Quantum graph with the Dirac operator and resonance state completeness

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Introduction

Closed resonator: the corresponding operator has purely discrete spectrum, the system of eigenfunctions in complete in L_2 inside the resonator.
Open resonator: the continuous spectrum appears, eigenvalues transform to quasi-eigenvalues (resonances). What about the completeness of the resonance states?

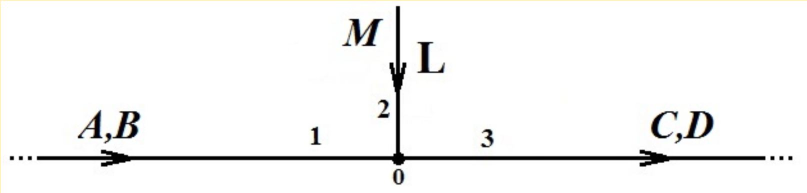

Рис. 1: Line with attached segment

The Schrödinger case (non-relativistic particle):
Popov, I.Y., Popov, A.I. J. King Saud Univ. - Science. 29, 133–136 (2017).
We deal with the Dirac case (relativistic particle)

Dirac operator

We consider the following operator at each edge of the metric graph Γ (E is the set of edges, V is the set of vertices):

$$D = i\hbar c \frac{d}{dx} \otimes \sigma_1 + mc^2 \otimes \sigma_3$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrix. The domain is as follows:

$$\mathcal{D}(D) = \{\psi = \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \psi_1, \psi_2 \in C^1(E), \psi_1 \in C(\Gamma), \quad \sum_j \pm \psi_2^j(v) = \frac{\imath\alpha}{c} \psi_1^j(v)\},$$

where the summation is over all edges including vertex v, sign "plus"is chosen for outgoing edge, sign "minus"for incoming edge, α characterizes the strength of point-like potential at the vertex.

Line with attached segment

The spectral problem reduces to the equation

$$D\psi = \lambda\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The system has the form

$$\begin{pmatrix} mc^2 & -i\hbar c \frac{d}{dx} \\ -i\hbar c \frac{d}{dx} & -mc^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

The system gives us:

$$\psi_2 = -\frac{i\hbar c}{\lambda + mc^2} \cdot \frac{\partial \psi_1}{\partial x}$$

$$-\frac{\hbar^2 c^2}{\lambda + mc^2} \cdot \frac{\partial^2 \psi_1}{\partial x^2} + (mc^2 - \lambda)\psi_1 = 0$$

Line with attached segment

The characteristic equation has the form

$$-\frac{\hbar^2 c^2}{\lambda + mc^2} \cdot k^2 + (mc^2 - \lambda) = 0.$$

Let

$$k_{1,2} = \pm\imath \frac{\sqrt{\lambda^2 - m^2 c^4}}{\hbar c}.$$

Here $k = \frac{\sqrt{\lambda^2 - m^2 c^4}}{\hbar c}$ is a wave number, $k_1 = \imath k$, $k_2 = -\imath k$. Finally, one comes to the solution:

$$\begin{cases} \psi_1 = C_1 e^{ikx} + C_2 e^{-ikx} \\ \psi_2 = \sqrt{\frac{\lambda - mc^2}{\lambda + mc^2}} (C_1 e^{ikx} - C_2 e^{-ikx}) \end{cases} \quad (1)$$

Lax-Phillips approach

Consider the Cauchy problem for the wave equation

$$\begin{cases} u''_{tt} = Hu, \\ u(x,0) = u_0(x), u'_t(x,0) = u_1(x), x \in \Gamma. \end{cases} \tag{2}$$

Let $\mathcal{E}$ be the Hilbert space of two-component functions (u_0, u_1) on the graph Γ with finite energy

$$\|(u_0, u_1)\|^2_{\mathcal{E}} = 2^{-1} \int_\Gamma (|u'_0|^2 + |u_1|^2) dx.$$

The pair (u_0, u_1) is called the Cauchy data. Operator giving the solution for problem (2), $U(t)$, $U(t)(u_0, u_1) = (u(x,t), u'_t(x,t))$, is unitary in $\mathcal{E}$. Unitary group $U(t)|_{t\in\mathbb{R}}$ has two orthogonal (in $\mathcal{E}$) subspaces, D_- and D_+, called, correspondingly, incoming and outgoing subspaces, which are defined as follows.

