrm{div};D)}} \le \left\| \frac{a^{-1}(\boldsymbol{\zeta}_z) - a^{-1}(\boldsymbol{\zeta})}{z} \right\|_{L^\infty(D)} \to \|\partial_m a^{-1}(\boldsymbol{\zeta})\|_{L^\infty(D)}$$

as $z \to 0$ due to the analyticity of a^{-1}. Thus, we have

$$\|(A_{\boldsymbol{\zeta}_z} - A_{\boldsymbol{\zeta}})/z\| \le \|(a^{-1}(\boldsymbol{\zeta}_z) - a^{-1}(\boldsymbol{\zeta}))/z\|_{L^\infty(D)} \to \|\partial_m a^{-1}(\boldsymbol{\zeta})\|_{L^\infty(D)}$$

as $z \to 0$. By linearity we obtain with the same argument

$$\left\| \frac{A_{\boldsymbol{\zeta}_z} - A_{\boldsymbol{\zeta}}}{z} - \partial_m A_{\boldsymbol{\zeta}} \right\| \le \left\| \frac{a^{-1}(\boldsymbol{\zeta}_z) - a^{-1}(\boldsymbol{\zeta})}{z} - \partial_m a^{-1}(\boldsymbol{\zeta}) \right\|_{L^\infty(D)} \to 0$$

as $z \to 0$, which finally yields $L_z \to L_0$ as $z \to 0$. Again, by the continuous dependence of the solution of (12) on the right hand side, we conclude

$$(q_z, \mathbf{v}_z) \to (\partial_m p(\boldsymbol{\zeta}), \partial_m \mathbf{u}(\boldsymbol{\zeta})) \quad \text{as } z \to 0,$$

where $(\partial_m p(\boldsymbol{\zeta}), \partial_m \mathbf{u}(\boldsymbol{\zeta}))$ solves (12) for the right hand side L_0. We have thus established that $(p(\boldsymbol{\zeta}), \mathbf{u}(\boldsymbol{\zeta}))$ possesses the partial derivative $(\partial_m p(\boldsymbol{\zeta}), \partial_m \mathbf{u}(\boldsymbol{\zeta}))$ in the m-th (complex) direction, which completes the proof. □

Example 2 (lognormal diffusion coefficient). We consider a coefficient

$$a(\mathbf{x},\boldsymbol{\zeta}) = \exp\left(\phi_0(\mathbf{x}) + \sum_{m=1}^{M} \phi_m(\mathbf{x})\zeta_m\right)$$

with (real-valued) $\phi_m \in L^\infty(D)$ for $m = 0,\ldots,M$. Let ρ be the M-dimensional standard normal probability density function. Setting $b_m := \|\phi_m\|_{L^\infty(D)}$ for $m = 0,\ldots,M$, then for all $\boldsymbol{\zeta} \in \Sigma$ with

$$\Sigma = \left\{\boldsymbol{\zeta} \in \mathbb{C}^M : \sum_{m=1}^{M} b_m\,|\mathrm{Im}\,\zeta_m| < \frac{\pi}{2}\right\} \tag{13}$$

there holds

$$a_{\min}(\boldsymbol{\zeta}) \geq \exp\Big(-b_0 - \sum_{m=1}^{M} b_m|\xi_m|\Big)\cos\Big(\sum_{m=1}^{M} b_m|\eta_m|\Big) > 0,$$
$$a_{\max}(\boldsymbol{\zeta}) \leq \exp\Big(b_0 + \sum_{m=1}^{M} b_m|\xi_m|\Big),$$

where $\boldsymbol{\zeta} = \boldsymbol{\xi} + i\boldsymbol{\eta}$. Furthermore, a then satisfies the assumptions of Lemma 3 for Σ as given in (13) and the weighting function $\sigma(\boldsymbol{\xi}) = \sigma_1(\xi_1)\cdots\sigma_M(\xi_M)$, where $\sigma_m(\xi_m) = \exp\big(-b_m|\xi_m|\big)$, $m = 1,\ldots,M$.

Remark 2. Note that if, in Example 2, the expansion functions $\{\phi_m\}_{m=1}^M$ in addition have disjoint supports, then a satisfies the assumptions of Lemma 3 for the larger domain

$$\Sigma = \{\boldsymbol{\zeta} \in \mathbb{C}^M : b_m\,|\mathrm{Im}\,\zeta_m| < \pi/2\},$$

since then

$$\begin{aligned}\mathrm{Re}\,a(\mathbf{x},\boldsymbol{\zeta}) &= \exp\Big(\phi_0(\mathbf{x}) + \sum_{m=1}^{M} \phi_m(\mathbf{x})\xi_m\Big)\cos\Big(\sum_{m=1}^{M} \phi_m(\mathbf{x})\eta_m\Big) \\ &\geq \exp\Big(-b_0 - \Big(\max_m b_m|\xi_m|\Big)\Big)\cos\Big(\max_m b_m|\eta_m|\Big).\end{aligned}$$

4 Sparse Grid Collocation

Stochastic collocation in the context of UQ or parametric problems can be described roughly as a method for approximating a function $u : \mathbb{R}^M \to W$ with values in, say, a separable Banach space W from the span of n linearly independent functions $\{u_j : \mathbb{R}^M \to W\}_{j=1}^n$ given only the values of u at certain distinct points in the parameter domain $\mathbb{R}^M$. Suitable finite-dimensional function spaces are determined by the smoothness of u as a function of the parameter. Since the solution of (11) depends smoothly on $\boldsymbol{\xi}$, as was shown in the previous section, we consider approximations by global interpolating polynomials as done in, e.g., [1, 3, 6, 18, 19, 23].

