Quantum channel entanglement producing and distance of the channel matrix from the tensor product

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Report plan

1. Introduction. Quantum computing motivation.
2. Matrix problem.
3. Quantum computing examples.

Qubit

Classical bit is a system which exists in one of two stable states (0, 1).
Quantum bit (qubit) is a quantum system which exists in a superposition of two stable states $|0>$ and $|1>$: $|\psi>=\alpha|0>+\beta|1>$. States are vectors in some Hilbert space.
If one performs a measurement, it obtains the system in the state $|0>$ with the probability $|\alpha|^2$ and in the state $|1>$ with the probability $|\beta|^2$. $<\psi|\psi>=|\alpha|^2+|\beta|^2=1$.
We use Dirac notation: $|\psi>$ is a vector, $<\psi|$ is an operator (in our space) of scalar multiplication by vector $|\psi>$. Correspondingly, the scalar product looks like $<\psi|\phi>$.
Matrix notation: $|0>=\binom{1}{0}$ and $|1>=\binom{0}{1}$.
Correspondingly: $|\psi>=\binom{\alpha}{\beta}$,
$<\psi|=(\overline{\alpha},\overline{\beta})$.

Tensor product

If a quantum system consists of two subsystems, then the state space of the system H is a tensor product $H_1 \otimes H_2$ of the state spaces H_1, H_2 of subsystems.
$H_1 \otimes H_2$ consists of linear combinations of all elements $f_1 \otimes f_2$, where $f_1 \in H_1$, $f_2 \in H_2$.
$< f_1 \otimes f_2 | g_1 \otimes g_2 >=< f_1 | g_1 >< f_2 | g_2 >$.
Tensor product of operators A_1, A_2:
$A_1 \otimes A_2 | f_1 \otimes f_2 >= | A_1 f_1 \otimes A_2 f_2 >$.

Tensor product, Matrix representation

Basis in one-qubit state space:
$|0>=\binom{1}{0}$ and $|1>=\binom{0}{1}$.
Basis in two-qubit state space:

$$|0>\otimes|0>=\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}$$

,

$$|0>\otimes|1>=\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}$$

,

$$|1>\otimes|0>=\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}$$

,

$$|1>\otimes|1>=\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}$$

.

Tensor product, Matrix representation

Tensor product of matrices (Kronecker product):

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix}$$

Tensor product, Matrix representation

$$|0>\otimes|0>=\binom{1}{0}\otimes\binom{1}{0}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ |0>\otimes|1>=\binom{1}{0}\otimes\binom{0}{1}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

$$|1>\otimes|0>=\binom{0}{1}\otimes\binom{1}{0}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},|1>\otimes|1>=\binom{0}{1}\otimes\binom{0}{1}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

To operate with a qubit means to apply an unitary operator (in more general case, a trace preserving operator) to the vector of this qubit. Tensor product of operators corresponds to Kronecker (tensor) product of their matrices in chosen basis.

Entanglement

Consider the state of two qubits A and B. There is a natural basis
$|0> \otimes |0>$, $|0> \otimes |1>$, $|1> \otimes |0>$, $|1> \otimes |1>$,
(for simplicity, let us mark it $|00>$, $|01>$, $|10>$, $|11>$).
Any state can be presented in the form:
$|\psi> = a|00> + b|01> + c|10> + d|11>.$,
where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.
Can one present this state in the form:

$$(\alpha|0> + \beta|1>) \otimes (\gamma|0> + \delta|1>)?$$

The necessary and sufficient condition for the factorization:

$$ad - bc = 0$$

. Definition. A multi-qubit state is called entangled if it can not be presented as a tensor product of single-qubit states.
The value of $|ad - bc|$ can be considered as a measure of entanglement.

