# Infinite Dimensional Preconditioners

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International Workshop on Operator Theory and its Applications
TU Chemnitz, Germany

# Contents

Consider the linear system $Ax = b$, where $A$ is an $n \times n$ matrix. There are several algorithms like Gauss elimination to solve this system. The computational complexity is a problem when $n$ is large.

Various iteration methods are used to obtain an approximate solution. There are several ways of iteration. For example, split the matrix $A = S - T$ and consider the iteration by $Sx_{k+1} = Tx_k + b, \;\; k = 1, 2, \ldots.$

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The choice of $S$ must be in such a way that it must be invertible so that we can compute $x_{k+1}$ from $x_k$. Also, the iteration must converge at a faster rate. That is the sequence $\{x_k\}$ must converge fast. We have to find an optimal choice of $S$ to meet both requirements.

Consider the following iteration

$$Cx_{j+1} = (C - A)x_j + b.$$

A preconditioner $C$ can be viewed as an approximation to $A$ that is efficiently invertible and can be used to obtain an approximate solution to the equation $Ax = b$.

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It was observed from numerical experiments that the convergence of this iteration process is much faster if the eigenvalues of $C^{-1}(C-A)$ are clustered around 0. That means the eigenvalues of $C^{-1}A$ must be clustered around 1.
From simple computations, we get if $\|A^{-1}(C-A)\| < 1$,

$$\rho(C^{-1}(C-A)) \leq \frac{\|A^{-1}(C-A)\|}{1-\|A^{-1}(C-A)\|},$$

where $\rho(C^{-1}(C-A))$ is the spectral radius.

**Remark**

Hence we have to choose $C$ such that $\|(C - A)\|$ is small.

**Definition (Frobenius Norm)**

For $A, B \in M_n(\mathbb{C})$ the Frobenius norm is defined by

$$\|A\|_F^2 = \sum_{j,k=1}^{n} |A_{j,k}|^2$$

induced by the classical Frobenius scalar product,

$$\langle A, B \rangle = \mathrm{trace}\,(B^* A)$$

For a given matrix $A \in M_n(\mathbb{C})$, our aim is to obtain a preconditioner $C$ such that the Frobenius norm $\|A - C\|_F$ is minimum.

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# Circulant Preconditioners

Here we discuss the circulant preconditioners for Toeplitz system obtained by Tony Chan in [9].

Consider the Toeplitz matrix with symbol $f$.

$$A_n(f) := \begin{bmatrix} a_0 & a_{-1} & \cdot & \cdot & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & \cdot & \cdots \\ \cdot & a_1 & a_0 & a_{-1} & \cdot & \cdot \\ \cdot & \cdot & a_1 & a_0 & a_{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{(n-1)} & \cdot & \cdots & \cdots & \cdot & a_0 \end{bmatrix} \quad (1)$$

where $a_j$ is the $j^{th}$ Fourier coefficient of $f$.

The circulant matrices are special types of Toeplitz matrices with the following form.

$$C_n := \begin{bmatrix} c_0 & c_1 & \cdot & \cdot & \cdots & c_{(n-1)} \\ c_{(n-1)} & c_0 & c_1 & \cdots & \cdot & c_{(n-2)} \\ \cdot & \cdot & c_0 & c_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & c_0 & c_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & \cdots & \cdots & \cdot & c_0 \end{bmatrix} \qquad (2)$$

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**Tony Chan** 1988

In [9], Tony Chan obtained explicitly the optimal circulant preconditioner. It is given by the circulant matrix $C_n$ where

$$c_i = \frac{(n-i)a_i + ia_{-(n-i)}}{n}; i = 0, 1, \ldots n-1.$$

In fact this is obtained by simply taking average of $n$ elements traveling along diagonal and parallel lines.

**Remark**

The above optimal preconditioners are more efficient, compared to the then existing preconditioners (due to Strang [8]) as observed in [9].

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After the circulant preconditioner obtained by Tony Chan [9], in the same spirit, several researchers have considered the problem to get efficient preconditioners such as Hartly, $\epsilon-$circulant etc. [1, 2]. The philosophy behind these developments is to identify the preconditioners as members in some matrix algebra.