Therefore, let $\chi_k = \{\xi_{k,1},\ldots,\xi_{k,n_k}\}$, $k = 1,2,\ldots$, be a given sequence of node sets in $\mathbb{R}$ and

$$(\mathcal{I}_k v)(\xi) := \sum_{j=0}^{n_k} v\left(\xi_{k,j}\right) \ell_{k,j}(\xi)$$

be the associated Lagrange interpolation operator with the Lagrange basis polynomials $\ell_{k,j}$. We further define the difference operators $\Delta_k = \mathcal{I}_k - \mathcal{I}_{k-1}$ for $k \geq 1$, where $\mathcal{I}_0 := 0$. Then the (Smolyak) sparse grid stochastic collocation operator is defined as

$$\mathcal{A}_{q,M} := \sum_{|\mathbf{i}| \leq q+M} \Delta_{i_1} \otimes \cdots \otimes \Delta_{i_M} = \sum_{q+1 \leq |\mathbf{i}| \leq q+M} c_{q,M}(\mathbf{i})\, \mathcal{I}_{i_1} \otimes \cdots \otimes \mathcal{I}_{i_M}, \tag{14}$$

where $|\mathbf{i}| := i_1 + \ldots + i_M$ and

$$c_{q,M}(\mathbf{i}) = (-1)^{q+M-|\mathbf{i}|} \binom{M-1}{q+M-|\mathbf{i}|},$$

cf. [22]. The sparse grid associated with $\mathcal{A}_{q,M}$ consists of the points

$$\mathcal{H}_{q,M} := \bigcup_{q+1 \leq |\mathbf{i}| \leq q+M} \chi_{i_1} \times \cdots \times \chi_{i_M} \subset \mathbb{R}^M. \tag{15}$$

One may choose $\mathcal{X}_k$ to be the roots of the n_k-th Hermite polynomial (w.r.t. to the weight $\rho_m(\xi) = e^{-\xi^2/2}/\sqrt{2\pi}$), since these nodal points yield maximally convergent interpolations (cf. [21]) and this choice also simplifies the computation of moments of $\mathcal{A}_{q,M} u$ w.r.t. the weight $\rho = \prod_m \rho_m$.

For bounded parameter domains and constant density $\rho_m \equiv$ const, popular sequences of nodal sets are Gauss-Legendre and Clenshaw-Curtis nodes. For sparse grid collocation based on these sequences a convergence analysis is given in [19], where it is indicated that a similar analysis applies to Gauss-Hermite nodes. We carry out this analysis in the following.

Assumption 4 *There exist constants $c > 0$ and $\varepsilon_m > 0$, $m = 1, \ldots, M$, such that*

$$\rho_m(\xi_m) = \frac{\exp(-\xi_m^2/2)}{\sqrt{2\pi}} \leq c \exp(-\varepsilon_m \xi_m^2) \sigma_m^2(\xi_m), \qquad m = 1, \ldots, M, \tag{16}$$

and the weighting function has the product structure $\sigma(\boldsymbol{\xi}) = \prod_{m=1}^M \sigma_m(\xi_m)$.

Note that Assumption 4 implies that $C_\sigma(\mathbb{R}^M; W)$ is continuously embedded in $L^2_\rho(\mathbb{R}^M; W)$, since for $v \in C_\sigma(\mathbb{R}^M; W)$ there holds

$$\begin{aligned}\int_{\mathbb{R}^M} \|v(\boldsymbol{\xi})\|_W^2 \, \rho(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} &\leq \|v\|^2_{C_\sigma(\mathbb{R}^M;W)} \int_{\mathbb{R}^M} \frac{\rho(\boldsymbol{\xi})}{\sigma^2(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi} \\ &\leq c \|v\|^2_{C_\sigma(\mathbb{R}^M;W)} \prod_{m=1}^M \int_{\mathbb{R}^M{}_m} \exp(-\varepsilon_m \xi_m^2) \, \mathrm{d}\boldsymbol{\xi} < \infty.\end{aligned}$$

The same is true of the restrictions of functions in $C_\sigma(\Sigma; W)$, since $\mathbb{R}^M \subset \Sigma \subset \mathbb{C}^M$ due to Assumption 2.

Theorem 1 (cf. [19, Theorem 3.18]). *Let W be a separable Banach space, let $u : \mathbb{R}^M \to W$ admit an analytic extension to the domain*

$$\Sigma_{\boldsymbol{\tau}} = \{\boldsymbol{\zeta} \in \mathbb{C}^M : |\operatorname{Im}\zeta_m| \le \tau_m, m = 1,\dots,M\}, \qquad \boldsymbol{\tau} = (\tau_1,\dots,\tau_M),$$

and, in addition, $u \in C_\sigma(\Sigma_{\boldsymbol{\tau}};W)$, i.e.,

$$\max_{\boldsymbol{\zeta}\in\Sigma_{\boldsymbol{\tau}}} \sigma(\operatorname{Re}\boldsymbol{\zeta})\,\|u(\boldsymbol{\zeta})\|_W < +\infty,$$

where $\rho(\boldsymbol{\xi}) = \prod_m \rho_m(\xi_m)$ and $\sigma(\boldsymbol{\xi}) = \prod_m \sigma_m(\xi_m)$ satisfy Assumption 4. Then the error of the sparse grid collocation approximation $\mathcal{A}_{q,M}u$ based on Gauss-Hermite nodes χ_k where

$$|\chi_k| = n_k = \begin{cases} 1, & k = 1, \\ 2^{k-1}+1, & k > 1, \end{cases}$$

can be bounded by

$$\|u - \mathcal{A}_{q,M}u\|_{L^2_\rho} \le \frac{C_r^{M+1} - C_r}{C_r - 1} \begin{cases} \exp\left(-q\frac{\log 2}{2}\left(R\frac{e}{\sqrt{2}} - 1\right)\right), & \text{if } 0 \le q \le \frac{2M}{\log 2}, \\ \exp\left(-R\frac{M}{\sqrt{2}}\sqrt{2^{q/M}} + \frac{q}{2}\log 2\right), & \text{otherwise,} \end{cases} \tag{17}$$

where $C_r = C(2 + \sqrt{8\pi/r}/\log 2)$ and

$$r := \min_{m=1,\dots,M} \tau_m, \qquad R := \sqrt[M]{\tau_1\cdots\tau_M}.$$

In particular, for $0 \le q \le \frac{2M}{\log 2}$ there holds

$$\|u - \mathcal{A}_{q,M}u\|_{L^2_\rho} \le \tilde{C}(r,R,M)\,N^{-\nu_1}, \qquad \nu_1 = \frac{\log 2}{2(2.1 + \log M)}\left(\frac{eR}{\sqrt{2}} - 1\right), \tag{18}$$

where $N = |\mathcal{H}_{q,M}|$ and $\tilde{C}(r,R,M) = C(r)\frac{1-C(r)^M}{1-C(r)}\sqrt{2^{eR/\sqrt{2}-1}}$.
Conversely, for $q > \frac{2M}{\log 2}$ there holds

$$\|u - \mathcal{A}_{q,M}u\|_{L^2_\rho} \le \frac{C_r^{M+1} - C_r}{C_r - 1}\,\frac{N^2}{M^2}\,e^{-\frac{R}{\sqrt{2}}MN^{\nu_2}}, \qquad \nu_2 = \frac{\log 2}{2M(2.1 + \log M)}. \tag{19}$$