Definition. Outgoing subspace D_+ is a subspace of $\mathcal{E}$ having the following properties:

(a) $U(t)D_+ \subset D_+$, $t > 0$;

(b) $\underline{\cap_{t>0} U(t) D_+} = \{0\}$,

(c) $\cup_{t<0} U(t) D_+ = \mathcal{E}$.

D_- is defined analogously (with the natural replacement $t > 0 \leftrightarrow t < 0$).

Lax-Phillips approach

Lemma 1. Unitary group $U(t)|_{t\in\mathbb{R}}$ has a pair of subspaces $D_\pm$. Particularly, one can choose $D_\pm$ by the following way:

$$D_+ = \{(u_0, u_1) : -u_1 = u_0', x \in \Omega_L; u_1 = u_0', x \in \Omega_R;$$

$$u_1 = u_0 = 0, x \in \Omega\},$$

$$D_- = \{(u_0, u_1) : u_1 = u_0', x \in \Omega_L; -u_1 = u_0', x \in \Omega_R;$$

$$u_1 = u_0 = 0, x \in \Omega\}.$$

For the proof, one should directly check properties a,b,c (see [**?**]).
Lemma 2. There is a pair of isometric maps $T_\pm : \mathcal{E} \to L_2(\mathbb{R}, \mathbb{C}^2)$ having the following properties:

$$T_\pm U(t) = \exp iktT_\pm, \quad T_+ D_+ = H^2_+(\mathbb{C}^2),$$

$$T_- D_- = H^2_-(\mathbb{C}^2),$$

where $H^2_\pm$ is the Hardy space in upper (lower) half-plane.

It is said that $T_+(T_-)$ gives one the outgoing (incoming) spectral representation of the unitary group $U(t), U(t) = \exp iAt$. Let $K = \mathcal{E} \ominus (D_+ \oplus D_-)$. Consider a semigroup $Z(t) = P_K U(t)|_K, \quad t > 0$, P_K is a projector to K. Let B be the generator of the semigroup $Z(t) : Z(t) = \exp iBt, t > 0$. Data which are eigenvectors of B are called resonance states. Operator $T_- T_+^{-1}$ is called the scattering operator. It acts as a multiplication by a matrix-function $S(k)$ which is the boundary value at the real axis of analytic matrix-function in the upper half-plane k such that $\|S(k)\| \leq 1$ for $\Im k > 0$ and $S^* S = I$ almost everywhere on the real axis. This analytic matrix-function $S(k)$ is called the scattering matrix.

Lemma 3. Map T_- gives one a spectral representation for the unitary group $U(t)$. The following relations take place.

$$T_- D_- = H_-^2(\mathbb{C}^2), \quad T_- D_+ = S H_+^2(\mathbb{C}^2),$$

$$T_- U(t) = \exp{(ikt)} T_-.$$

Matrix-function S is an inner function in $\mathbb{C}_+$ and

$$K_- = T_- K = H_+^2 \ominus S H_+^2, T_- Z(t)|_K = P_{K_-} e^{(ikt)} T_-.$$

Functional model

Sz.-Nagy, B., Foias, C., Bercovici, H., Kerchy, L.: Harmonic Analysis of Operators on Hilbert Space, 2nd edition. Springer, Berlin (2010)
Nikol'skii, N.: Treatise on the shift operator: spectral function theory. Springer Science & Business Media, Berlin (2012).
Khrushchev, S.V., Nikol'skii, N.K., Pavlov, B.S.: Unconditional bases of exponentials and of reproducing kernels, Complex Analysis and Spectral Theory (Leningrad, 1979/1980). Lecture Notes in Math. 864, 214Џ-335 (1981)

As an inner function, S can be represented in the form $S = \Pi\Theta$, where Π is the Blaschke-Potapov product and Θ is a singular inner function. We are interesting in the completeness of the system of resonance states. It is related to the factorization of the scattering matrix.

Theorem 1 (Completeness criterion) [Nikol'skii]. Let S be an inner function, $H^2_+(N) \ominus SH^2_+(N)$, $B = P_K A|_K$. The following statements are equivalent:
1. Operator B is complete;
2. Operator B^* is complete;
3. S is a Blaschke-Potapov product.

Here N is an auxiliary space (in our case it is $\mathbb{C}^2$).

