The coupling method and operator relations

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Outline

- Application of the Coupling Method
- Formalization of the Coupling Method

Three Banach space operator relations: MC, EAE and SC

- Question 1: Do MC, EAE and SC coincide
- Question 2: When are two operators MC/EAE/SC?

Integral operators with semi-separable kernel

Define

$$K : L^2_n[0,\tau] \to L^2_n[0,\tau], \quad (Kf)(t) = \int_0^\tau k(t,s)f(s)\,ds, \quad (f \in L^2_n[0,\tau]).$$

Here

$$k(s,t) = \begin{cases} C(t)(I-P)B(s), & s < t; \\ -C(t)PB(s), & s > t, \end{cases}$$

with $P \in \mathrm{Mat}^{n\times n}_{\mathbb{C}}$ a projection and $C, B \in L^2_{n\times n}[0,\tau]$.
Then K is Hilbert-Schmidt, so $I - K$ is Fredholm.

Integral equation: Given $g \in L^2_n[0,\tau]$, find $f \in L^2_n[0,\tau]$ with

$$g = (I-K)f, \quad i.e., \quad g(t) = f(t) - \int_0^\tau k(t,s)f(s)\,ds.$$

Integral operators with semi-separable kernel

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Associated system

With B and C we associate the differential equation:

$$\dot{x}(t) = B(t)C(t)x(t) \quad (t \in [0,\tau]).$$

Write $U : [0,\tau] \to \mathrm{Mat}_{\mathbb{C}}^{n\times n}$ for the associated fundamental matrix.

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Bart-Gohberg-Kaashoek '84

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$$H : L_n^2[0,\tau] \to L_n^2[0,\tau], (Hf)(t) = \int_0^\tau C(t)B(s)f(s)\,ds;$$

$$Q : L_n^2[0,\tau] \to \operatorname{Im} P, Qf = P\int_0^\tau B(s)f(s)\,ds$$

$$R : \operatorname{Im} P \to L_n^2[0,\tau], (Qx)(t) = C(t)Px.$$

Then $I - H$ is invertible and

$$\begin{bmatrix} I-K & -R \\ -Q & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-H)^{-1} & (I-H)^{-1}R \\ Q(I-H)^{-1} & S_\tau \end{bmatrix}. \qquad \text{(MC)}$$

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Moreover, there exist invertible operators E and F such that

$$\begin{bmatrix} I-K & 0 \\ 0 & I_{\operatorname{Im} P} \end{bmatrix} = E \begin{bmatrix} S_\tau & 0 \\ 0 & I_{L^2_n[0,\tau]} \end{bmatrix} F. \qquad \text{(EAE)}$$

The Schur complements of $\left[\begin{smallmatrix} I & -R \\ Q & I-H \end{smallmatrix}\right]$ are given by

$$I - K = (I-H) + RQ \quad \text{and} \quad S_\tau = I + Q(I-H)^{-1}R. \qquad \text{(SC)}$$

Fredholm properties

The identity

$$\begin{bmatrix} I-K & 0 \\ 0 & I_{\operatorname{Im} P} \end{bmatrix} = E \begin{bmatrix} S_\tau & 0 \\ 0 & I_{L_n^2[0,\tau]} \end{bmatrix} F$$

with E and F invertible yields:

$I - K$ (on $L_n^2[0,\tau]$) and S_τ (on $\operatorname{Im} P$) have the same 'Fredholm properties.'

And one can show:

$$\operatorname{Ker}(I-K) = (I-H)^{-1} R \operatorname{Ker} S_\tau \quad \text{and} \quad \operatorname{Im}(I-K) = \{f : Q(I-H)^{-1} f \in \operatorname{Im} S_\tau\}.$$

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Generalized inverse

Expressing the Moore-Penrose generalized inverse of $\left[\begin{smallmatrix} I & -R \\ Q & I-H \end{smallmatrix}\right]$ in terms of its Schur complements one can compute the MP generalized inverse of $I-K$:

$$(I+K)^+ = (I-H)^{-1} - (I-H)^{-1} R S_\tau^+ Q (I-H)^{-1},$$

and solve the integal equation:

$$f = (I+K)^+ g, \quad \text{if } g \in \operatorname{Im}(I-K).$$

The Coupling Method: Formalization

Two Banach space operators $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ are called *matricially coupled* (MC), *equivalent after extension* (EAE) resp. *Schur coupled* (SC) if:

