One-sided invertibility of infinite band-dominated matrices

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One-sided and two-sided invertible operators

Let $\mathcal{B}(X, Y)$ be the Banach space of all bounded linear operators acting from a Banach space X to a Banach space Y. We abbreviate $\mathcal{B}(X, X)$ to $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X, Y)$ is called left invertible (resp. right invertible) if there exists an operator $B \in \mathcal{B}(Y, X)$ such that $BA = I_X$ (resp. $AB = I_Y$) where $I_X \in \mathcal{B}(X)$ and $I_Y \in \mathcal{B}(Y)$ are the identity operators on X and Y, respectively. The operator B is called a left (resp. right) inverse of A. An operator $A \in \mathcal{B}(X, Y)$ is said to be invertible if it is left invertible and right invertible simultaneously. We say that A is strictly left (resp. right) invertible if it is left (resp. right) invertible, but not invertible. If the operator A is invertible only from one side, then the corresponding inverse is not uniquely defined.

A function $a : \mathbf{Z} \to \mathbf{C}$ with uniformly bounded values $a(n)$ is called slowly oscillating, $a \in SO(\mathbf{Z})$, if

$$\lim_{n \to \pm\infty} |a(n+1) - a(n)| = 0.$$

Discrete operators on the spaces l^p, $p \in [1, \infty]$

Given $p \in [1, \infty]$, we consider the Banach space $l^p = l^p(\mathbf{Z})$ consisting of all functions $f : \mathbf{Z} \to \mathbf{C}$ equipped with the norm

$$\|f\|_{l^p} = \begin{cases} \left(\sum_{n\in\mathbf{Z}} |f(n)|^p\right)^{1/p} & \text{if } \ p \in [1, \infty), \\ \sup_{n\in\mathbf{Z}} |f(n)| & \text{if } \ p = \infty. \end{cases}$$

We establish criteria of the one-sided invertibility of discrete operators of the Wiener type

$$\mathcal{A} := \sum_{k\in\mathbf{Z}} a_k V^k, \quad \|\mathcal{A}\|_W := \sum_{k\in\mathbf{Z}} \|a_k\|_{l^\infty} < \infty, \qquad (1)$$

on the spaces l^p with $p \in [1, \infty]$, where $a_k \in SO(\mathbf{Z}) \subset l^\infty$ for all $k \in \mathbf{Z}$, and the isometric shift operator V is given on functions $f \in l^p$ by $(Vf)(n) = f(n+1)$ for all $n \in \mathbf{Z}$. Clearly, V is invertible on each space l^p. Thus, for every $f \in l^p$, we have

$$(\mathcal{A}f)(n) = \sum_{k\in\mathbf{Z}} a_k(n) f(n+k) \ \text{ for all } \ n \in \mathbf{Z}.$$

Let $\mathcal{W}$ be the Banach algebra of operators (1) with norm $\|\cdot\|_W$.

Maximal ideal space of the unital commutative C^*-algebra $SO(\mathbf{Z})$

The set $SO(\mathbf{Z})$ of all slowly oscillating (at $\pm\infty$) functions in l^∞ is a unital commutative C^*-algebra properly containing the C^*-algebra $C(\overline{\mathbf{Z}})$, where $\overline{\mathbf{Z}} := \mathbf{Z} \cup \{\pm\infty\}$. Let $M(SO(\mathbf{Z}))$ be the maximal ideal space of the algebra $SO(\mathbf{Z})$. Identifying the points $n \in \overline{\mathbf{Z}}$ with the evaluation functionals $n(f) = f(n)$ for $f \in C(\overline{\mathbf{Z}})$, we get $M(C(\overline{\mathbf{Z}})) = \overline{\mathbf{Z}}$. Consider the fibers

$$M_s(SO(\mathbf{Z})) := \{\xi \in M(SO(\mathbf{Z})) : \xi|_{C(\overline{\mathbf{Z}})} = s\}$$

of the maximal ideal space $M(SO(\mathbf{Z}))$ over points $s \in \{\pm\infty\}$.