Proof. The proof follows closely the procedure for showing convergence of $\mathcal{A}_{q,M}$ w.r.t. Clenshaw-Curtis nodes given in [19]. Since only certain steps need to be modified we only mention these here and refer to [19] for further details.

Step 1: Show $\|u - \mathcal{A}_{q,M}u\|_{L^2_\rho} \le \sum_{k=1}^M R(q,k)$.
According to the proof of [19, Lemma 3.4], there holds

$$I - \mathcal{A}_{q,M} = \sum_{k=2}^{M} \left[\tilde{R}(q,k) \bigotimes_{m=k+1}^{M} I \right] + \left(I - \mathcal{A}_{q,1}\right) \bigotimes_{m=2}^{M} I,$$

where

$$\tilde{R}(q,k) = \sum_{\substack{\mathbf{i} \in \mathbb{N}^{k-1} \\ |\mathbf{i}| \le q+k-1}} \bigotimes_{m=1}^{k-1} \Delta_{i_m} \otimes \left(I - \mathcal{I}_{n_{\hat{i}_k}} \right)$$

and $\hat{i}_k = 1 + q - \sum_{m=1}^{k-1}(i_m - 1)$. Further, the term $\tilde{R}(q,k)$ can bounded using the results given in the Appendix:

$$\begin{aligned}
\|\tilde{R}(q,k)u\|_{L^2_\rho} &\le \sum_{\substack{\mathbf{i} \in \mathbb{N}^{k-1} \\ |\mathbf{i}| \le q+k-1}} \left\| \bigotimes_{m=1}^{k-1} \Delta_{i_m} \otimes \left(I - \mathcal{I}_{n_{\hat{i}_k}} \right) u \right\|_{L^2_\rho} \\
&\le \sum_{\substack{\mathbf{i} \in \mathbb{N}^{k-1} \\ |\mathbf{i}| \le q+k-1}} C^k \left(\sqrt{2^{\hat{i}_k}} \right) e^{-\frac{1}{2}\left(\sum_{m=1}^{k-1} \tau_m \sqrt{2^{i_m}} + \tau_k \sqrt{2^{\hat{i}_k + 1}} \right)} \prod_{m=1}^{k-1} \left(\sqrt{2^{i_m}} + 1 \right) \\
&= C^k \sum_{\substack{\mathbf{i} \in \mathbb{N}^{k} \\ |\mathbf{i}| = q+k}} \exp\left(-\frac{1}{2} h(\mathbf{i},k) \right),
\end{aligned}$$

where $h(\mathbf{i},k) = \sum_{m=1}^{k} \tau_m \sqrt{2^{i_m}} - (\log 2) i_m$. Moreover, we obtain by applying results from [1, Section 4]

$$\begin{aligned}
\| \left(I - \mathcal{A}_{q,1}\right) u \|_{L^2_\rho} &= \| (I - \mathcal{I}_{n_{q+1}}) u \|_{L^2_\rho} \le C \left(\sqrt{2^{q+1}} \right) \exp\left(-\frac{\tau_1}{\sqrt{2}} \sqrt{2^{q+1}} \right) \\
&= \sum_{\substack{i \in \mathbb{N}^1 \\ |i| = q+1}} C \left(\sqrt{2^i} \right) \exp\left(-\frac{\tau_1}{\sqrt{2}} \sqrt{2^i} \right).
\end{aligned}$$

Therefore, setting

$$R(q,k) := C^k \sum_{\substack{\mathbf{i} \in \mathbb{N}^{k} \\ |\mathbf{i}| = q+k}} \exp\left(-\frac{1}{2} h(\mathbf{i},k) \right),$$

we arrive at the bound $\left\| (I - \mathcal{A}_{q,M}) u \right\|_{L^2_\rho} \le \sum_{k=1}^{M} R(q,k)$.

Step 2: Estimate $R(q,k)$.
Computing the minimum of $h(\cdot,k)$ on the set $\{\mathbf{x} \in \mathbb{R}^k : x_1 + \cdots + x_k = q+k\}$ yields the optimal point $\mathbf{i}^* = (i_1^*, \ldots, i_k^*)$ with

$$i_m^* = 1 + q/k + \frac{2}{k} \sum_{n=1}^{k} \log_2(\tau_n / \tau_m), \qquad m = 1, \ldots, k.$$

Moreover, expanding $h(\cdot,k)$ at $\mathbf{i}^*$ up to second order yields for any $\mathbf{i}\in\mathbb{N}^k$ with $|\mathbf{i}| = q+k$

$$\begin{aligned}
h(\mathbf{i},k) &= h(\mathbf{i}^*,k) + \underbrace{\nabla h(\mathbf{i}^*,k)\cdot(\mathbf{i}-\mathbf{i}^*)^T}_{=0} + \frac{1}{2}(\mathbf{i}-\mathbf{i}^*)\cdot\nabla^2 h(\boldsymbol{\iota},k)\cdot(\mathbf{i}-\mathbf{i}^*)^T \\
&= k2^{(q+k)/(2k)}\prod_{m=1}^{k}\sqrt[k]{\tau_m} - (\log 2)(q+k) + \frac{1}{2}\sum_{m=1}^{k}\tau_m\frac{(\log 2)^2}{4}2^{\iota_m/2}(i_m-i_m^*)^2 \\
&\geq k2^{(q+k)/(2k)}\prod_{m=1}^{k}\sqrt[k]{\tau_m} - (\log 2)(q+k) + r\frac{(\log 2)^2}{8}\sum_{m=1}^{k}(i_m-i_m^*)^2,
\end{aligned}$$

where $\iota_m\in[\min(i_m,i_m^*),\max(i_m,i_m^*)]$ for $m=1,\ldots,M$.