Functional model

There is simple criterion for absence of the singular inner factor for the case $\dim N < \infty$ (for general operator case there is no simple criterion):

Theorem 2 [Nikol'skii]. Let $\dim N < \infty$. The following statements are equivalent:

1. S is a Blaschke-Potapov product;

2.
$$\lim_{r\to 1} \int_{C_r} \ln|\det S(k)| \frac{2\mathrm{i}}{(k+i)^2} dk = 0, \qquad (3)$$

where C_r is an image of $|\zeta| = r$ under the inverse Cayley transform.

The integration curve can be parameterized as $C_r = \{R(r)e^{\mathrm{i}t} + \mathrm{i}C(r) \mid t \in [0, 2\pi)\}$ (see (5)). Let $s(k) = |\det S(k)|$. Then ($R \to \infty$ corresponds to $r \to 0$):

$$\lim_{r\to 1} \int_0^{2\pi} \frac{R \ln s(R(r)e^{\mathrm{i}t} + \mathrm{i}C(r))}{(R(r)e^{\mathrm{i}t} + \mathrm{i}C(r) + i)^2} dt = 0. \qquad (4)$$

$$C(r) = \frac{1+r^2}{1-r^2}, \quad R(r) = \frac{2r}{1-r^2}. \qquad (5)$$

Scattering matrix

Consider a system consisting of a subgraph playing the role of resonator and two semi-infinite wires Ω_1, Ω_4. The wave functions for Ω_1 is marked as $\psi_1^{(1)}$ and $\psi_2^{(1)}$ with the corresponding coefficients A and B. The wave functions for Ω_4 is marked as $\psi_1^{(4)}$ and $\psi_2^{(4)}$ with the corresponding coefficients C and D. S-matrix states the relation between A, B, C, D:

$$\begin{pmatrix} B \\ C \end{pmatrix} = S \begin{pmatrix} A \\ D \end{pmatrix}.$$

The scattering matrix has the form:

$$S = \begin{pmatrix} R & T \\ T & R \end{pmatrix}$$

Let $A = 1$, $D = 0$, Then

$$\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R \\ T \end{pmatrix}$$

Hence, $A = 1$, $B = R$, $C = T$, $D = 0$.

Line with attached segment

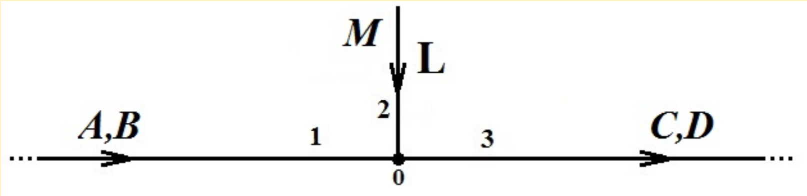

Рис. 2: Line with attached segment

Consider a segment as a model of resonator (Fig. **??**). Wave function at each edge has the form:

$$\begin{cases} \psi_1^{(1)} = Ae^{ikx} + Be^{-ikx}, \\ \psi_2^{(1)} = \sqrt{\frac{\lambda - mc^2}{\lambda + mc^2}}(Ae^{ikx} - Be^{-ikx}), \\ \psi_1^{(2)} = iM \sin kx \\ \psi_2^{(2)} = \sqrt{\frac{\lambda - mc^2}{\lambda + mc^2}} M \cos kx, \\ \psi_1^{(3)} = Ce^{ikx} + De^{-ikx} \\ \psi_2^{(3)} = \sqrt{\frac{\lambda - mc^2}{\lambda + mc^2}}(Ce^{ikx} - De^{-ikx}), \\ k = \frac{\sqrt{\lambda^2 - m^2c^4}}{\hbar c}. \end{cases} \tag{6}$$

Line with attached segment

The boundary condition at the vertex is as follows:

$$\begin{cases} \psi_1^{(1)}(0) = \psi_1^{(2)}(L) = \psi_1^{(3)}(0), \\ -\psi_2^{(1)}(0) - \psi_2^{(2)}(L) + \psi_2^{(3)}(0) = \frac{i\alpha}{c}\psi_1^{(1)}(0). \end{cases} \tag{7}$$

Let $\gamma = \sqrt{\frac{\lambda+mc^2}{\lambda-mc^2}}\frac{i\alpha}{c}$, $A = 1, B = R, C = T, D = 0$, then

$$s(k) = |R^2 - T^2| = |\frac{2+\gamma - i\cot kL}{2-\gamma + i\cot kL}|.$$

If $\gamma = 0$, then $s(k) = |\frac{-2\sin kL + i\cos kL}{2\sin kL + i\cos kL}|$.