The fibers $M_{\pm\infty}(SO(\mathbf{Z}))$ are connected compact Hausdorff spaces. The set

$$\Delta := M_{-\infty}(SO(\mathbf{Z})) \cup M_{+\infty}(SO(\mathbf{Z})) = \mathrm{clos}_{SO^*}\, \mathbf{Z} \setminus \mathbf{Z},$$

where $\mathrm{clos}_{SO(\mathbf{Z})^*}\, \mathbf{Z}$ is the weak-star closure of $\mathbf{Z}$ in the dual space of $SO(\mathbf{Z})$. Then $M(SO(\mathbf{Z})) = \Delta \cup \mathbf{Z}$. We write $a(\xi) := \xi(a)$ for every $a \in SO(\mathbf{Z})$ and every $\xi \in \Delta$.

Application of limit operators

Discrete operators $\mathcal{A} \in \mathcal{W}$ are operators of multiplication by infinite band-dominated matrices $\big(a_{k-n}(n)\big)_{n,k\in\mathbf{Z}}$.

Lemma

Let $p \in [1,\infty)$ and let $\mathcal{A} = \sum_{k\in\mathbf{Z}} a_k V^k \in \mathcal{W} \subset \mathcal{B}(l^p)$, where $a_k \in SO(\mathbf{Z})$ for all $k \in \mathbf{Z}$. Then for every $\xi \in \Delta$ there exists a sequence $\{k_n\}_{n\in\mathbf{N}}$ of numbers $k_n \in \mathbf{N}$ such that $k_n \to \infty$ as $n \to \infty$, and

$$\operatorname*{s-lim}_{n\to\infty} \left(V^{\pm k_n} \mathcal{A} V^{\mp k_n}\right) = A_\xi := \sum_{k\in\mathbf{Z}} a_k(\xi) V^k \ \in \mathcal{W} \ \text{ if } \ s = \pm\infty.$$

Corollary

If $p \in [1,\infty)$ and $\mathcal{A} = \sum_{k\in\mathbf{Z}} a_k V^k \in \mathcal{W}$ is left invertible on l^p, then for every $\xi \in \Delta$ the operators $A_\xi = \sum_{k\in\mathbf{Z}} a_k(\xi) V^k \in \mathcal{W}$ possess the properties: $\operatorname{Ker} A_\xi = \{0\}$, $\operatorname{Im} A_\xi$ *is a closed subspace of l^p, and A_ξ are invertible from l^p onto* $\operatorname{Im} A_\xi$.

Invertibility of limit operators

Corollary

If $p \in (1, \infty)$ and the operator $\mathcal{A} = \sum_{k \in \mathbf{Z}} a_k V^k \in \mathcal{W}$ is invertible on the space l^p, then for every $\xi \in \Delta$ the limit operators $A_\xi = \sum_{k \in \mathbf{Z}} a_k(\xi) V^k$ are also invertible on l^p.

Lemma

The spectrum of the isometric operator V coincides with the unit circle $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$.

Consider the unital commutative Banach algebra $\mathcal{W}_{\mathbf{C}}$ consisting of all operators $\mathcal{A} = \sum_{k \in \mathbf{Z}} a_k V^k \in \mathcal{W}$ with constant coefficients $a_k \in \mathbf{C}$ on l^p. The maximal ideal space of $\mathcal{W}_{\mathbf{C}}$ can be identified with $\mathbf{T}$, and the Gelfand transform of $\mathcal{A} = \sum_{k \in \mathbf{Z}} a_k V^k \in \mathcal{W}_{\mathbf{C}}$ is given by $A(z) := \sum_{k \in \mathbf{Z}} a_k z^k$ for all $z \in \mathbf{T}$, where $A(\cdot)$ belongs to the algebra W of absolutely convergent Fourier series on $\mathbf{T}$.