(MC) There exist an invertible operator $\widehat{U} : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ such that

$$\widehat{U} = \begin{bmatrix} U & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad \text{and} \quad \widehat{U}^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V \end{bmatrix}.$$

(EAE) There exist Banach spaces $\mathcal{X}_0$ and $\mathcal{Y}_0$ and invertible operators E and F s.t.

$$\begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{X}_0} \end{bmatrix} = E \begin{bmatrix} V & 0 \\ 0 & I_{\mathcal{Y}_0} \end{bmatrix} F.$$

(SC) There exists an operator matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with A and D invertible and

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In the example

$I - K$ and S_τ are MC $\Rightarrow$ $I - K$ and S_τ are EAE $\Rightarrow$ $I - K$ and S_τ are SC

EAE $\Downarrow$ Fredholm properties

SC $\Downarrow$ generalized inverse

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More recent applications

- Diffraction theory (Castro, Duduchava, Speck, e.g., 2014)
- Truncated Toeplitz operators (Câmara, Partington, 2016)
- Connection with Paired Operators approach (Speck, 2017)
- Wiener-Hopf factorization (Groenewald, Kaashoek, Ran, 2017)

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More recent applications

- Completeness theorems in dynamical systems (Kaashoek, Verduyn Lunel)
- Unbounded operator functions (Engström, Torshage, Arxiv)

Do MC, EAE and SC coincide?

Question (Bart-Tsekanovskii '92)

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$$\text{MC} \iff \text{EAE} \impliedby \text{SC}$$

Early results

- Bart-Gohberg-Kaashoek '84: MC $\Rightarrow$ EAE
- Bart-Tsekanovskii '92: EAE $\Rightarrow$ MC (so EAE $\Leftrightarrow$ MC)
- Bart-Tsekanovskii '94: SC $\Rightarrow$ EAE

Proof MC $\Longrightarrow$ EAE

$\left[\begin{smallmatrix} U & 0 \\ 0 & I_{\mathcal{Y}} \end{smallmatrix}\right] = E \left[\begin{smallmatrix} V & 0 \\ 0 & I_{\mathcal{X}} \end{smallmatrix}\right] F$ holds with $E = \left[\begin{smallmatrix} U_{12} & U \\ U_{22} & U_{21} \end{smallmatrix}\right]$ and $F = \left[\begin{smallmatrix} -U_{21} & I_{\mathcal{Y}} \\ V_{11}U & V_{12} \end{smallmatrix}\right]$ and

$$E^{-1} = \begin{bmatrix} V_{21} & V \\ V_{11} & V_{12} \end{bmatrix}, \quad F^{-1} = \begin{bmatrix} -V_{12} & I \\ U_{22}V & U_{21} \end{bmatrix}.$$

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  - BT'92: Yes if U and V are Fredholm (Banach space: + index = 0)
  - BGKR'05: Yes if SC is an equivalence relation (this is true for EAE)

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$$\begin{array}{ccccc} & & & & \text{EAOE} \\ & & & & \Downarrow \\ \text{MC} & \Longleftrightarrow & \text{EAE} & \Longleftarrow & \text{SC} \\ & & & & \Updownarrow \\ & & & & \text{SEAE} \end{array}$$

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  - BT'92: Yes if U and V are SEAE (SEAE $\Leftrightarrow$ SC)
    (SEAE = Strong EAE = EAE with E_{21} and F_{12} invertible)
  - BGKR'05: Yes if U and V are EAOE (EAOE $\Rightarrow$ SC)
    (EAOE= EAE with $\mathcal{X}_0 = \{0\}$ or $\mathcal{Y}_0 = \{0\}$ (one-sided extension))

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Approximation by invertibles

Theorem (tH-Ran '13) *Let U and V be EAE operators that can be approx. by invertible operators (in norm). Then U and V are SEAE, and hence SC.*