Without loss of generality we may assume that $\tau_1\geq\tau_2\geq\ldots\geq\tau_M$. Thus, we have for any $k=1,\ldots,M$

$$\prod_{m=1}^{k}\sqrt[k]{\tau_m}\geq\prod_{m=1}^{M}\sqrt[M]{\tau_m} =: \sqrt[M]{\boldsymbol{\tau}}$$

and there holds furthermore

$$\begin{aligned}
R(q,k) &\leq C^k\exp\left(\frac{q}{2}\log 2 - k\frac{\sqrt[M]{\boldsymbol{\tau}}}{2}2^{(q+k)/(2k)}\right)\sum_{\substack{\mathbf{i}\in\mathbb{N}^k\\|\mathbf{i}|=q+k}}\prod_{m=1}^{k}e^{r\log^2 2/8(i_m-i_m^*)^2} \\
&\leq C^k\exp\left(\frac{q}{2}\log 2 - k\frac{\sqrt[M]{\boldsymbol{\tau}}}{2}2^{(q+k)/(2k)}\right)\prod_{m=1}^{k}\sum_{i=1}^{q+1}e^{r\log^2 2/8(i-i_m^*)^2} \\
&\leq C^k\exp\left(\frac{q}{2}\log 2 - k\frac{\sqrt[M]{\boldsymbol{\tau}}}{2}2^{(q+k)/(2k)}\right)\left(2+\sqrt{\frac{8\pi}{r\log^2 2}}\right)^k \\
&= C_r^k\exp\left(\frac{q}{2}\log 2 - k\frac{\sqrt[M]{\boldsymbol{\tau}}}{2}2^{(q+k)/(2k)}\right),
\end{aligned}$$

where we have used $\{\mathbf{i}\in\mathbb{N}^k : |\mathbf{i}| = q+k\}\subset\{\mathbf{i}\in\mathbb{N}^k : |\mathbf{i}|\leq q+k\}$ in the second and [19, Lemma A.1] in the next-to-last last line.

Step 3: Combine previous steps.
The remaining steps are analogous to the proof of [19, Theorem 3.7] and [19, Theorem 3.10], respectively, using the bound for $N = |\mathcal{H}_{q,M}|$ from [19, Lemma 3.17]

$$\frac{\log N}{2.1+\log M} - 1 \leq q \leq \log_2(N/M-1).$$

□

Remark 3. Note that Theorem 1 states algebraic convergence of $\mathcal{A}_{q,M}u$ w.r.t. the number of collocation nodes N in the regime $q\leq 2M/\log 2$ and subexponential convergence in the regime $q > 2M/\log 2$. Typically, in applications with $M \geq 3$ a level $q > 2M/\log 2$ is seldom feasible.

Remark 4. Note, that our proof takes into account different widths τ_m of the strips of analyticity for different dimensions ξ_m in contrast to the corresponding proofs of [19, Lemma 3.4 & Lemma 3.16]. Moreover, we would like to mention that the proofs in [19], in particular the estimates of the term $\|\tilde{R}(q,k)u\|_{L^2_\rho}$ given there, require also that u possesses an analytic continuation to a product subdomain $\prod_{m=1}^M \Sigma_0$ of $\mathbb{C}^M$, $\Sigma_0 \subset \mathbb{C}$. This condition is, however, never explicitly assumed or shown to hold in [19]. Rather, the authors only state one-dimensional regularity results, i.e., results on the domain of analytic continuation of u w.r.t. each ζ_m, $m = 1,\dots,M$, separately, with the remaining coordinates $\xi_n \in \mathbb{R}$, $n \neq m$, kept fixed and *real*. However, this type of one-dimensional regularity is not sufficient for concluding analyticity of u in a product domain in $\mathbb{C}^M$. As we have seen in the proof of Lemma 3, the results on the one-dimensional complex domain of analytic continuation of u w.r.t. ζ_m, $m = 1,\dots,M$, need to hold for all fixed, *complex* coordinates $\zeta_n \in \Sigma_0$, $n \neq m$.

Combining the result above with our investigations of the previous section, we conclude

Corollary 1 (Convergence in case of lognormal diffusion). *Let a problem* (11) *with a diffusion coefficient of the form*

$$a(\mathbf{x},\boldsymbol{\xi}) = \exp\left(\phi_0(\mathbf{x}) + \sum_{m=1}^M \phi_m(\mathbf{x})\xi_m\right)$$

be given and let the assumption of Lemma 3 be satisfied. Then there holds for $0 \leq q \leq \frac{2M}{\log 2}$ *and* $N = |\mathcal{H}_{q,M}|$

$$\|u - \mathcal{A}_{q,M}u\|_{L^2_\rho} \leq \tilde{C}(r,R,M)\, N^{-\frac{\log 2}{2(2.1+\log M)}\left(\frac{eR}{\sqrt{2}}-1\right)}$$

where $\tilde{C}(r,R,M)$ *is according to Theorem 1 and where*

$$R \geq \frac{\pi - \varepsilon}{2M\left(\|\phi_1\|_{L^\infty(D)} \cdots \|\phi_M\|_{L^\infty(D)}\right)^{1/M}}$$

for any $\varepsilon > 0$.

