Desch-Schappacher Perturbations and Extrapolation Spaces for Bi-Continuous Semigroups

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Outline

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- Applications to Feller semigroups generated by (non)-autonomous second-order differential equations with unbounded coefficients in $C_b(\mathbb{R}^n)$

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- $(X_0, \tau)'$ *is norming for* $(X_0, \|\cdot\|)$, *i.e. for all* $x \in X_0$:

$$\|x\| = \sup_{\substack{\varphi\in(X_0,\tau)' \\ \|\varphi\|\le 1}} |\varphi(x)|$$

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$$\mathcal{D}(A) := \left\{ x \in X_0 : \exists\ \tau\text{-}\lim_{t\to 0} \frac{T(t)x - x}{t},\ \sup_{t\in(0,1]} \frac{\|T(t)x - x\|}{t} < \infty \right\}$$

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- *$(A, \mathcal{D}(A))$ is bi-closed, i.e for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(A)$ such that $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$, with $(x_n)_{n\in\mathbb{N}}$ and $(Ax_n)_{n\in\mathbb{N}}$ $\|\cdot\|$-bounded, we have $x \in \mathcal{D}(A)$ and $Ax = y$.*

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- *$\mathcal{D}(A)$ is bi-dense in X_0, i.e. for each $x \in X_0$ there exists a $\|\cdot\|$-bounded sequence in $\mathcal{D}(A)$ which τ-converges to x.*

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Example

For $(T(t))_{t\geq 0}$ the left translation semigroup on $X_0 := C_b(\mathbb{R})$ with generator $Af := f'$ on $\mathcal{D}(A) := \{f \in C_b(\mathbb{R}) : \; f' \in C_b(\mathbb{R})\}$ we obtain $\underline{X}_0 := \mathrm{BUC}(\mathbb{R})$.

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Extrapolation Spaces

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- Norm on X_{-1} comes from $\underline{X}_{-2}$, i.e. $\|x\|_{-1} = \|A_{-2}y\|_{-2}$.
- $\widetilde{p}_i(x) := p_i(A_{-2}^{-1}x)$ $(i \in I)$ forms locally convex topology.

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Theorem

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Application: Desch-Schappacher Perturbation Result for bi-continuous semigroups (Result for locally convex spaces by Jacob, Wegner, Wintermayr (2015))

Outline

Desch-Schappacher Perturbation

Summary for C_0-semigroups

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup

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- $\mathcal{S}_{t_0}^{DS} := \{B \in \mathcal{L}(X, X_{-1}) : \; V_B \in \mathcal{L}(\mathcal{X}_{t_0}), \; \|V_B\| < 1\}$

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Theorem

Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X.

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Theorem

Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X. If $B \in \mathcal{S}_{t_0}^{DS}$ for some $t_0 > 0$, then the operator $(A_{-1} + B)_{|X}$ with domain

$$D\left((A_{-1} + B)_{|X}\right) := \{x \in X : \; A_{-1}x + Bx \in X\}$$

generates a strongly continuous semigroup on X.

Bi-Continuous Desch-Schappacher Perturbation

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$$\mathfrak{X}_{t_0} := \left\{ \begin{array}{c} F : [0, t_0] \to \mathcal{L}(X) : \tau\text{-strongly continuous, norm bounded} \\ \text{and } \{F(t) : t \in [0, t_0]\} \text{ is bi-equicontinuous} \end{array} \right\}$$

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Admissible Operators

Definition

We say that an continuous linear operator $B : (X, \tau) \to (X, \tau_{-1})$ is an element of the set $\mathcal{S}_{t_0}^{DS,\tau}$, if B satisfies all of the following conditions:

- $V_B F(t)x \in X$ for all $t \in [0, t_0]$ and $x \in X$
- The map $t \mapsto V_B F(t)$ is τ-continuous, norm-bounded and bi-equicontinuous (in particular: $V_B : \mathfrak{X}_{t_0} \to \mathfrak{X}_{t_0}$)
- $\|V_B\| < 1$

The Perturbation Result

Theorem

Let $(A, \mathcal{D}(A))$ be a generator of a bi-continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X. If $B \in \mathcal{S}_{t_0}^{DS,\tau}$ for some $t_0 > 0$, then the operator $(A_{-1} + B)_{|X}$ with domain $\mathcal{D}((A_{-1} + B)_{|X}) := \{x \in X : A_{-1}x + Bx \in X\}$ generates a bi-continuous semigroup on X under the assumption that $\mathcal{D}((A_{-1} + B)_{|X})$ is bi-dense.

Admissibility Conditions

For C_0-semigroups the following holds:

Corollary

Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$.

Admissibility Conditions

For C_0-semigroups the following holds:

Corollary

Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$. Moreover, assume that there exists $t_0 > 0$ and $q \in [0, 1)$ such that

- $\int_0^{t_0} T_{-1}(t-r)Bf(r)\, dr \in X$
- $\left\| \int_0^{t_0} T_{-1}(t-r)Bf(r)\, dr \right\| \leq q \left\| f \right\|_\infty$

for all continuous functions $f \in C\left([0, t_0], X\right)$. Then $B \in \mathcal{S}^{DS}_{t_0}$.

Admissibility Conditions

By changing the conditions we can get the following result:

Corollary

Let $(T(t))_{t\geq 0}$ *be a bi-continuous semigroup with generator* $(A, \mathcal{D}(A))$*.* *Let* $B : (X, \tau) \to (X_{-1}, \tau_{-1})$ *be a linear and continuous operator and* $t_0 > 0$ *a number such that:*

- $V_B F(t)x \in X$ *for all* $t \in [0, t_0]$ *and* $x \in X$

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- $V_B F(t)x \in X$ *for all* $t \in [0, t_0]$ *and* $x \in X$
- $\exists M \in \left(0, \frac{1}{2}\right) \ \forall f \in C_b\left([0, t_0], (X, \tau)\right) :$
  $\left\| \int_0^{t_0} T_{-1}(t_0 - r) B f(r) \, dr \right\| \leq M \left\| f \right\|_\infty$

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- $\forall f \in C_b\left([0, t_0], (X, \tau)\right) \ \forall p \in \mathscr{P} \ \exists q \in \mathscr{P} \ \exists K > 0 :$
  $$p\left(\int_0^{t_0} T_{-1}(t_0 - r) B f(r)\, dr \right) \leq K \cdot \sup_{r \in [0, t_0]} \left| q(f(r)) \right|$$

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Then $B \in \mathcal{S}_{t_0}^{DS,\tau}$*.*

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Thank you for listening!