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Proof *Go for SEAE (F_{12} and E_{21} invertible).*
By concrete formulas for EAE $\Rightarrow$ MC $\Rightarrow$ EAE, WLOG

$$E = \begin{bmatrix} E_{11} & U \\ E_{21} & E_{22} \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} \widetilde{E}_{11} & V \\ \widetilde{E}_{21} & \widetilde{E}_{22} \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & I_{\mathcal{Y}} \\ F_{21} & F_{22} \end{bmatrix}, \quad F^{-1} = \begin{bmatrix} \widetilde{F}_{11} & I_{\mathcal{X}} \\ \widetilde{F}_{21} & \widetilde{F}_{22} \end{bmatrix}.$$

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In particular, $E_{21}V + E_{22}\widetilde{E}_{22} = I$. Take an invertible $\widetilde{V}$ close to V s.t.

$$N := E_{21}\widetilde{V} + E_{22}\widetilde{E}_{22} \text{ is invertible.}$$

Then also

$$E_{21} - E_{22}\widetilde{E}_{22}\widetilde{V}^{-1} = N\widetilde{V}^{-1} \text{ is invertible.}$$

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Then note that $\left[\begin{smallmatrix} U & 0 \\ 0 & I \end{smallmatrix}\right] = \widehat{E}\left[\begin{smallmatrix} V & 0 \\ 0 & I \end{smallmatrix}\right]\widehat{F}$ holds with

$$\widehat{E} = E \begin{bmatrix} I & 0 \\ \widetilde{E}_{22}\widetilde{V}^{-1} & I \end{bmatrix} = \begin{bmatrix} * & * \\ E_{21} - E_{22}\widetilde{E}_{22}\widetilde{V}^{-1} & * \end{bmatrix},$$

$$\widehat{F} = \begin{bmatrix} I & 0 \\ -\widetilde{E}_{22}\widetilde{V}^{-1}V & I \end{bmatrix} F = \begin{bmatrix} * & I \\ * & * \end{bmatrix}.$$

Thus U and V are SEAE. □

Question Which operators can be approximated by invertibles?

EAE and SC on separable Hilbert spaces

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Banach space operators: Not much seems to be known.
Hilbert space operators:

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Theorem (Feldman-Kadison '54) *Let $W : \mathcal{Z} \to \mathcal{Z}$, with $\mathcal{Z}$ a separable Hilbert space. Then W cannot be approximated by invertible operators if and only if W has closed range and* $\dim \operatorname{Ker} W \neq \dim \operatorname{Ker} W^*$.

Thus a separable Hilbert space operator can be approximated by invertibles or has closed range. (Not true on non-separable Hilbert spaces.)

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Theorem (tH-Ran '13) *Let U and V be closed range Hilbert space operators. Then U and V are EAE if and only if U and V are SC if and only if*

$$\dim \operatorname{Ker} U = \dim \operatorname{Ker} V \text{ and } \dim \operatorname{Ker} U^* = \dim \operatorname{Ker} V^*.$$

Theorem (tH-Ran '13) *Assume U and V are EAE operators on separable Hilbert spaces. Then U and V are SEAE, and hence SC.*

When are operators EAE?

Question When are operators U and V EAE?

Known: Assume U and V are closed range Hilbert space operators. Then:

$$U \text{ and } V \text{ are EAE} \quad \iff \quad \dim \operatorname{Ker} U = \dim \operatorname{Ker} V \text{ and } \dim \operatorname{Ker} U^* = \dim \operatorname{Ker} V^*.$$

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U and V are EAE $\iff$ $\dim \operatorname{Ker} U = \dim \operatorname{Ker} V$ and $\dim \operatorname{Ker} U^* = \dim \operatorname{Ker} V^*$.

Definition (generated operator ideal) For a Banach space operator $U : \mathcal{X} \to \mathcal{X}$ and Banach spaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$ we define

$$\mathfrak{I}_U(\mathcal{Z}_1, \mathcal{Z}_2) := \left\{ \sum_{j=1}^{n} R_j U R_j' : R_j : \mathcal{X} \to \mathcal{Z}_2,\ R_j' : \mathcal{Z}_1 \to \mathcal{X},\ n \in \mathbb{N} \right\}$$

and the operator ideal generated by U: $\mathfrak{I}_U = \bigcup_{\mathcal{Z}_1, \mathcal{Z}_2} \mathfrak{I}_U(\mathcal{Z}_1, \mathcal{Z}_2)$.