The key observation is that circulant matrices are precisely elements in the commutative algebra $M_{U_n}$ of matrices defined as follows. $M_{U_n} = \{A \in M_n(\mathbb{C}) ; {U_n}^* A U_n \text{ complex diagonal}\}$, where $U_n = \left(\frac{1}{\sqrt{n}} e^{\frac{2\pi ijk}{n}}\right) \quad , \quad j, k = 0, 1, \ldots n-1.$

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Corresponding to each $A \in M_n(\mathbb{C})$, there exists a unique matrix $P_{U_n}(A)$ in $M_{U_n}$ such that

$$\|A - X\|_F^2 \geq \|A - P_{U_n}(A)\|_F^2 \text{ for every } X \in M_{U_n}.$$

In fact the circulant preconditioner $C_n$ obtained by Tony Chan in [9] is precisely the Frobenius optimal preconditioner $P_{U_n}(A_n)$.

**R.H. Chan, Dario Bini, F. Paola** 1991 – 1993

The other important cases are obtained for different choices of $U_n'$s. Some of them are listed below.

$$U_n = G_n = \left(\sqrt{\frac{2}{n+1}} sin\frac{(j+1)(i+1)\pi}{n+1}\right), \quad i,j = 0,1,\dots n-1,$$

$$U_n = H_n = \left(\frac{1}{\sqrt{n}}\left[sin(\frac{2ij\pi}{n}) + cos(\frac{2ij\pi}{n})\right]\right), \quad i,j = 0,1,\dots n-1.$$

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**Stefano Serra-Capizzano** 1999

In [7], Stefano Serra-Capizzano unified these techniques. Considering an arbitrary sequence $\{U_n\}$ of unitary matrices of growing order, we shall define $M_{U_n}$ and $P_{U_n}(A_n(f))$ as above with $\{U_n\}$ being arbitrary. The important question is when does $P_{U_n}(A_n(f))$ become an efficient preconditioner for $A_n(f)$. That is when does the matrix $P_{U_n}(A_n(f))^{-1}(A_n(f))$ has eigenvalues clustered around 1. Here is an important result in this regard.

**Theorem**

*If $f$ is positive, and $P_{U_n}(A_n(f))$ converges to $A_n(f)$ in the strong cluster sense, then for any $\epsilon > 0$, for n large enough, the matrix $P_{U_n}(A_n(f))^{-1}(A_n(f))$ has eigenvalues in $(1-\epsilon, 1+\epsilon)$ except $N_\epsilon$ outliers, at most.*

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Here the convergence notion is defined as follows.

### Definition

Let $\{A_n\}$ and $\{B_n\}$ be two sequences of Hermitian matrices of growing order. We say that $A_n - B_n$ converges to 0 in ***strong cluster*** if for any $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all the eigenvalues $\lambda_j(A_n - B_n)$ lie in the interval $(-\epsilon, \epsilon)$ except for at most $N_{1,\epsilon}$ (independent of the size $n$) eigenvalues, for all $n > N_{2,\epsilon}$.

If the number $N_{1,\epsilon}$ does not depend on $\epsilon$, we say that $A_n$ converges to $B_n$ in ***uniform cluster***. And if $N_{1,\epsilon}$ depends on $\epsilon, n$ and is of $o(n)$, we say that $A_n$ converges to $B_n$ in ***weak cluster***.

**VBK, MNN, S. 2013**

The convergence in the eigenvalue cluster sense shall be characterized as follows.

**Theorem**

*Let $\{A_n\}$ and $\{B_n\}$ be two sequences of Hermitian matrices of growing order. Then $\{A_n\} - \{B_n\}$ converges to 0 in strong cluster (weak or uniform cluster respectively) if and only if for every given $\epsilon > 0$, there exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that*

$$A_n - B_n = R_n + M_n, \quad n > N_{2,\epsilon},$$

*where the rank of $R_n$ is at most $N_{1,\epsilon}$ and $\|M_n\| < \epsilon$.*

**Remark**

This result can be extended to the case of normal matrices by defining the convergence notion appropriately.

# 2013

The following result is a kind of generalization of Theorem 4 into the case arbitrary Hermitian matrices obtained in [4].

