Rank one perturbations of linear relations with applications to DAE's

Carsten Trunk

TU Ilmenau

(together with J. Behrndt, L. Leben F. Martinez Peria, F. Philipp & H. Winkler)

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Cauchy problem

$$\dot{x} = Sx \quad \curvearrowright \quad (\lambda - S)x = 0 \quad \curvearrowright \quad JC$$

$$\text{DAE} \quad E\dot{x} = Ax \quad \curvearrowright \quad (\lambda E - A)x = 0 \quad \curvearrowright \quad JC$$

Questions: What happens with JC under a rank one perturbation?

$$\lambda - S \quad \to \quad \lambda - (S + \Delta S) \qquad \text{(known)}$$

$$\lambda E - A \quad \to \quad \lambda(E + \Delta E) - (A + \Delta A)$$

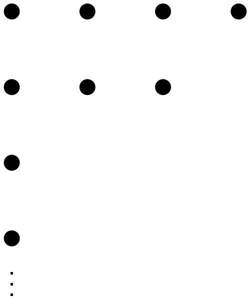
1. Movement of eigenvalues? (in general quite arbitrary)
2. Change of the algebraic eigenspace? (TODAY)

Why are rank one perturbation of DAE interesting?

(collaboration wth Institut für Mikroelektronik- und Mechatronik (IMMS), Ilmenau)

Given $S : \operatorname{dom} S \to X$, $\{x_0, \dots, x_{n-1}\} \subset \operatorname{dom} S$ is Jordan chain of length n at λ if

- $(S - \lambda)x_0 = 0$, and;
- $(S - \lambda)x_i = x_{i-1}$, $i = 1, \dots, n-1$.



$\ker(S - \lambda)$ (Ferrers diagram)

$\ker(S - \lambda)^2 \setminus \ker(S - \lambda)$

$\ker(S - \lambda)^3 \setminus \ker(S - \lambda)^2$

$\ker(S - \lambda)^4 \setminus \ker(S - \lambda)^3$

Note: $\dim \left(\frac{\ker(S-\lambda)^n}{\ker(S-\lambda)^{n-1}} \right)$ is the number of Jordan chains of length $\geq n$.

Recent result

Theorem (J. Behrndt, L. Leben F. Martinez Peria & CT, Lin. Alg. Appl. '15)

Let S and T be linear operators which are rank 1-perturbations and $n \in \mathbb{N}$:

1. *If* $\dim\left(\frac{\ker(S-\lambda)^n}{\ker(S-\lambda)^{n-1}}\right) < \infty$, *then*

$$\left|\dim\left(\frac{\ker(S-\lambda)^n}{\ker(S-\lambda)^{n-1}}\right) - \dim\left(\frac{\ker(T-\lambda)^n}{\ker(T-\lambda)^{n-1}}\right)\right| \leq 1.$$

2. *The above estimates are sharp.*

Remark

The above statement was shown by S. Savchenko '05 for matrices.

Hypothesis

Definition

S, T are rank 1-perturbations (of each other) if ex. $M \subseteq \operatorname{dom} S \cap \operatorname{dom} T$ with

- $Sx = Tx$ for every $x \in M$,
- $\max\{\dim(\operatorname{dom} S/M), \dim(\operatorname{dom} T/M)\} = 1$.

Three typical situations:

1. S, T matrices with $rk(S - T) = 1$.
2. S, T bounded operators with $\dim(\operatorname{ran}(S - T)) = 1$.
3. Exists $\mu_0 \in \rho(S) \cap \rho(T)$ with

$$\dim\big(\operatorname{ran}\big((S - \mu_0)^{-1} - (T - \mu_0)^{-1}\big)\big) = 1.$$

Plan for today

- Generalize to DAE $sE - A$.
- But from now on we restrict to square matrices E, A in X.
- And also for simplicity only for $\lambda = 0$.

Jordan chains for DAE $sE - A$

Definition

$\{x_0, \dots, x_{n-1}\}$ is Jordan chain of length n at 0 if

$$Ax_0 = 0, \quad Ax_1 = Ex_0, \dots, \quad Ax_{n-1} = Ex_{n-2}.$$

Definition

Denote by $\mathcal{A}$ the subspace in $X \times X$:

$$\mathcal{A} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times X : Ax = Ey \right\}$$

We have $\mathcal{A} = E^{-1}A$ if E is invertible or in the sense of *linear relations*.

Define

$$\mathcal{A}^2 := \left\{ \begin{pmatrix} x \\ z \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{A},\ \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{A} \text{ for some } y \right\}.$$

By induction, $\mathcal{A}^k$. Define $\ker \mathcal{A} := \{x : (x\ 0)^\top \in \mathcal{A}\}$.

Proposition

The following two statements are equivalent.

(i) $(x_0, \ldots, x_{n-1})$ *is a Jordan chain of the DAE* $sE - A$ *at* 0.

(ii)
$$\begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}, \begin{pmatrix} x_{n-2} \\ x_{n-3} \end{pmatrix}, \ldots, \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \in \mathcal{A}.$$

(iii) $x_{n-1} \in \ker \mathcal{A}^n$, $x_{n-2} \in \ker \mathcal{A}^{n-1}$, $\ldots$, $x_0 \in \ker \mathcal{A}$.

That is: Jordan chains of the DAE $sE - A$ and the linear relation $\mathcal{A}$ coincide.

Perturbation

Now we perturb $sE - A$. Choose u, v, w from X the (1-dim) pencil:

$$swu^* + wv^*$$

and consider the new (perturbed) DAE

Definition

$$\mathcal{B} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times X : (A + wv^*)x = (E + wu^*)y \right\}$$

It is easy to see: $\max\{\dim(\mathcal{A}/M), \dim(\mathcal{B}/M)\} \leq 1$ for $M := (\mathcal{A} \cap \mathcal{B})$.

Theorem

$\mathcal{A}$ and $\mathcal{B}$ as above.

1. *If* $\dim\left(\frac{\ker \mathcal{A}^n}{\ker \mathcal{A}^{n-1}}\right) < \infty$, *then*

$$\left|\dim\left(\frac{\ker \mathcal{A}^n}{\ker \mathcal{A}^{n-1}}\right) - \dim\left(\frac{\ker \mathcal{B}^n}{\ker \mathcal{B}^{n-1}}\right)\right| \leq n.$$

2. *The above estimates are sharp.*

Thank you!