Theorem (tH-Messerschmidt-Ran '15) *Let $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ be compact Banach space operators that are EAE. Then $\mathfrak{I}_U = \mathfrak{I}_V$.*

Timotin's approach to the general Hilbert space case, Pt I

Let $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ be Hilbert space operators. Define

$$|U| = (U^* U)^{1/2} \quad \text{and} \quad |V| = (V^* V)^{1/2}.$$

Theorem (Timotin '14) *The operators U and V are EAE if and only if $|U|$ and $|V|$ are EAE and*

$$\dim \ker U = \dim \ker V, \quad \dim \ker U^* = \dim \ker V^*. \tag{$*$}$$

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$$\dim\ker U = \dim\ker V, \quad \dim\ker U^* = \dim\ker V^*. \qquad (*)$$

For any interval $I \subset \mathbb{R}$, let $E_{|U|}[I]$ and $E_{|V|}[I]$ be the spectral projections of $|U|$ and $|V|$ on I.

Theorem (Fillmore-Williams '71) *The operators U and V are equivalent if and only if $(*)$ holds and there is a $\delta > 0$ so that for all $0 < \alpha \leq \beta < \infty$ we have*

$$\dim\operatorname{ran} E_{|U|}([\alpha, \beta)) \leq \dim\operatorname{ran} E_{|V|}([\alpha\delta, \beta/\delta))$$
$$\dim\operatorname{ran} E_{|V|}([\alpha, \beta)) \leq \dim\operatorname{ran} E_{|U|}([\alpha\delta, \beta/\delta)).$$

Timotin's approach to the general Hilbert space case, Pt II

Theorem (Timotin '14) *For Hilbert space operators U and V TFAE:*

- *U and V are EAE;*
- *U and V satisfy*

$$\dim\ker U = \dim\ker V, \quad \dim\ker U^* = \dim\ker V^*. \qquad (*)$$

and there exist $0 < \delta < 1$ and $\mathbf{a} > \mathbf{0}$ such that for al $0 < \alpha \le \beta < \mathbf{a}$

$$\dim\operatorname{ran} E_{|U|}([\alpha,\beta)) \le \dim\operatorname{ran} E_{|V|}([\alpha\delta,\beta/\delta))$$
$$\dim\operatorname{ran} E_{|V|}([\alpha,\beta)) \le \dim\operatorname{ran} E_{|U|}([\alpha\delta,\beta/\delta)).$$

- *U and V are EAOE, and hence SC.*
  (Recall: EAOE= EAE with $\mathcal{X}_0 = \{0\}$ or $\mathcal{Y}_0 = \{0\}$ (one-sided extension))

Timotin's approach to the general Hilbert space case, Pt II

Theorem (Timotin '14) *For Hilbert space operators U and V TFAE:*

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$$\dim \operatorname{ran} E_{|U|}([\alpha, \beta]) \le \dim \operatorname{ran} E_{|V|}([\alpha\delta, \beta/\delta])$$
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This result specializes to compact operators in the following form.

Theorem (Timotin '14) *Assume U and V are compact (and of infinite rank) with singular values $u_n \searrow 0$ and $v_n \searrow 0$, respectively. Then U and V are EAE=MC=SC if and only if $(*)$ holds and their singular values are* comparable after a shift*: There exist $0 < \delta < 1$ and $m \in \mathbb{N}$ such that*

$$\delta \le \frac{u_n}{v_{n+m}} \le \frac{1}{\delta} \text{ for all } n \ge 0 \quad \text{or} \quad \delta \le \frac{v_n}{u_{n+m}} \le \frac{1}{\delta} \text{ for all } n \ge 0.$$

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Which shows U and V EAE is much stronger than generating the same operator ideal.

Theorem (Schatten '60) *Let U and V be compact Hilbert space operators with with singular values $u_n \searrow 0$ and $v_n \searrow 0$. Then $\mathfrak{I}_U = \mathfrak{I}_V$ if and only if there exist $M > 0$ and $m \in \mathbb{N}$ so that*

$$u_{m(n-1)+j} \leq M v_n \quad \text{and} \quad v_{m(n-1)+j} \leq M u_n \quad (n \in \mathbb{N}, j = 1, \ldots, m-1).$$

Definition Banach spaces $\mathcal{X}$ a $\mathcal{Y}$ are called *essentially incomparable* if for any operators $T : \mathcal{X} \to \mathcal{Y}$ and $S : \mathcal{Y} \to \mathcal{X}$

$$I - TS \quad \text{and} \quad I - ST \quad \text{are Fredholm.}$$

Pitt-Rosenthal Theorem

Any operator $T : \ell^p \to \ell^q$, for $1 \leq q < p < \infty$, is compact. (Hence ℓ^p and ℓ^q are essentially incomparable.)