The Gelfand transform and the Cauchy index

Hence, for each $\xi \in \Delta$ the operator $A_\xi = \sum_{k\in\mathbf{Z}} a_k(\xi)V^k \in \mathcal{W}_{\mathbf{C}}$ is invertible on the space l^p with $p \in [1,\infty)$ if and only if

$$A_\xi(z) := \sum_{k\in\mathbf{Z}} a_k(\xi)z^k \neq 0 \quad \text{for all} \quad z \in \mathbf{T}.$$

Since this is true for all $\xi \in \Delta$, we infer, by the continuity of the function $\xi \mapsto A_\xi(\cdot) \in W$ on the connected Hausdorff compact $M_s(SO(\mathbf{Z}))$ for every $s \in \{\pm\infty\}$, that the numbers

$$\operatorname{ind} A_\xi(\cdot) := \frac{1}{2\pi}\{\arg A_\xi(z)\}_{z\in\mathbf{T}}$$

do not depend on $\xi \in M_s(SO(\mathbf{Z}))$ and can only depend on $s \in \{\pm\infty\}$. Put

$$N_\pm := \operatorname{ind} A_\xi(\cdot) \;\; \text{for all} \;\; \xi \in M_{\pm\infty}(SO(\mathbf{Z})).$$

While all limit operators A_ξ are invertible for each invertible operator $\mathcal{A} \in \mathcal{W}$ by the last corollary, this fact for strictly one-sided invertible operators $\mathcal{A} \in \mathcal{W}$ we still need to prove.

Necessary conditions at fixed points

Theorem

Let $p \in (1, \infty)$. If the discrete operator $\mathcal{A} = \sum_{k \in \mathbf{Z}} a_k V^k \in \mathcal{W}$ with coefficients $a_k \in SO(\mathbf{Z})$ is left or right invertible on the space l^p, then

$$A_\xi(z) = \sum_{k \in \mathbf{Z}} a_k(\xi) z^k \neq 0 \quad \textit{for all} \quad \xi \in \Delta \quad \textit{and all} \quad z \in \mathbf{T},$$

and the Cauchy indices $\operatorname{ind} A_\xi(\cdot)$ *coincide, respectively, for every $\xi \in M_{-\infty}(SO(\mathbf{Z}))$ and for every $\xi \in M_{+\infty}(SO(\mathbf{Z}))$.*

Thus, for the one-sided invertible operators $\mathcal{A} \in \mathcal{W} \subset \mathcal{B}(l^p)$, we again can uniquely define the numbers $N_\pm := \operatorname{ind} A_\xi(\cdot)$.

Let $\mathcal{A} \in \mathcal{W}$. Take in $\mathcal{B}(l^p)$ the projections $P_n^\pm := \operatorname{diag}\{P_{s,n}^\pm\}_{s \in \mathbf{Z}}$, $P^0_{n-N_-, n+N_+} := I - P^-_{n-N_-} - P^+_{n+N_+}$, $P_n^0 := I - P_n^- - P_n^+$, where

$$P_{s,n}^+ = \begin{cases} 1 & \text{if } s \geq n, \\ 0 & \text{if } s < n, \end{cases} \qquad P_{s,n}^- = \begin{cases} 1 & \text{if } s \leq -n, \\ 0 & \text{if } s > -n. \end{cases}$$

Invertibility of outermost blocks for discrete operators

Consider the operators $\mathcal{A}_n^+ := P_n^+ \mathcal{A} P_{n+N_+}^+$, $\mathcal{A}_n^- := P_n^- \mathcal{A} P_{n-N_-}^-$.

Theorem

If the discrete operator $\mathcal{A} = \sum_{k\in\mathbf{Z}} a_k V^k \in \mathcal{W}$ is left or right invertible on the space l^p with $p \in (1, \infty)$, then there exists a number $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ the operators

$$\mathcal{A}_n^+ : P_{n+N_+}^+ l^p \to P_n^+ l^p, \quad \mathcal{A}_n^- : P_{n-N_-}^- l^p \to P_n^- l^p$$

are invertible.

Let $\mathcal{W}^\pm$ denote the unital Banach subalgebras of $\mathcal{W}$ given by

$$\mathcal{W}^\pm := \left\{ \sum_{k\in\mathbf{Z}_+} a_k^\pm V^{\pm k} \in \mathcal{W} : \ a_k^\pm \in SO(\mathbf{Z}) \right\},$$

where $\mathbf{Z}_+ := \mathbf{N} \cup \{0\}$. Let $W^\pm$ be the unital Banach subalgebras of the algebra W of absolutely convergent Fourier series on $\mathbf{T}$,

$$W^\pm := \left\{ f = \sum_{k\in\mathbf{Z}_+} a_k^\pm z^{\pm k} \in W : \ a_k^\pm \in \mathbf{C}, \ z \in \mathbf{T} \right\}.$$