### Theorem

*Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices such that $\{A_n\} - \{B_n\}$ converges to 0 in strong cluster (weak cluster respectively). Assume that $\{B_n\}$ is positive definite and invertible such that there exists a $\delta > 0$, with*

$$B_n \geq \delta I_n > 0, \quad \textit{for all } n.$$

*Then for a given $\epsilon > 0$, there will exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all eigenvalues of ${B_n}^{-1}A_n$ lie in the interval $(1-\epsilon, 1+\epsilon)$ except possibly for $N_{1,\epsilon} = O(1)$ ($N_{1,\epsilon} = o(n)$ respectively) eigenvalues for every $n > N_{2,\epsilon}$.*

**Stefano Serra-Capizzano** 1999

Naturally now we are interested to know when does $P_{U_n}(A_n(f))$ converges to $A_n(f)$. Here is a result in this regard.

**Theorem**

*Let $f$ be a continuous periodic real-valued function. Then $P_{U_n}(A_n(f))$ converges to $A_n(f)$ in strong cluster, if $P_{U_n}(A_n(p))$ converges to $A_n(p)$ in strong cluster for all the trigonometric polynomials $p$.*

## 2013, 2017

This result can be improved and as a consequence we obtain the same conclusion with assumption only on 3 polynomials (test set) instead of every polynomials. Here is the result.

### Theorem

*Let $\{g_1, g_2, \ldots g_m\}$ be a finite set of real-valued continuous $2\pi$ periodic functions such that $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster, for $f$ in $\{g_1, g_2, \ldots g_m, \sum_{i=1}^{m} {g_i}^2\}$. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster for all $f$ in the $C^*$-algebra $\mathbb{A}$ generated by $\{g_1, g_2, g_3, \ldots g_m\}$.*

### Remark

Notice that the above theorem is in the spirit of the classical theorem due to P.P. Korovkin where the approximation is reduced to a finite set called test set (see [3]).

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Notice that the above theorem is in the spirit of the classical theorem due to P.P. Korovkin where the approximation is reduced to a finite set called test set (see [3]).

We have already observed that in some special situations, a sequence of matrices $B_n$ is an efficient preconditioner of $A_n$ if $A_n$ converges to $B_n$ in the strong eigenvalue cluster. Hence we accept such types of convergence as a technique to obtain efficient preconditioners.

This is the idea behind generalizing the notion of preconditioners to the inifinite dimensional operators context. We will introduce such types of convergence for infinite dimensional operators sequence and obtain Korovkin-type theorems.

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**P.P. Korovkin** 1960

First we recall the classical approximation theorem due to P.P. Korovkin.

**Theorem**

*Let $\{\Phi_n\}$ be a sequence of positive linear maps on $C[0,1]$. If $\Phi_n(f) \to f$ for every $f$ in the set $\{1, x, x^2\}$, then $\Phi_n(f) \to f$ for every $f$ in $C[0,1]$.*

This classical approximation theorem unified many approximation processes such as Bernstein polynomial approximation of continuous real functions. This discovery inspired several mathematicians to extend the Korovkin's theorem in many ways and to several settings including function spaces, abstract Banach lattices, Banach algebras, Banach spaces, and so on. Such developments are referred to as KOROVKIN-TYPE APPROXIMATION THEORY.

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In [4], noncommutative Korovkin-type theorems were proved under the modes of convergence induced by strong, weak and uniform eigenvalue clustering of matrix sequences with growing order.

### Definition

Let $\{\Phi_n\}$ be a sequence of positive linear maps on $\mathcal{B}(\mathcal{H})$ and $P_n$ be a sequence of projections on $\mathcal{H}$ with rank $n$ that converges strongly to the identity. We say that $\{\Phi_n(A)\}$ converges to $A$ in the ***strong distribution sense***, if the sequence of matrices $\{P_n\Phi_n(A)P_n\} - \{P_nAP_n\}$ converges to 0 in strong cluster as per Definition 5.

Similarly we can define convergence in the ***weak distribution sense (uniform distribution sense respectively)***.

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Similarly we can define convergence in the ***weak distribution sense (uniform distribution sense respectively)***.

**VBK, MNN, S.** 2013

Here is the Korovkin-type theorem obtained in [4].