Density matrix

Pure quantum state is described by a vector in a Hilbert state. If the system interacts with the surrounding systems it can be described only by a positive Hermitian operator with the unit trace in the Hilbert space (density operator or density matrix): $\rho = \sum_k p_k |\psi_k >< \psi_k|$, where $|\psi_k >$ is an orthogonal and normalized basis, $p_k \geq 0$ and $\sum_k p_k = 1$. p_k is the probability to observe the system in the state $|\psi_k >$ under measurement.

Gates and entanglement

Does a gate produce an entanglement? In which case is it important? If different qubits transform independently by the quantum gate or quantum channel then there is no entanglement producing. This takes place if the transformation matrix is a tensor product of matrices corresponding to different qubits.
Yang Xiang and Shi-Jie Xiong. Entanglement fidelity and measurement of entanglement preservation in quantum processes. Phys. Rev. A 76, (2007) 014301.
The problem of qubits' independent transformation is especially important in connection with recent results concerning free space teleportation.
Herbst, T., Scheidl, T., Fink, M., Handsteiner, J., Wittmann, B., Ursin, R., Zeilinger, A.: Teleportation of entanglement over 143 km. PNAS, 112 (2015) 1420214205.
In this case, the transmission matrix plays the role of a quantum gate matrix (transformation matrix). If there is qubit dependence during the transformation then the entanglement perturbation takes place (an additional entanglement can appear or, conversely, the entanglement can be destroyed). This leads to the destruction of the teleportation.

Gates and entanglement

The degree of independence for the qubit's transformation can be estimated using a distance from the transformation matrix to the subspace of matrices which are tensor products.

It is related to well known Eckart-Young-Mirsky theorem dealing with approximation of a matrix by low-rank matrices.

I. Markovsky, Low-Rank Approximation: Algorithms, Implementation, Applications, Springer, 2012

A. O. Pittenger, M. H. Rubin. Convexity and the separability problem of quantum mechanical density matrices. Linear Algebra and its Applications 346 (2002) 4771

G. Dahl, J.M. Leinaas, J.Myrheim, E.Ovrum. A tensor product matrix approximation problem in quantum physics. Linear Algebra and its Applications 420 (2007) 711725.

Our theorem is a modification of EYM theorem. We suggest another proof and, correspondingly, a way for the distance calculations. Examples of quantum computing applications are given.

Theorem

Definition. Vectorization operator vec is the operator which transforms a matrix M into vector $\mathrm{vec}M$ by the following way:

$$\mathrm{vec}M = \left(m_1', ..., m_p'\right)',$$

where vectors $m_1, ..., m_p$ are the columns of the matrix M, prime marks the transposition operation.
Let us take two matrices $B_{m\times m}$, $C_{n\times n}$ and their tensor (Kronecker) product $R_{mn\times mn} = B \otimes C$.
Definition. $\widetilde{R}$ is the matrix obtained from the matrix R by the following manner:

$$\widetilde{R}_{m^2\times n^2} = \mathrm{vec}B(\mathrm{vec}C)'.$$

Remark. Matrix $\widetilde{R}$ consists of the same entries as the matrix $R_{mn\times mn} = B \otimes C$, arranged in another order. As for arbitrary matrix $A_{mn\times mn}$, one divides it into blocks of sizes $n \times n$ and obtains the matrix $\widetilde{A}_{m^2\times n^2}$, following the same procedure as for R.

Theorem 1.
For given real matrix $A_{mn\times mn}$ the norm

$$\|A - B\otimes C\|$$

is minimal if the matrices $B_{m\times m}$ and $C_{n\times n}$ are such that

$$\mathrm{vec}B(\mathrm{vec}C)' = kbc'.$$

Here $k=\sigma_1$ is the maximal singular value of the matrix $\widetilde{A}$ ($\widetilde{A}_{m^2\times n^2}$), $b=u_1$ and $c=v_1$ are the right and the left singular vectors of the matrix $\widetilde{A}$ corresponding to singular value σ_1.