Proof. Given the statement of Theorem 1 and the observations made in Example 2, we simply maximize $\sqrt[M]{\tau_1 \cdots \tau_M}$ under the constraint

$$\sum_{m=1}^M \tau_m \|\phi_m\|_{L^\infty(D)} = \frac{\pi - \varepsilon}{2}$$

for an arbitrary $\varepsilon > 0$. This yields the optimal point

$$\tau_m^* = \frac{\pi - \varepsilon}{2M\|\phi_m\|_{L^\infty(D)}}, \qquad m = 1, \dots, M,$$

and, furthermore, $\sqrt[M]{\tau_1^* \cdots \tau_M^*} = (\pi - \varepsilon)/\big(2M(\|\phi_1\|_{L^\infty(D)} \cdots \|\phi_M\|_{L^\infty(D)})^{1/M}\big)$. □

5 Numerical Example

We illustrate the theoretical results of the previous sections for a simple elliptic boundary value problem in mixed form as arises, e.g., in groundwater flow modeled by Darcy's law. In addition, we examine how the convergence of stochastic collocation is affected by properties of the lognormal diffusion coefficient a. Although the results given in the previous section are valid for a general random field with a representation of the form given in (3), we wish to relate common properties of Gaussian random fields such as their mean, variance etc. to the convergence of the stochastic collocation method.

The Gaussian random field $\log a$ is uniquely determined by its mean and covariance functions. The mean ϕ_0 of $\log a$ does not affect the convergence of the stochastic collocation approximation as Corollary 1 shows, but the covariance plays a more important role, since it determines the representation (3). Generally speaking, covariance functions are characterized by a variance parameter, correlation length and its degree of smoothness. The latter may also be expressed in terms of a parameter, as is the case for the Matérn family of covariance functions (see, e.g., [7]). However, since the smoothness of the covariance function controls the asymptotic decay of the eigenvalues of the associated covariance operator and the correlation length determines the length of a preasymptotic plateau preceding the asymptotic decay, both will affect the length M of a truncated Karhunen-Loève expansion with sufficiently small truncation error. Hence, by relating the smoothness and the correlation length to M, we will illustrate the effect of increasing M and increasing σ on the convergence of the stochastic collocation approximations in the following.

A Simple Groundwater Flow Model.

The PDE under consideration is of the form (1) with source term $f \equiv 0$, boundary data $g(x_1, x_2) = 3\big(x_1^2 + (1 - x_2)^2\big)^{1/2}$, and lognormal coefficient a on the unit square $D = [0,1]^2$ in $\mathbb{R}^2$. In particular, we assume for the Gaussian random field $\log a$ a mean $\phi_0(\mathbf{x}) \equiv 1$ and a stationary and isotropic two-point covariance function given by

$$\mathrm{Cov}\left(\log a(\mathbf{x}), \log a(\mathbf{y})\right) = \sigma^2 \exp\left(-\|\mathbf{x} - \mathbf{y}\|^2\right).$$

Thus, the approximation $\log a_M(\mathbf{x}, \boldsymbol{\xi})$ is the truncated KLE of this Gaussian random field, i.e.,

$$a_M(\mathbf{x}, \boldsymbol{\xi}) = \exp\left(1 + \sum_{m=1}^{M} \phi_m(\mathbf{x})\, \xi_m\right), \tag{20}$$

where $\phi_m(\mathbf{x}) = \sigma^2 \sqrt{\lambda_m} \psi_m(\mathbf{x})$ and $\{(\lambda_m, \psi_m)\}_{m=1,\dots,M}$ are the first M eigenpairs (in order of decreasing eigenvalues) of

$$\lambda\, \psi(\mathbf{x}) = \int_{[0,1]^2} \exp\left(-\|\mathbf{x} - \mathbf{y}\|^2\right) \psi(\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

Figure 1 displays the exponential decay of the eigenvalues λ_m and the norms $\|\phi_m\|_{L^\infty(D)}$ of the corresponding lognormal diffusion coefficient.

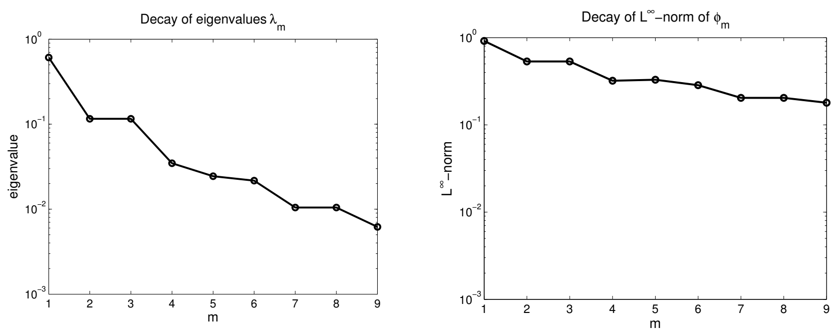

Fig. 1 Left: Decay of the eigenvalues of the covariance operators associated with $\log a$. Right: Decay of the $L^\infty(D)$-norm of the ϕ_m in (20).

Remark 5. Note that, in geophysical applications such as hydrogeology, one usually encounters ‘rougher’ random fields with a covariance function, e.g., of Matérn type, see [7]. However, the above model for $\log a$ is sufficient for our purpose of illustrating how the convergence of stochastic collocation depends on M and σ as explained above.

Note that, as the variance parameter σ of the random field $\log a$ increases, so does the L^∞-norm of the expansion functions ϕ_m, and therefore the rate of convergence for the stochastic collocation should decrease according to Corollary 1. We will demonstrate this in the following.

For the spatial discretization we use Raviart-Thomas finite elements of lowest order [5] for the flux and piecewise constants for the head variable. Thus, $p(\cdot, \boldsymbol{\xi})$ is approximated as a piecewise constant and $\mathbf{u}(\cdot, \boldsymbol{\xi})$ as a piecewise linear function. Moreover, the domain D is decomposed into 4206 triangles resulting in 10595 spatial degrees of freedom. Hence, the space $\mathcal{V} = L^2(D) \times H(\mathrm{div}; D)$ is replaced by the cartesian product $\mathcal{V}_h \subset \mathcal{V}$ of the finite dimensional approximation spaces and the continuous solution pair $(p, \mathbf{u})$ by the semidiscrete pair $(p_h, \mathbf{u}_h)$. Note that this does not influence the analysis of the previous sections, we merely apply the statements of Lemma 3 and Theorem 1 to the finite-dimensional subspaces. The full—i.e., collocation and finite element approximation—error can be obtained by appealing to standard finite element approximation theory, (cf. e.g., [2, 20]).

Solution and Convergence.

In Figure 2 we show for illustration the computational domain D, the triangular mesh and the mean head (left) and streamlines of the mean flux (right) of the solution obtained by sparse grid collocation with level $q = 5$ for a truncated KLE containing $M = 9$ terms.