**Theorem**

*Let $\{A_1, A_2, \ldots A_m\}$ be a finite set of self-adjoint operators on $\mathcal{H}$ and $\Phi_n$ be a sequence of contractive positive maps on $\mathcal{B}(\mathcal{H})$, such that $\Phi_n(A)$ converges to $A$ in the strong (or weak respectively) distribution sense, for $A$ in $\left\{A_1, A_2, \ldots A_m, {A_1}^2, {A_2}^2, \ldots {A_m}^2\right\}$. In addition, if we assume that the difference $P_n({A_k}^2)P_n - (P_n(A_k)P_n)^2$ converges to the 0 matrix in strong cluster (weak cluster respectively), for each k, then $\Phi_n(A)$ converges to $A$ in the strong (or weak respectively) distribution sense, for all $A$ in the $J^*$- sub algebra $\mathbb{A}$ generated by $\{A_1, A_2, A_3, \ldots .A_m\}$.*

Now we mention the technique to identify preconditioners as contractive positive maps on $\mathcal{B}(\mathcal{H})$. Let $\{P_n\}$ be a sequence of orthogonal projections on $\mathcal{H}$ such that

$$\dim(P_n(\mathcal{H})) = n < \infty, \text{ for each } n = 1, 2, 3 \ldots,$$

$$\lim_{n\to\infty} P_n(x) = x, \text{ for every } x \text{ in } \mathcal{H}.$$

Let $\{U_n\}$ be a sequence of unitary matrices over $\mathbb{C}$ of growing order. For each $A \in \mathcal{B}(\mathcal{H})$, consider the following truncations $A_n = P_n A P_n$, which can be regarded as $n \times n$ matrices in $M_n(\mathbb{C})$, by restricting the domain to the range of $P_n$.

For each $n$, we define the commutative algebra $M_{U_n}$ and the preconditioner $P_{U_n}(A_n)$ as we did earlier.

$$M_{U_n} = \{A \in M_n(\mathbb{C})\,;\, {U_n}^* A U_n \text{ complex diagonal}\}$$

$P_{U_n}(A_n)$ be the unique matrix in $M_{U_n}$ such that

$$\|A - X\|_2^2 \geq \|A - P_{U_n}(A)\|_2^2 \text{ for every } X \in M_{U_n}.$$

Now, we introduce a completely positive map on $\mathcal{B}(\mathcal{H})$ as follows.

**Definition**

For each $A \in \mathcal{B}(\mathcal{H})$, $\Phi_n : \mathcal{B}(\mathcal{H}) \to M_n(\mathbb{C})$ is defined as

$$\Phi_n(A) = P_{U_n}(A_n),$$

The maps $\{\Phi_n\}$ is a sequence of completely positive maps such that

- $\|\Phi_n\| = 1, \text{ for each } n.$
- $\Phi_n$ is continuous in the strong topology of operators for each $n$.
- $\Phi_n(I) = I_n$ for each $n$ where $I$ is the identity operator on $\mathcal{H}$.

# Spectral Approximation Problem

Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $\mathcal{H}$ separable. Consider the orthonormal basis $\mathcal{B} = \{e_1, e_2, \ldots\}$ and the projections $P_n$'s of $\mathcal{H}$ on to the finite dimensional subspace spanned by first $n$ elements of $\mathcal{B}$.

Many researchers used the sequence of eigenvalues of the finite dimensional truncations $A_n = P_n A P_n$ to obtain information about spectrum of $A$. But in many situations, these $A_n'$s need not be simple enough to make the computations easier. The natural question "*Can we use some simpler sequence of matrices $B_n$ instead of $A_n$*' is addressed in [5]. The usage of preconditioners in the spectral gap prediction problems are also interesting.

**VBK** 2016

Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators. Then the operator $R = A - B$ is compact if and only if the truncations $A_n - B_n$ converges to the zero matrix in the strong cluster.

**Remark**

Since a compact perturbation may change the discrete eigenvalues, the above result shows that the convergence of preconditioners in the sense of eigenvalue clustering, is not sufficient to use them to approximate eigenvalues. Nevertheless one can use it in the spectral gap prediction problems, since the compact perturbation preserves the essential spectrum. In particular, it can be used to compute the upper and lower bound of the essential spectrum.

### 2017

Presently, we consider the preconditioners associated with non-self-adjoint operators.

The following is a list of problems to be addressed in future.

1. The application of preconditioners in operator equations has to be investigated in detail.
2. Korovkin-Shadow has to be investigated with respect to the modes of convergence introduced.
3. When does $\Phi_n(A)$ become a useful preconditioner to $A$? That is when does the spectrum of $\Phi_n(A)^{-1}A$ cluster around 1?

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Thank You !