Theorem

Proof.

$$R_{mn\times mn} = B_{m\times m} \otimes C_{n\times n} = \begin{bmatrix} b_{11}C \ b_{12}C \ \ldots. \ b_{1m}C \\ b_{21}C \ b_{22}C \ \ldots. \ b_{2m}C \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ b_{m1}C \ b_{m2}C \ \ldots. \ b_{mm}C \\ \end{bmatrix}.$$

Theorem

$$\widetilde{R}_{m^2\times n^2} = \mathrm{vec}B(\mathrm{vec}C)' =$$

$$\begin{bmatrix}
b_{11}c_{11} & \dots & b_{11}c_{n1} & b_{11}c_{12} & \dots & b_{11}c_{n2} & \dots & b_{11}c_{n1} & b_{11}c_{1n} & \dots & b_{11}c_{nn} \\
b_{21}c_{11} & \dots & b_{21}c_{n1} & b_{21}c_{12} & \dots & b_{21}c_{n2} & \dots & b_{21}c_{n1} & b_{21}c_{1n} & \dots & b_{21}c_{nn} \\
\vdots & & & & & & & & & & \\
b_{m1}c_{11} & \dots & b_{m1}c_{n1} & b_{m1}c_{12} & \dots & b_{m1}c_{n2} & \dots & b_{m1}c_{n1} & b_{m1}c_{1n} & \dots & b_{m1}c_{nn} \\
b_{12}c_{11} & \dots & b_{12}c_{n1} & b_{12}c_{12} & \dots & b_{12}c_{n2} & \dots & b_{12}c_{n1} & b_{12}c_{1n} & \dots & b_{12}c_{nn} \\
b_{22}c_{11} & \dots & b_{22}c_{n1} & b_{22}c_{12} & \dots & b_{22}c_{n2} & \dots & b_{22}c_{n1} & b_{22}c_{1n} & \dots & b_{22}c_{nn} \\
\vdots & & & & & & & & & & \\
b_{m2}c_{11} & \dots & b_{m2}c_{n1} & b_{m2}c_{12} & \dots & b_{m2}c_{n2} & \dots & b_{m2}c_{n1} & b_{m2}c_{1n} & \dots & b_{m2}c_{nn} \\
\vdots & & & & & & & & & & \\
b_{1m}c_{11} & \dots & b_{1m}c_{n1} & b_{1m}c_{12} & \dots & b_{1m}c_{n2} & \dots & b_{1m}c_{n1} & b_{1m}c_{1n} & \dots & b_{1m}c_{nn} \\
b_{2m}c_{11} & \dots & b_{2m}c_{n1} & b_{2m}c_{12} & \dots & b_{2m}c_{n2} & \dots & b_{2m}c_{n1} & b_{2m}c_{1n} & \dots & b_{2m}c_{nn} \\
\vdots & & & & & & & & & & \\
b_{mm}c_{11} & \dots & b_{mm}c_{n1} & b_{mm}c_{12} & \dots & b_{mm}c_{n2} & \dots & b_{mm}c_{n1} & b_{mm}c_{1n} & \dots & b_{mm}c_{nn}
\end{bmatrix}.$$

Theorem

$$\widetilde{A} =$$

$$\begin{bmatrix}
a_{1,1} & \cdots & a_{m,1} & \cdots & a_{1,m} & \cdots & a_{m,m} \\
a_{(m+1),1} & \cdots & a_{2m,1} & \cdots & a_{(m+1),n} & \cdots & a_{2m,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{mn-m+1,1} & \cdots & a_{mn,1} & \cdots & a_{mn-m+1,m} & \cdots & a_{mn,m} \\
a_{1,m+1} & \cdots & a_{m,m+1} & \cdots & a_{1,2m} & \cdots & a_{m,2m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{mn-m+1,m+1} & \cdots & a_{mn,m+1} & \cdots & a_{mn-m+1,2m} & \cdots & a_{mn,2m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1,mn-m+1} & \cdots & a_{m,mn-m+1} & \cdots & a_{1,mn} & \cdots & a_{m,mn} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{mn-m+1,mn-m+1} & \cdots & a_{mn,mn-m+1} & \cdots & a_{mn-m+1,mn} & \cdots & a_{mn,mn}
\end{bmatrix}.$$