Essentially incomparable Banach spaces and EAE

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Theorem (tH-Messerschmidt-Ran '15) *Assume $\mathcal{X}$ and $\mathcal{Y}$ are infinite dimensional, essentially incomparable Banach spaces. Then operators $U:\mathcal{X}\to\mathcal{X}$ and $V:\mathcal{Y}\to\mathcal{Y}$ with U or V compact cannot be EAE.*

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Theorem (tH-Messerschmidt-Ran '15) *Assume $\mathcal{X}$ and $\mathcal{Y}$ are infinite dimensional, essentially incomparable Banach spaces. Then no operators $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ are ever EAOE.*

Corollary *In general the operator relation EAOE cannot coincide with and SC/EAE/MC.*

Implications of EAE + compact

Proposition (tH-Messerschmidt-Ran '15) *Let $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ be EAE with V compact. Then there exists a closed subspace of $\mathcal{Y}$ of finite co-dimension that is topologically isomorphic to a closed subspace of $\mathcal{X}$.*

Corollary (tH-Messerschmidt-Ran '15) *Let $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ be EAE with V compact. Assume $\mathcal{Y}$ is prime, i.e., every infinite dimensional complementable subspace of $\mathcal{Y}$ is topologically isomorphic to $\mathcal{Y}$. Then $\mathcal{X}$ contains a copy of $\mathcal{Y}$.*

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Corollary (tH-Messerschmidt-Ran '15) *Let $U : \mathcal{X} \to \mathcal{X}$ and $V : \mathcal{Y} \to \mathcal{Y}$ be EAE with V compact. Then:*

- *If $\mathcal{X}$ is isomorphic to a Hilbert space, then so is $\mathcal{Y}$;*
- *If $\mathcal{X}$ is separable, then so is $\mathcal{Y}$;*
- *If $\mathcal{X}$ is reflexive, then so is $\mathcal{Y}$;*
- *If $\mathcal{X}$ has the Radon-Nikodym property, then so does $\mathcal{Y}$;*
- *If $\mathcal{X}$ has the Hereditary Dunford-Pettis property, then so does $\mathcal{Y}$.*

EAE $\Rightarrow$ SC for compact operators

Theorem (tH-Messerschmidt-Ran-Roelands-Wortel '15) *Let $U \in \mathcal{B}(\mathcal{X})$ and $V \in \mathcal{B}(\mathcal{Y})$ be compact. Then*

$$U \text{ and } V \text{ are EAE} \iff U \text{ and } V \text{ are EAOE} \iff U \text{ and } V \text{ are SC.}$$

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Sketch of proof

By the constructions from EAE $\Rightarrow$ MC $\Rightarrow$ EAE, WLOG $\mathcal{X}_0 = \mathcal{Y}$, $\mathcal{Y}_0 = \mathcal{X}$ and E and F have the form

$$F = \begin{bmatrix} F_{11} & I_{\mathcal{Y}} \\ F_{21} & F_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_{11} & U \\ E_{21} & -F_{11} \end{bmatrix}$$

$$F^{-1} = \begin{bmatrix} -F_{22} & I_{\mathcal{X}} \\ I + F_{11}F_{22} & -F_{11} \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} \widehat{E}_{11} & V \\ \widehat{E}_{21} & F_{22} \end{bmatrix}.$$

Then $U \otimes I_{\mathcal{Y}} = E(V \otimes I_{\mathcal{X}})F$ yields

(i) $I = F_{21} - F_{22}F_{11}$, (ii) $U = E_{11}VF_{11} + UF_{21}$, (iii) $E_{21}VF_{11} = F_{11}F_{21}$,

(iv) $E_{11}V = -UF_{22}$, (v) $F_{11}F_{22} = E_{21}V - I$, (vi) $\widehat{E}_{11}U = VF_{11}$,

(vii) $\widehat{E}_{21}U = F_{21}$, $(viii)$ $E_{11}\widehat{E}_{11} = I - U\widehat{E}_{21}$, (ix) $E_{21}\widehat{E}_{11} = F_{11}\widehat{E}_{21}$,

(x) $\widehat{E}_{11}E_{11} = I - VE_{21}$, (xi) $\widehat{E}_{21}E_{11} = -F_{22}E_{21}$.