Proof of completeness

Using the criterion (4). We should estimate the following integral

$$\int_0^{2\pi} F(t)dt = \int_0^{2\pi} \frac{R \ln s(R(r)e^{\mathrm{i}t} + \mathrm{i}C(r))}{(R(r)e^{\mathrm{i}t} + \mathrm{i}C(r) + i)^2} dt.$$

Here C, R is given by (5), s is as follows

$$s(k) = \left| \frac{(3+\gamma)e^{\imath x}e^{-y} - (1+\gamma)e^{-\imath x}e^{y}}{(3-\gamma)e^{\imath x}e^{-y} - (1-\gamma)e^{-\imath x}e^{y}} \right|,$$

where $k = x + \imath y$, $x = R\cos t$, $y = R\sin t + C$. The integral curve is divided into several parts. The first part is that inside a strip $0 < y < \delta$. Taking into account that at the real axis ($y = 0$) one has $s(k) = 1$. Correspondingly, $|\ln s(Re^{\imath t} + C\imath)| < \delta$. The length of the corresponding part of the circle is of order $\sqrt{2R\delta}$. As a result, the integral over this part of the curve is $o(\frac{1}{\sqrt{R}})$ and tends to zero if $R \to \infty$.

Proof of completeness

The second part of the integral is related to singularities of F, i.e., roots of $s(k)$ (resonances). These values are roots of an analytic function. Correspondingly, the number of roots at the integration curve is finite. Let t_0 be the value of the parameter corresponding to a resonance. Let one take a neighbourhood $(t_0 - \delta_1', t_0 + \delta_1$ such that outside it one has

$$|\ln s(Re^{\imath t} + C\imath)| < c_1. \tag{8}$$

One can find such δ_1', δ_1, because if $e^{2y} > \frac{3+\gamma}{1+\gamma}$ then $s(k)$ has no roots. Let us take δ_1', δ_1 such that $e^{2y} > 4\frac{3+\gamma}{1+\gamma}$ outside the interval, correspondingly, $|s(k)| \leq c_3$.
Inside the interval, one has

$$|F| \leq c_2 R^{-1} \ln t.$$

The corresponding integral is estimated as

$$I_2 = |\int_{t_0-\delta_1'}^{t_0+\delta_1} F(t)dt| \leq c_2 R^{-1} \delta_1 \ln \delta_1.$$

For the remain part of the integration curve one has $|F| \leq c_1 R^{-1}$, and the length of integration interval is not greater than 2π.

Proof of completeness

Thus, the procedure of estimation is as follows. Choose δ_1', δ_1 to separate the root (or roots) of $s(k)$. If $t_0 - \delta_1 > 0$ then consider $(0, t_0 - \delta_1]$ separately (for the second semi-circle $\pi \leq t < 2\pi$ the consideration is analogous). For this part of the curve with small t (i.e. small y), the estimation of the integral is $O(\frac{1}{\sqrt{R}}$. For the part of the curve outside these intervals, the estimation of the integral is $O(R^{-1})$. Correspondingly, the full integral is estimated as $O(\frac{1}{\sqrt{R}}$, i.e. the integral tends to zero if $R \to \infty$. In accordance with the completeness criterion we come to the theorem

Theorem 3. The system of resonance states is complete in $L_2(\Omega_2)$.