Theorem

One can see that

$$\|A - B \otimes C\| = \left\| \widetilde{A} - \mathrm{vec}B(\mathrm{vec}C)' \right\|.$$

Let $\mathrm{vec}B(\mathrm{vec}C)' = kbc'$, where $\|b_{m^2\times 1}\| = \|c_{n^2\times 1}\| = 1$, k is a normalizing factor. Consider the singular value decomposition for the matrix $\widetilde{A}$:

$$\widetilde{A} = \sum_{i=1}^{s} \sigma_i u_i v_i',$$

where $s = \mathrm{rank}\widetilde{A}$, $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_s > 0$ are the matrix singular values arranged in decreasing order, u_i are orthogonal and normalized vectors of size m^2, $\|u_i\| = 1$, v_i are orthogonal and normalized vectors of size n^2, $\|v_i\| = 1$.

Theorem

Definition. The Frobenius norm of a matrix M is as follows:

$$\|M\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}.$$

Consider

$$\left\|\sum_{i=1}^{s}\sigma_i u_i v_i' - kbc'\right\|^2 = \operatorname{tr}\left(\left(\sum_{i=1}^{s}\sigma_i u_i v_i' - kbc'\right)'\left(\sum_{i=1}^{s}\sigma_i u_i v_i' - kbc'\right)\right) =$$
$$= \operatorname{tr}\left(\sum_{i=1}^{s}\sum_{j=1}^{s}\sigma_i\sigma_j v_i u_i' u_j v_j' \; - kcb'\sum_{i=1}^{s}\sigma_i u_i v_i' - \left(\sum_{i=1}^{s}\sigma_i v_i u_i'\right) kbc' + k^2 cb'bc'\right) =$$
$$= \operatorname{tr}\left(\sum_{i=1}^{s}\sigma_i^2 \; - kcb'\sum_{i=1}^{s}\sigma_i u_i v_i' - \left(\sum_{i=1}^{s}\sigma_i v_i u_i'\right) kbc' + k^2\right).$$

Theorem

$$\left\|\sum_{i=1}^{s}\sigma_i u_i v_i' - kbc'\right\|^2 =$$
$$= \sum_{i=1}^{s}\operatorname{tr}\left(\sigma_i^2\right) - k\sum_{i=1}^{s}\sigma_i \operatorname{tr}\left(cb'u_i v_i'\right) - k\sum_{i=1}^{s}\sigma_i \operatorname{tr}\left(v_i u_i' bc'\right) + \operatorname{tr}\left(k^2\right) =$$
$$= \sum_{i=1}^{s}\sigma_i^2 - k\sum_{i=1}^{s}\sigma_i \operatorname{tr}\left(cb'u_i v_i'\right) - k\sum_{i=1}^{s}\sigma_i \operatorname{tr}\left(v_i u_i' bc'\right) + k^2.$$

Theorem

Consider the expression:

$$\operatorname{tr}\left(v_i u_i' b c'\right) = \operatorname{tr}\left(v_i u_i' b c'\right)' = \operatorname{tr}\left(\left(b c'\right)'\left(v_i u_i'\right)'\right) = \operatorname{tr}\left(c b' u_i v_i'\right).$$

One has:

$$\left\|\sum_{i=1}^{s} \sigma_i u_i v_i' - k b c'\right\|^2 =$$
$$\sum_{i=1}^{s} \sigma_i^2 - k \sum_{i=1}^{s} \sigma_i \operatorname{tr}\left(c b' u_i v_i'\right) - k \sum_{i=1}^{s} \sigma_i \operatorname{tr}\left(v_i u_i' b c'\right) + k^2 = \sum_{i=1}^{s} \sigma_i^2 -$$
$$-2k \sum_{i=1}^{s} \sigma_i \operatorname{tr}\left(c b' u_i v_i'\right) + k^2.$$