Sketch of proof II

Since U and V are compact

$$-F_{22}F_{11} = I - \widehat{E}_{21}U \quad \text{and} \quad -F_{11}F_{22} = I - E_{21}V \quad \text{are Fredholm.}$$

Atkinson's Theorem: F_{11} and F_{22} are Fredholm and $\mathrm{Ind}(F_{11}) = -\mathrm{Ind}(F_{22})$.
Similar argument: E_{11} and $\widehat{E}_{11}$ are Fredholm and $\mathrm{Ind}(E_{11}) = -\mathrm{Ind}(\widehat{E}_{11})$.
We can decompose

$$F_{22} = \begin{bmatrix} F_{22}' & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \mathrm{Ker}\, F_{22} \end{bmatrix} \to \begin{bmatrix} \mathrm{Im}\, F_{22} \\ \mathcal{H}_2 \end{bmatrix},$$

$$E_{11} = \begin{bmatrix} E_{11}' & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{F}_1 \\ \mathrm{Ker}\, E_{11} \end{bmatrix} \to \begin{bmatrix} \mathrm{Im}\, E_{11} \\ \mathcal{G}_1 \end{bmatrix},$$

with F_{22}' and E_{11}' invertible, and decompose U and V accordingly:

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} : \begin{bmatrix} \mathrm{Im}\, F_{22} \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{Im}\, E_{11} \\ \mathcal{G}_1 \end{bmatrix},$$

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \mathrm{Ker}\, F_{22} \end{bmatrix} \to \begin{bmatrix} \mathcal{F}_1 \\ \mathrm{Ker}\, E_{11} \end{bmatrix}.$$

Use further identities:

$$U_{21} = 0,\ V_{12} = 0 \text{ and } U_{11} \text{ and } V_{11} \text{ are equivalent: } U_{11}F_{22}' = -E_{11}'V_{11}.$$

Sketch of proof III

We have U_{11} and V_{11} equivalent ($U_{11}F'_{22} = -E'_{11}V_{11}$) and

$$U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} : \begin{bmatrix} \text{Im } F_{22} \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \text{Im } E_{11} \\ \mathcal{G}_1 \end{bmatrix},$$
$$V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \text{Ker } F_{22} \end{bmatrix} \to \begin{bmatrix} \mathcal{F}_1 \\ \text{Ker } E_{11} \end{bmatrix}.$$

After many more manipulations of the identities: U_{22} and V_{22} are invertible. Then U and V are equivalent to

$$\widetilde{U} := \begin{bmatrix} U_{11} & 0 \\ 0 & I_{\mathcal{H}_2} \end{bmatrix} \quad \text{and} \quad \widetilde{V} := \begin{bmatrix} V_{11} & 0 \\ 0 & I_{\text{Ker } E_{11}} \end{bmatrix}$$

and $\mathcal{H}_2$ and Ker E_{11} are finite dimensional. Say $\dim \mathcal{H}_2 < \dim \text{Ker } E_{11}$. Let $T : \mathcal{H}_2 \to \text{Ker } E_{11}$ be injective and $\mathcal{Z}'$ a complement of $\mathcal{Z} := J\mathcal{H}_2$ in Ker E_{11}. Then $\widetilde{U}$ and $\widetilde{V}$ are EAOE via

$$\begin{bmatrix} U_{11} & 0 & 0 \\ 0 & I_{\mathcal{H}_2} & 0 \\ 0 & 0 & I_{\mathcal{Z}'} \end{bmatrix} = \begin{bmatrix} E'_{11} & 0 & 0 \\ 0 & T^+T_{\mathcal{Z}} & 0 \\ 0 & 0 & I_{\mathcal{Z}'} \end{bmatrix} \begin{bmatrix} V_{11} & 0 & 0 \\ 0 & I_{\mathcal{Z}} & 0 \\ 0 & 0 & I_{\mathcal{Z}'} \end{bmatrix} \begin{bmatrix} -F_{22}'^{-1} & 0 & 0 \\ 0 & \Pi_{\mathcal{Z}}T & 0 \\ 0 & 0 & I_{\mathcal{Z}'} \end{bmatrix}$$