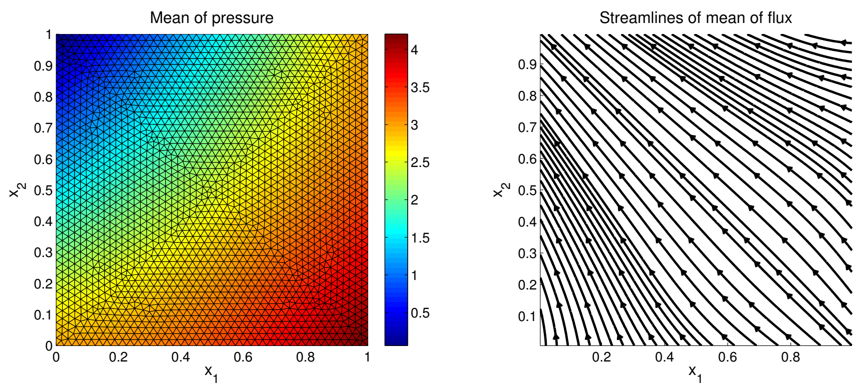

Fig. 2 Left: The mean of the pressure head approximation $\mathcal{A}_{5,9}p_h$; right: Streamlines of the mean of the flux approximation $\mathcal{A}_{5,9}\mathbf{u}_h$.

We observe (at least algebraic) convergence of the stochastic collocation approximation for head and flux in the left plot in Figure 3. Here and in the following we estimate the $L^2(\mathbb{R}^M;W)$-error of the stochastic collocation approximations by a sparse quadrature method applied to the error $\|p_h(\boldsymbol{\xi}) - \mathcal{A}_{q,M}p_h(\boldsymbol{\xi})\|_{L^2(D)}$ and $\|\mathbf{u}_h(\boldsymbol{\xi}) - \mathcal{A}_{q,M}\mathbf{u}_h(\boldsymbol{\xi})\|_{H(\mathrm{div};D)}$, respectively. We have chosen the sparse Smolyak quadrature operator corresponding to a stochastic collocation approximation of a high level q^*, i.e.,

$$\mathbb{E}\left[\|p_h - \mathcal{A}_{q,M}p_h\|^2_{L^2(D)}\right] \approx \sum_{\boldsymbol{\xi}_j \in \mathcal{H}_{q^*,M}} w_j \|p_h(\boldsymbol{\xi}_j) - \mathcal{A}_{q,M}p_h(\boldsymbol{\xi}_j)\|^2_{L^2(D)},$$

where $\{w_j : \boldsymbol{\xi}_j \in \mathcal{H}_{q^*,M}\}$ are the weights of the sparse quadrature operator associated with $\mathcal{A}_{q^*,M}$. For the results shown in Figure 3 we used $q^* = 5$. Note that we have also applied a Monte Carlo integration for the error estimation above for comparison which showed no substantial difference to the quadrature procedure above. The error estimation for $\mathcal{A}_{q,M}\mathbf{u}_h$ was obtained in the same way. We observe that the relative error for the flux does not immediately decay at the asymptotic rate. This is due to a

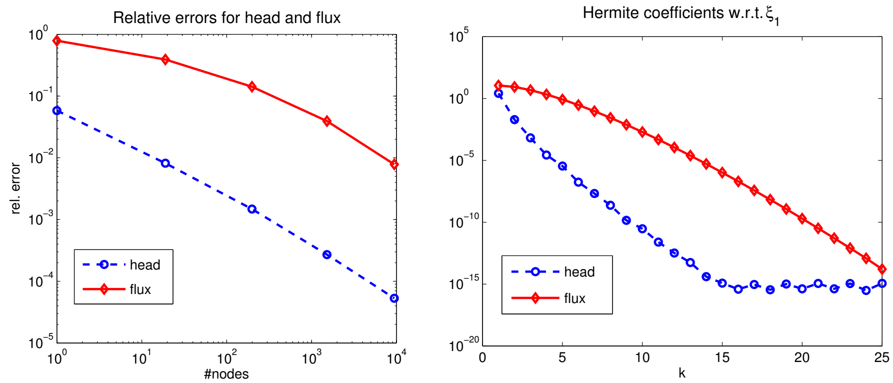

Fig. 3 Left: Relative errors for head and flux for $M = 9$ and $q = 0,\dots,4$. Right: Estimated Hermite coefficients of p_h and $\mathbf{u}_h$ for the first 25 Hermite polynomials w.r.t. ξ_1.

preasymptotic phase of slower decay of the Hermite coefficients of $\mathbf{u}_h$. We display the Hermite coefficients for the first 25 Hermite polynomials in ξ_1 for p_h and $\mathbf{u}_h$ on the right hand side of Figure 3. The preasymptotic slow decay of the coefficients in case of the flux is clearly visible. However, both errors apparently decay at a much greater rate than the estimate in Corollary 1 would suggest.

Influence of the Input Variance σ.

We fix $M = 5$ and vary the variance parameter $\sigma \in \{1/2, 1, 2\}$. For all three values of σ we choose a quadrature level of $q^* = 6$ for the error estimation. The results are shown in Figure 5. We observe the expected behaviour that for increased σ the convergence rate is reduced.

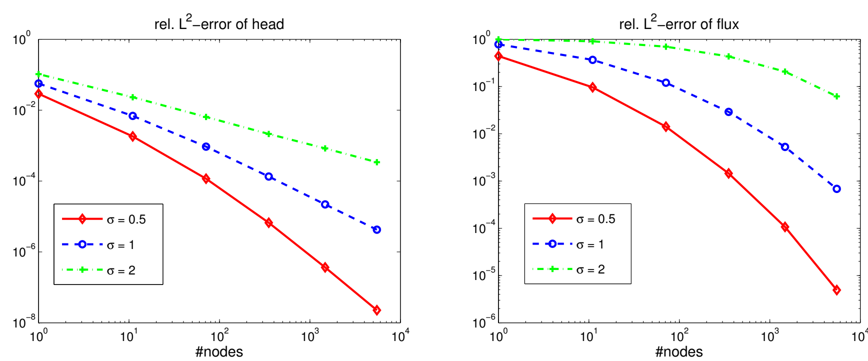

Fig. 4 Estimated relative $L^2(\mathbb{R}^M;W)$ errors of the sparse grid stochastic collocation approximations for pressure head p_h and flux $\mathbf{u}_h$ for $M = 5$ but different values of σ. The level q of $\mathcal{A}_{q,5}$ varies from 0 to 5.