Theorem

One comes to the following expression

$$\begin{aligned}
&\sum_{i=1}^{s}\sigma_i^2 - 2k\sum_{i=1}^{s}\sigma_i \mathrm{tr}\left(cb'u_i v_i'\right) + k^2 = \\
&\sum_{i=1}^{s}\sigma_i^2 - 2k\sum_{i=1}^{s}\left(\sigma_i\sum_{l=1}^{n^2}\left(v_l^i c_l\right)\sum_{j=1}^{m^2}\left(b_j u_j^i\right)\right) + k^2 = \\
&\sum_{i=1}^{s}\sigma_i^2 - 2k\sum_{i=1}^{s}\left(\sigma_i v_i' c b' u_i\right) + k^2.
\end{aligned} \tag{1}$$

Consider the obtained expression as a function of k other arguments being fixed. The function has a minimal value for

$$k = \sum_{i=1}^{s}(\sigma_i v_i' c b' u_i).$$

Theorem

Inserting this value of k into (1), one obtains:

$$\sum_{i=1}^{s} \sigma_i^2 - \left(\sum_{i=1}^{s} (\sigma_i v_i' c b' u_i) \right)^2 .$$

This expression takes its minimal value if

$$\sum_{i=1}^{s} \sigma_i \left| v_i' c b' u_i \right|$$

is maximal. Due to the ordering of the singular values, one has

$$\sum_{i=1}^{s} \sigma_i \left| v_i' c b' u_i \right| \leq \sum_{i=1}^{s} \sigma_1 \left| v_i' c b' u_i \right|.$$

Theorem

The Hölder inequality leads to the following inequalities

$$|v_i'c|^2 \leq \|v_i'\|^2 \|c\|^2 = 1. \, |b'u_i|^2 \leq \|b'\|^2 \|u_i\|^2 = 1.$$

Hence, $|v_i'cb'u_i| = 1$ for $c = v_i$,$b = u_i$. σ_1 is the maximal singular value, consequently, $\sum\limits_{i=1}^{s} \sigma_i |v_i'cb'u_i|$ is maximal for $k = \sigma_1$, $c = v_1$,$b = u_1$.

Thus, for given matrix $A_{mn \times mn}$, the norm $\|A - B \otimes C\|$ is minimal if the matrices $B_{m \times m}$ and $C_{n \times n}$ are such that

$$\text{vec}B(\text{vec}C)' = \sigma_1 u_1 v_1'.$$

This norm is as follows

$$\|A - B \otimes C\| = \left\| \widetilde{A} - \text{vec}B(\text{vec}C)' \right\| = \left\| \widetilde{A} - \sigma_1 u_1 v_1' \right\|.$$

It is the norm that gives us the distance from $A_{mn \times mn}$ to the subspace of matrices that are tensor products of matrices having sizes $m \times m$ and $n \times n$.

CNOT gate

CNOT gate is controlled NOT. The first qubit is control qubit, the second one is target qubit. The operator does not change the state of the control qubit $|a>$ and transform the target qubit: if $|a>=|1>$ then $|0>$ is replaced by $|1>$ and $|1>$ is replaced by $|0>$. If $|a>=|0>$ then $|b>$ does not change. Here $\oplus$ means modulo 2 ($0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$). CNOT is an analog of classical XOR.