with T^+ a left inverse of T, $J_{\mathcal{Z}} : \mathcal{Z} \to \text{Ker } E_{11}$ and $\Pi_{\mathcal{Z}} : \text{Ker } E_{11} \to \mathcal{Z}$ the canonical embedding and projection. Then U and V are also EAOE, and hence SC.

Beyond compact operators

Observation: The arguments involving compact operators only use that the invertible elements in the Calkin algebra of the compacts are the Fredholm operator.

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Definition Let $T : \mathcal{X} \to \mathcal{Y}$ be a Banach space operator. We we call T:

- *inessential* if $I_{\mathcal{Y}} - TS$ is Fredholm for any $S : \mathcal{Y} \to \mathcal{X}$ (equiv. $I_{\mathcal{X}} - ST$ is Fredholm for any $S : \mathcal{Y} \to \mathcal{X}$) (Kleinecke, 1963).
- *strictly singular* if for no infinite dimensional, closed, complementable subspace $\mathcal{M}$ of $\mathcal{X}$ the operator $T|_{\mathcal{M}} : \mathcal{M} \to \mathcal{Y}$ is an isomorphism.
- *strictly co-singular* if for no infinite codimensional, closed, complementable subspace $\mathcal{N}$ of $\mathcal{Y}$ the operator $P_{\mathcal{N}} T : \mathcal{X} \to \mathcal{N}$ is surjective.

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Then for operators $T : \mathcal{X} \to \mathcal{X}$:

$$\{\text{compacts}\} \subset \frac{\{\text{strictly singular}\}}{\{\text{strictly co-singular}\}} \subset \{\text{inessentials}\} \subset \mathcal{B}(\mathcal{X})$$

are all closed operator ideals in $\mathcal{B}(\mathcal{X})$ and the inessential operators $\text{In}(\mathcal{X})$ is the largest closed ideal in $\mathcal{B}(\mathcal{X})$ s.t. in the Calkin algebra $\mathcal{B}(\mathcal{X})/\text{In}(\mathcal{X})$ the Fredholm operators in $\mathcal{B}(\mathcal{X})$ coincide with the invertible operators.

In all results above "compact" can be replaced by "inessential".

Exotic Banach spaces

We now know EAE and SC coincide:

- for Hilbert space operators
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Definition A Banach space $\mathcal{X}$ has

- *few operators* if every operator on $\mathcal{X}$ is of the form $\lambda I_{\mathcal{X}} + S$ with $\lambda \in \mathbb{C}$ and S strictly singular;
- *very few operators* if every operator on $\mathcal{X}$ is of the form $\lambda I_{\mathcal{X}} + K$ with $\lambda \in \mathbb{C}$ and K compact.

In both cases the Calkin algebra is one dimensional.
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Existence of such spaces:

- Few operators: Gowers-Maurey 1997; All hereditarily indecomposable Banach spaces have few operators
- Very few operators: Argyros-Heydon 2011

An application: Multiplication operators

For $f \in L^\infty$ over the unit circle $\mathbb{T}$, define the multiplication operator

$$M_f : L^p \to L^p, \quad (M_f g)(e^{it}) = f(e^{it})g(e^{it})$$

which decomposes w.r.t. the direct sum $L^p = K^p \dot{+} H^p$ as

$$M_f = \begin{bmatrix} \widetilde{T}_f & \widetilde{H}_f \\ H_f & T_f \end{bmatrix} : \begin{bmatrix} K^p \\ H^p \end{bmatrix} \to \begin{bmatrix} K^p \\ H^p \end{bmatrix}$$

with H_f and T_f the Hankel and Toeplitz operators of f and $\widetilde{H}_f$ and $\widetilde{T}_f$ associated with the Hankel and Toeplitz operators of $\widetilde{f}(z) = \overline{f(\overline{z})}$.