A loop with two semi-infinite lines attached

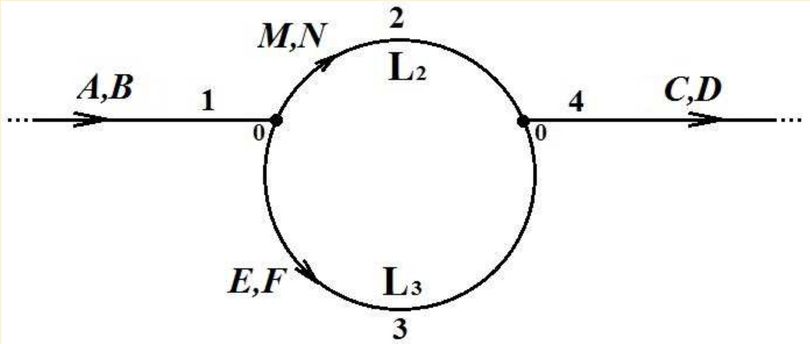

Рис. 3: Graph structure: A loop with two semi-infinite lines attached

Let $L_2 < L_3$, $\beta_1 = i\cot kL_2 + i\cot kL_3 + 1 - \gamma$, $\beta_2 = \frac{1}{i\sin kL_2} + \frac{1}{i\sin kL_3}$. Finally,

$$s(k) = \left|\frac{-2-(1+\gamma)^2 - \tan\frac{kL_2}{2}\cot\frac{kL_3}{2} - \tan\frac{kL_3}{2}\cot\frac{kL_2}{2} + 2i(1+\gamma)(\cot kL_2 + \cot kL_3)}{-2-(1-\gamma)^2 - \tan\frac{kL_2}{2}\cot\frac{kL_3}{2} - \tan\frac{kL_3}{2}\cot\frac{kL_2}{2} - 2i(1-\gamma)(\cot kL_2 + \cot kL_3)}\right|.$$

If $\gamma = 0$, then

$$s(k) = \left|\frac{-3 - \tan\frac{kL_2}{2}\cot\frac{kL_3}{2} - \tan\frac{kL_3}{2}\cot\frac{kL_2}{2} + 2i(\cot kL_2 + \cot kL_3)}{-3 - \tan\frac{kL_2}{2}\cot\frac{kL_3}{2} - \tan\frac{kL_3}{2}\cot\frac{kL_2}{2} - 2i(\cot kL_2 + \cot kL_3)}\right|.$$

A loop with two semi-infinite lines attached

For equal edges $L_2 = L_3 = L$, one has

$$s(k) = |\frac{-5 + 4i\cot kL}{-5 - 4i\cot kL}| = |\frac{4i\cos kL - 5\sin kL}{-4i\cos kL - 5\sin kL}|.$$

The investigation of the integral from the completeness criterion is analogous to the previous section. The result is in the following theorem.
Theorem 4. The system of resonance states is complete in $L_2(\Omega_2 \cup \Omega_3)$.
Remark. The obtained result can be compared with the Schrödinger quantum graph. For the Dirac and the Schrödinger operators on graphs of the identical structures, the completeness takes place for the same subgraphs.
Gerasimov, D.A., Popov, I.Y.: Completeness of resonance states for quantum graph with two semi-infinite edges. Complex Variables and Elliptic Equations. 62 (2017) DOI: 10.1080/17476933.2017.1289517.

Loop touched a line

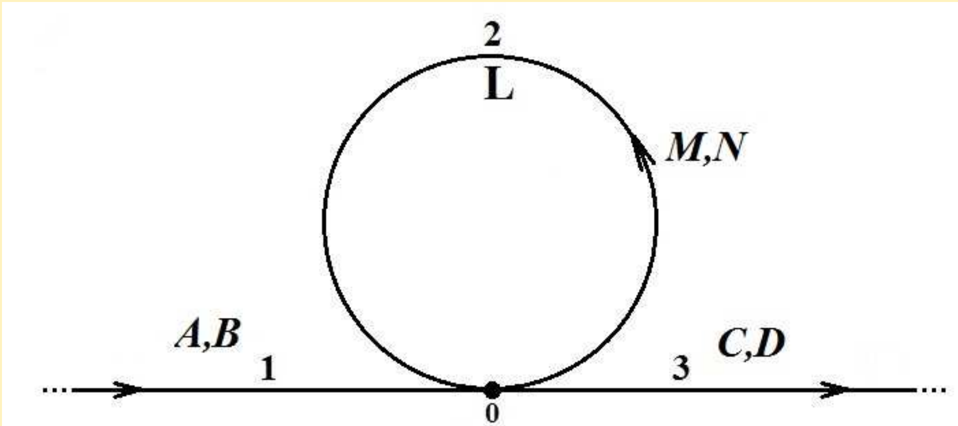

Рис. 4: Graph structure: Loop coupled to a line at one point.