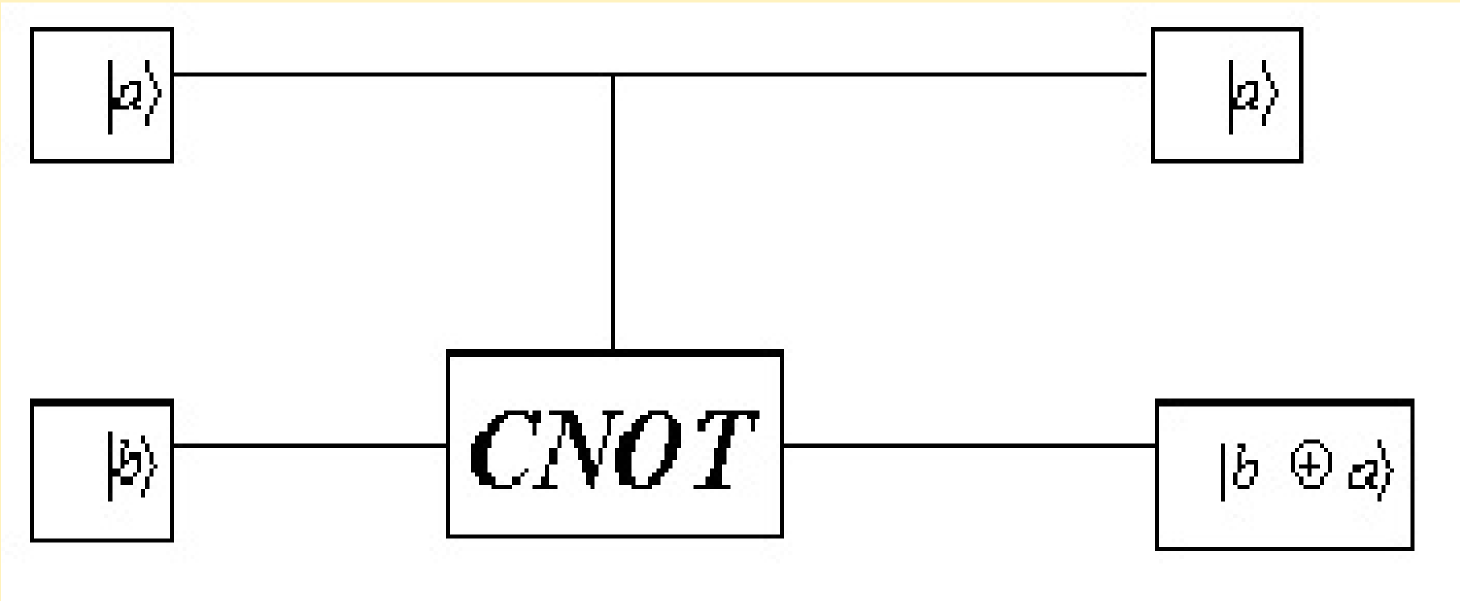

Figure 1: CNOT gate for basic vectors

CNOT gate

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Correspondingly,

$$\widetilde{U}_{CNOT} = \begin{bmatrix} 1\,0\,0\,1 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,1\,1\,0 \end{bmatrix}.$$

CNOT gate

$$\widetilde{U}'_{CNOT}\widetilde{U}_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Correspondingly, the eigenvalues are as follows: $y_1 = y_2 = \sigma_1^2 = \sigma_2^2 = 2$, $y_3 = y_4 = \sigma_3^2 = \sigma_4^2 = 0$. The corresponding normalized eigenvectors are:

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

$$\left\| \widetilde{U}_{CNOT} - \sigma_1 u_1 v_1' \right\| = \sqrt{2}.$$

SWAP operator

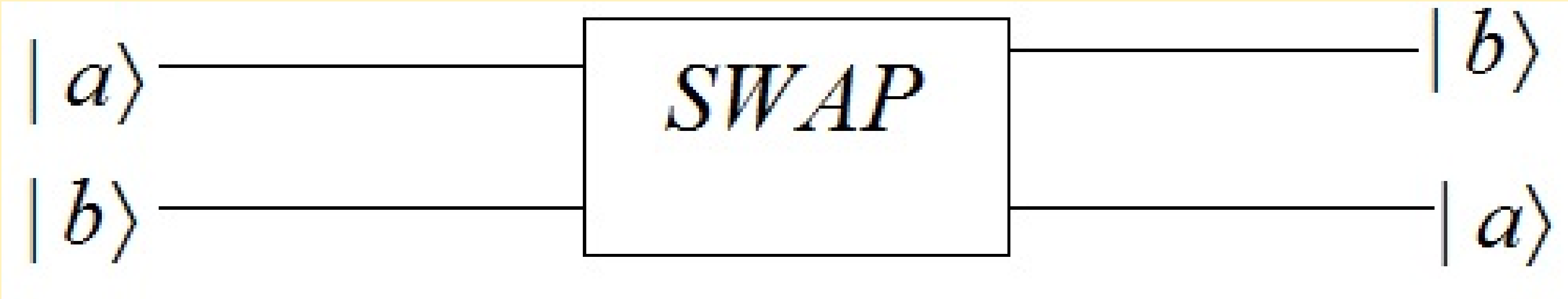

Figure 2: Action of SWAP operator on basic vectors

$$U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{U}_{SWAP} = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix}.$$

$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1.$

$$\left\| \widetilde{U}_{SWAP} - \sigma_1 u_1 v_1' \right\| = \sqrt{3}.$$

SWAP operator

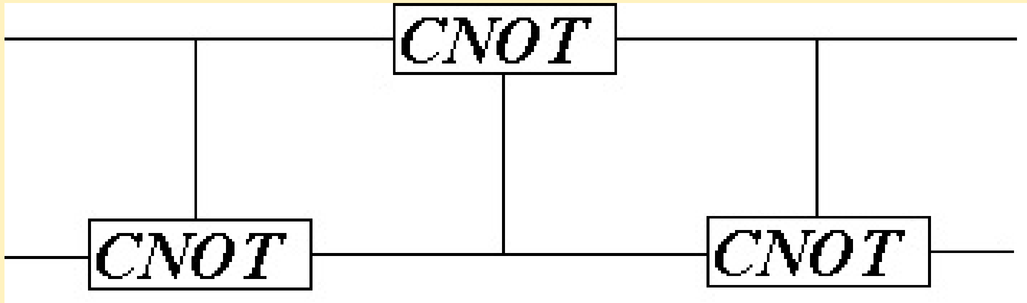

Figure 3: SWAP operator as three CNOT operator

The SWAP operator does not produce an entanglement, but it is not a tensor product of 2×2 matrices.

Density operator

Any two-qubit pure state can be presented in the form:

$$|\psi >= a|00 > +b|01 > +c|10 > +d|11 >,$$

where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. The necessary and sufficient condition for the factorization:

$$ad - bc = 0.$$

The value of $|ad - bc|$ can be considered as a measure of entanglement.
Consider the density matrix S

$$S = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} . \begin{bmatrix} a & b & c & d \end{bmatrix}$$

Density operator

Following the described procedure, one obtains

$$\widetilde{S}\widetilde{S}^* = \left[\begin{array}{cc} A\begin{bmatrix} A & B \\ \overline{B} & C \end{bmatrix} & \overline{B}\begin{bmatrix} A & B \\ \overline{B} & C \end{bmatrix} \\ B\begin{bmatrix} A & B \\ \overline{B} & C \end{bmatrix} & C\begin{bmatrix} A & B \\ \overline{B} & C \end{bmatrix} \end{array}\right] = \begin{bmatrix} A & \overline{B} \\ B & C \end{bmatrix} \otimes \begin{bmatrix} A & B \\ \overline{B} & C \end{bmatrix},$$

where

$$A = |a|^2 + |b|^2, \quad C = |c|^2 + |d|^2, \quad B = a\bar{c} + b\bar{d}, \quad \overline{B} = c\bar{a} + d\bar{b}.$$

Correspondingly, the eigenvalues σ_i of the 4×4- matrix are products of eigenvalues λ_j of 2×2-matrices, the eigenvectors are tensor products of the corresponding eigenvectors. One can see that

$$\lambda_{1,2} = 2^{-1} \pm \sqrt{4^{-1} - AC + |B|^2} = 2^{-1} \pm \sqrt{4^{-1} - |ad - bc|^2},$$

i.e. our measure is correlated with the measure mentioned above.

Thank you for your attention