Influence of the Parameter Dimension M.

We set $\sigma = 1$ and let $M \in \{3,6,9\}$. As quadrature levels for the error estimation we choose $q^* = 8$ for $M = 3$, $q^* = 7$ for $M = 6$, and $q^* = 6$ for $M = 9$. The results are shown in Figure 4. Again, the results are according to the conjecture that for increased dimension M the convergence rate decreases.

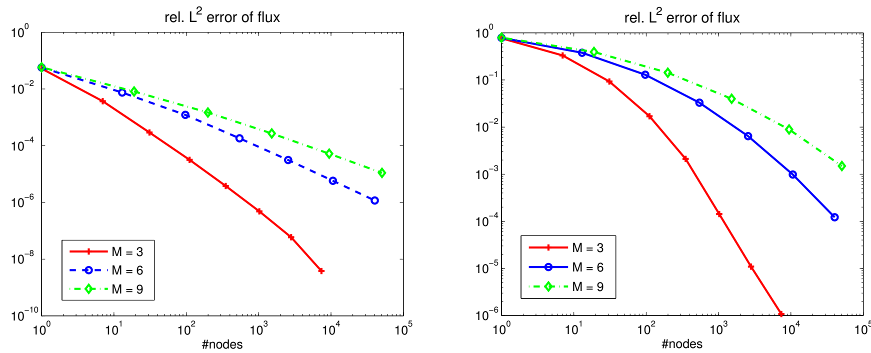

Fig. 5 Estimated relative $L^2(\mathbb{R}^M;W)$ errors of the sparse grid stochastic collocation approximations for pressure head p_h and flux $\mathbf{u}_h$ for $\sigma = 1$ but different values of M. The level q of $\mathcal{A}_{q,M}$ varies from 0 to 7 for $M = 3$, from 0 to 6 for $M = 6$ and from 0 to 5 for $M = 9$.

Remark 6. In view of the decelerating effect of large variance σ and roughness of a random field a (requiring large M for small truncation error) on the convergence rate of stochastic collocation, certain advanced Monte Carlo methods (such as quasi- or multilevel Monte Carlo) might be preferable for certain applications in subsurface physics where such rough random fields of high variance are common. We refer to the results in [7] for a comparison of the Monte Carlo and stochastic collocation method in case of a real-world subsurface flow problem. However, while efficient for estimating moments, probabilities or other statistical properties (so-called *quantities of interest*), Monte Carlo methods do not yield an approximate solution function of the PDE problem with random data as does stochastic collocation, which may serve as a cheap, sufficiently accurate surrogate model in many situations.

6 Conclusions

In this paper we have filled some remaining theoretical gaps for the application of sparse grid stochastic collocation to diffusion equations with a random, lognormally distributed diffusion coefficient. In particular, we have shown the smooth dependence of the solution of the associated parametric variational problems on the parameter under natural assumptions. This extends previous work [4] on random mixed elliptic problems to a broader and practically relevant class of diffusion coefficients. In addition, we have given a complete convergence proof for sparse grid stochastic collocation using basic random variables with unbounded supports, which was previously only hinted at in the literature as a remark [19]. Both results combine to form the theoretical foundation for applying stochastic collocation to interesting real world problems [7, 12] which we have illustrated for a simple groundwater flow model. The qualitative behavior of the approximation bounds indicate the limitations of stochastic collocation when applied to problems with diffusion coefficients displaying roughness or short correlation length.

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Appendix

We will here prove some auxiliary results used in the previous sections. In particular, we want to generalize some results from [1, Section 4] to multi-dimensional interpolation. The first concerns the uniform boundedness of the operator

$$\mathcal{I}_{k_1} \otimes \cdots \otimes \mathcal{I}_{k_M} : C_\sigma(\mathbb{R}^M;W) \to L^2_\rho(\mathbb{R}^M;W),$$

where σ and ρ are according to (16). It can be shown by an obvious generalization of [1, Lemma 4.2] that

$$\|\mathcal{I}_{k_1} \otimes \cdots \otimes \mathcal{I}_{k_M} v\|_{L^2_\rho(\mathbb{R}^M;W)} \leq C(\rho,\sigma)\, \|v\|_{C_\sigma(\mathbb{R}^M;W)},$$

where the constant $C(\rho,\sigma)$ is independent of $\mathbf{k} = (k_1,\ldots,k_M)$. In the following, let $\mathcal{P}_n$ denote the space of all univariate polynomials up to degree n. We state

Lemma 4 (cf. [1, Lemma 4.3]). *For every function $v \in C_\sigma(\mathbb{R}^M;W)$ there holds*

$$\|v - \Delta_{\mathbf{k}} v\|_{L^2_\rho(\mathbb{R}^M;W)} \leq C^M \inf_{w \in \mathcal{P}_{\mathbf{n_k}-1} \otimes W} \|v - w\|_{C_\sigma(\mathbb{R}^M;W)}$$

where $\Delta_{\mathbf{k}} = \Delta_{k_1} \otimes \cdots \otimes \Delta_{k_M}$ and $\mathcal{P}_{\mathbf{n_k}-1} = \mathcal{P}_{n_{k_1}-1} \otimes \cdots \otimes \mathcal{P}_{n_{k_M}-1}$.