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Now assume f is in the Wiener space $\mathcal{W}$ (abs. summable Fourier coeffs.). Then H_f and $\widetilde{H}_f$ are compact and by Wiener's $1/f$ theorem:

$$f(z) \neq 0 \ (z \in \mathbb{T}) \quad \Longleftrightarrow \quad 1/f \in \mathcal{W}.$$

and in that case

$$\begin{bmatrix} \widetilde{T}_{1/f} & \widetilde{H}_{1/f} \\ H_{1/f} & T_{1/f} \end{bmatrix} = M_{1/f} = M_f^{-1} = \begin{bmatrix} \widetilde{T}_f & \widetilde{H}_f \\ H_f & T_f \end{bmatrix}^{-1}.$$

An application: Multiplication operators

Reconfigure $M_{1/f}$ and M_f^{-1} as:

$$\begin{bmatrix} H_{1/f} & T_{1/f} \\ \widetilde{T}_{1/f} & \widetilde{H}_{1/f} \end{bmatrix} = \begin{bmatrix} \widetilde{H}_f & \widetilde{T}_f \\ T_f & H_f \end{bmatrix}^{-1} : \begin{bmatrix} K^p \\ H^p \end{bmatrix} \to \begin{bmatrix} H^p \\ K^p \end{bmatrix}^{-1}.$$

Conclusion: H_f and $H_{1/f}$ are MC, and $\widetilde{H}_f$ and $\widetilde{H}_{1/f}$ are MC.

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Theorem (tH-Messerschmidt-Ran-Roelands-Wortel) *Let $f \in \mathcal{W}$ with $f(z) \neq 0$, $z \in \mathbb{T}$. Then H_f and $H_{1/f}$ are EAE and hence H_f and $H_{1/f}$ generate the same operator ideal. In particular, H_f is in the q-th Schatten-von Neumann class $\mathfrak{C}_q$ if and only if $H_{1/f}$ is in $\mathfrak{C}_q$.*

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Let $\mathbb{P}$ denote the Riesz projection from L^p onto H^p. By Peller's theorem.

Corollary *Let $f \in \mathcal{W}$ with $f(z) \neq 0$, $z \in \mathbb{T}$. Then $\mathbb{P}f$ is in the Besov space $B_q^{1/q}$ if and only if $\mathbb{P}(1/f)$ is in $B_q^{1/q}$.*

Corollary *Let $p = 2$. Let $f \in \mathcal{W}$ with $f(z) \neq 0$, $z \in \mathbb{T}$. Let $\alpha_n \searrow 0$ and $\beta_n \searrow 0$ be the singular values of H_f and $H_{1/f}$. Then there exists a positive integer k and a $c > 0$ such that*

$$c < \frac{\alpha_n}{\beta_{n+k}} < 1/c \quad (n \in \mathbb{N}) \quad or \quad c < \frac{\beta_n}{\alpha_{n+k}} < 1/c \quad (n \in \mathbb{N}). \qquad (*)$$

It is not clear if $(*)$ holds with approx. numbers in case $p \neq 2$.

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- The operator relations MC, EAE and SC play an important role in today's application of operator theory.

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- While in many applications MC, EAE and SC coincide, the implication

$$\text{MC/EAE} \implies \text{SC}$$

remains open in general, but is proved affirmatively for

  - Hilbert space operators
  - Fredholm operators with index 0
  - Inessential operators (and hence compact and strictly singular operators)
  - operators that can be approx. by invertibles

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  remains open in general, but is proved affirmatively for
  - Hilbert space operators
  - Fredholm operators with index 0
  - Inessential operators (and hence compact and strictly singular operators)
  - operators that can be approx. by invertibles
- What does EAE of U and V mean?
  - Full answer for Hilbert space operators in terms of spectral projections
  - For Hilbert space compact operators: singular values comparable after a shift
  - Banach space compact operators: Generate the same ideals
  - Banach space compact operators: Banach space structure cannot be too different

Summary

- The operator relations MC, EAE and SC play an important role in today's application of operator theory.
- While in many applications MC, EAE and SC coincide, the implication

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- Hopefully at a future IWOTA: full proof for EAE $\Rightarrow$ SC, and many more applications.

THANK YOU FOR YOUR ATTENTION