The solution is:

$$\begin{cases} R = \frac{2e^{ikL}-2+\gamma(1+e^{ikL})}{4-\gamma(1+e^{ikL})}, \\ T = \frac{2(1+e^{ikL})}{4-\gamma(1+e^{ikL})}. \end{cases}$$

The $S-$matrix determinant for this case takes the form:

$$s(k) = |\frac{4e^{ikL} + \gamma(1 + e^{ikL})}{4 - \gamma(1 + e^{ikL})}|.$$

Loop touched a line

If $\gamma \neq 0$, then the integral estimation is similar to the previous section. If $\gamma = 0$, then $s(k) = |e^{ikL}|$. In this case, the result differs from the the previous one. It is clear that $\ln s(k)$ has a linear growth in upper half-plane, and the corresponding integral does not tend to zero for $R \to \infty$. We come to the theorem.

Theorem 5. If $\gamma \neq 0$, then the system of resonance states is complete in $L_2(\Omega_2)$; If $\gamma = 0$, then the system of resonance states is not complete in $L_2(\Omega_2)$.

Loop coupled to a line through a segment

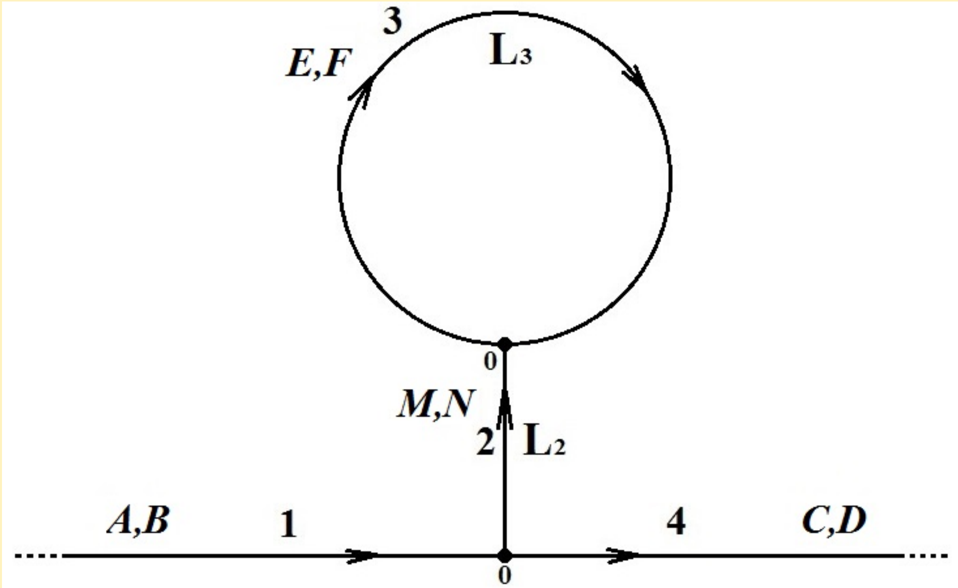

Рис. 5: Graph structure: A loop coupled to a line through a segment)

$$s(k) = \left|\frac{e^{ikL_2}(\gamma+3)\left(1-3e^{ikL_3}-\gamma(1+e^{ikL_3})\right)-e^{-ikL_2}(\gamma+1)\left(3-e^{ikL_3}-\gamma(1+e^{ikL_3})\right)}{e^{ikL_2}(\gamma-1)\left(1-3e^{ikL_3}-\gamma(1+e^{ikL_3})\right)+e^{-ikL_2}(3-\gamma)\left(3-e^{ikL_3}-\gamma(1+e^{ikL_3})\right)}\right|.$$

If $\gamma = 0$ then

$$s(k) = \left|\frac{3e^{ikL_2}(1-3e^{ikL_3})-e^{-ikL_2}(3-e^{ikL_3})}{-e^{ikL_2}(1-3e^{ikL_3})+3e^{-ikL_2}(3-e^{ikL_3})}\right|.$$

Loop coupled to a line through a segment

If $L_2 = 0$ then one has a natural answer $s(k) = |e^{ikL_3}|$
The integral estimation is analogous to the previous cases. We have a completeness of the resonance states in $L_2(\Omega_3)$ Thus, only the case $L_2 = 0$ leads to incompleteness. Any perturbation (small coupling segment or point-like potential at the vertex) restores the completeness.

Thank you for your attention