In particular, there holds

$$\left\| v - \bigotimes_{m=1}^{M-1} \Delta_{k_m} \otimes (I - \mathcal{I}_{k_M}) v \right\|_{L^2_\rho(\mathbb{R}^M;W)} \leq C^M \inf_{w \in \mathcal{P}_{\mathbf{n_k}-1} \otimes W} \|v - w\|_{C_\sigma(\mathbb{R}^M;W)}.$$

Proof. We consider a separable function $v(\boldsymbol{\xi}) = v_1(\xi_1)\cdots v_M(\xi_M) \in C_\sigma(\mathbb{R}^M;W)$. Note that the set of separable functions is dense in $C_\sigma(\mathbb{R}^M;W)$. Further, let $w \in \mathcal{P}_{\mathbf{n_{k-1}}} \otimes W$ be arbitrary. There holds $\mathcal{I}_{\mathbf{k}-1} w = w$ and

$$
\begin{aligned}
\|v-\Delta_{\mathbf{k}}v\|^2_{L^2_\rho(\mathbb{R}^M;W)} &= \prod_{m=1}^{M} \|v_m-\Delta_{k_m}v_m\|^2_{L^2_{\rho_m}(\mathbb{R};W)} \\
&\leq \prod_{m=1}^{M} 2\Big(\|v_m-\mathcal{I}_{k_m}v_m\|^2_{L^2_{\rho_m}(\mathbb{R};W)} + \|v_m-\mathcal{I}_{k_m-1}v_m\|^2_{L^2_{\rho_m}(\mathbb{R};W)}\Big) \\
&\leq \prod_{m=1}^{M} 4\Big(\|v_m-w_m\|^2_{L^2_{\rho_m}(\mathbb{R};W)} + \|\mathcal{I}_{k_m}(v_m-w_m)\|^2_{L^2_{\rho_m}(\mathbb{R};W)} \\
&\qquad + \|v_m-w_m\|^2_{L^2_{\rho_m}(\mathbb{R};W)} + \|\mathcal{I}_{k_m-1}(v_m-w_m)\|^2_{L^2_{\rho_m}(\mathbb{R};W)}\Big) \\
&\leq 4^M \prod_{m=1}^{M} C^2 \|v_m-w_m\|^2_{C_{\sigma_m}(\mathbb{R};W)} \\
&= C^M \|v-w\|^2_{C_\sigma(\mathbb{R}^M;W)}.
\end{aligned}
$$

The statement follows by density. □

Lemma 5 ([16]). *Let $v(\zeta)$ be an analytic function in the strip $\Sigma_\tau = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < \tau+\varepsilon\}$, $\varepsilon > 0$. A necessary and sufficient condition that the Fourier-Hermite series*

$$
v(\zeta) = \sum_{n=0}^{\infty} v_n h_n(\zeta), \quad v_n = \int_{\mathbb{R}} v(\xi) h_n(\xi)\,\mathrm{d}\xi,
$$

where $h_n(\xi) = e^{-\xi^2/2} H_n(\xi)$ and $H_n(\xi) = \frac{(-1)^n}{\sqrt{\pi^{1/2} 2^n n!}} e^{\xi^2} \partial^n (e^{-\xi^2})$, converge, is that for every $\beta \in [0, \tau+\varepsilon)$ there exists $C(\beta)$ such that

$$
|v(\xi+i\eta)| \leq C(\beta) e^{-|\xi|\sqrt{\beta^2-\eta^2}}, \qquad y \in \mathbb{R},\ |\eta| \leq \beta.
$$

In this case the Fourier coefficients satisfy

$$
v_n \leq C e^{-\tau\sqrt{2n+1}}.
$$

Following the proofs in [15, 16], it is clear that if a multivariate function $v : \mathbb{R}^M \to W$ admits an analytic extension to the domain $\Sigma_{\boldsymbol{\tau}} = \{\boldsymbol{\zeta} \in \mathbb{C}^M : |\operatorname{Im}\zeta_m| < \tau_m + \varepsilon, m = 1,\dots,M\}$, $\varepsilon > 0$, and satisfies

$$
|v(\xi_1+i\eta_1,\dots,\xi_M+i\eta_M)| \leq C(\beta_1,\dots,\beta_M) e^{-\sum_{m=1}^{M}|\xi_m|\sqrt{\beta_m^2-\eta_m^2}}, \qquad \xi_m \in \mathbb{R},\ |\eta_m| \leq \beta_m,\ \forall m,
$$

for all $\beta_m \in [0, \tau_m]$, $m = 1,\dots,M$, then we have

$$
v(\boldsymbol{\zeta}) = \sum_{\mathbf{n}} v_{\mathbf{n}} \prod_{m=1}^{M} h_{n_m}(\zeta_m), \quad v_{\mathbf{n}} = \int_{\mathbb{R}^M} v(\boldsymbol{\xi}) \prod_{m=1}^{M} h_{n_m}(\xi_m)\,\mathrm{d}\boldsymbol{\xi},
$$

for all $\boldsymbol{\zeta} \in \Sigma_{\boldsymbol{\tau}}$, and, in particular,

$$v_{\mathbf{n}} \leq C \exp\left(-\sum_{m=1}^{M} \tau_m \sqrt{2n_m+1}\right).$$

Thus, we can generalize [1, Lemma 4.6] by an obvious modification to

Lemma 6 (cf. [1, Lemma 4.6]). *Let $v : \mathbb{R}^M \to W$ admit an analytic extension to*

$$\Sigma_{\boldsymbol{\tau}} = \{z \in \mathbb{C}^M : |\operatorname{Im} \zeta_m| < \tau_m + \varepsilon, m = 1, \dots, M\},$$

$\varepsilon > 0$*, and satisfy*

$$\max_{\boldsymbol{\zeta} \in \Sigma_{\boldsymbol{\tau}}} \sigma(\operatorname{Re} \boldsymbol{\zeta}) \|v(\boldsymbol{\zeta})\|_W \leq +\infty.$$

Then there holds

$$\min_{w \in \mathcal{P}_{\mathbf{n}}} \max_{\boldsymbol{\xi} \in \mathbb{R}^M} \left| \|v(\boldsymbol{\xi}) - w(\boldsymbol{\xi})\|_W e^{-\|\boldsymbol{\xi}\|^2/8} \right| \leq C\Theta(\mathbf{n}) \exp\left(-\frac{1}{\sqrt{2}} \sum_{m=1}^{M} \tau_m \sqrt{n_m}\right),$$

where $\Theta(\mathbf{n}) = C(\boldsymbol{\tau})(n_1 \cdots n_M)^{1/2}$.

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