Invertibility of outermost blocks: a scheme of the proof

It suffices to prove the invertibility of the operator $\mathcal{A}_n^+$, assuming that $N_+ = 0$. Since $A_\xi(z) \ne 0$ for all $\xi \in M_{+\infty}(SO(\mathbf{Z}))$ and all $z \in \mathbf{T}$, and since $\operatorname{ind} A_\xi(\cdot) = 0$ for these ξ, we conclude that for every $\xi \in M_{+\infty}(SO(\mathbf{Z}))$ the function $z \mapsto A_\xi(z)$ admits a unique canonical factorization

$$A_\xi(z) = A_\xi^+(z)\, A_\xi^-(z) \ \text{ for all } \ z \in \mathbf{T},$$

where $A_\xi^\pm(\cdot), \big(A_\xi^\pm(\cdot)\big)^{-1} \in W^\pm$ and $\int_{\mathbf{T}} A_\xi^+(z)|dz| = 2\pi$.

Using the functions $\big(A_\xi^\pm(\cdot)\big)^{-1} \in W^\pm$ for all $\xi \in M_{+\infty}(SO(\mathbf{Z}))$, it is possible to construct discrete operators

$$\mathcal{C}^\pm = \sum\nolimits_{k\in\mathbf{Z}_+} c_k^\pm V^{\pm k} \in \mathcal{W}^\pm$$

such that the operators $P_n^+\mathcal{C}^\pm P_n^+$ are invertible in the Banach algebras $P_n^+\mathcal{W}^\pm P_n^+$ for all sufficiently large $n \in \mathbf{N}$, and the operator $P_n^+(\mathcal{C}^+\mathcal{A}\mathcal{C}^-)P_n^+$ is close to the identity operator on the space $P_n^+ l^p$, which leads to the invertibility of the operators $\mathcal{A}_n^+$.

One-sided invertibility of modified central block

Representing the operator $\mathcal{A} \in \mathcal{W}$ acting from the direct sum of spaces $P^-_{n-N_-} l^p \dot{+} P^0_{n-N_-,n+N_+} l^p \dot{+} P^+_{n+N_+} l^p$ to the direct sum of spaces $P^-_n l^p \dot{+} P^0_n l^p \dot{+} P^+_n l^p$ as the operator matrix

$$\mathcal{A} := \begin{bmatrix} P^-_n \mathcal{A} P^-_{n-N_-} & P^-_n \mathcal{A} P^0_{n-N_-,n+N_+} & P^-_n \mathcal{A} P^+_{n+N_+} \\ P^0_n \mathcal{A} P^-_{n-N_-} & P^0_n \mathcal{A} P^0_{n-N_-,n+N_+} & P^0_n \mathcal{A} P^+_{n+N_+} \\ P^+_n \mathcal{A} P^-_{n-N_-} & P^+_n \mathcal{A} P^0_{n-N_-,n+N_+} & P^+_n \mathcal{A} P^+_{n+N_+} \end{bmatrix}, \qquad (2)$$

we infer that the operator

$$\mathcal{D}_{n,\infty} := \begin{bmatrix} P^-_n \mathcal{A} P^-_{n-N_-} & P^-_n \mathcal{A} P^+_{n+N_+} \\ P^+_n \mathcal{A} P^-_{n-N_-} & P^+_n \mathcal{A} P^+_{n+N_+} \end{bmatrix}, \qquad (3)$$

acting from the space $P^-_{n-N_-} l^p \dot{+} P^+_{n+N_+} l^p$ onto the space $P^-_n l^p \dot{+} P^+_n l^p$, is invertible along with operators $P^-_n \mathcal{A} P^-_{n-N_-}$ and $P^+_n \mathcal{A} P^+_{n+N_+}$. As (3) is invertible, the one-sided invertibility of (2) is equivalent to the one-sided invertibility of a modified central block.

Two-sided invertibility of Wiener type discrete operators

Theorem

The operator $\mathcal{A} = \sum_{k\in\mathbf{Z}} a_k V^k \in \mathcal{W}$ *with coefficients* $a_k \in SO(\mathbf{Z})$ *is invertible on the space* l^p *with* $p \in [1,\infty]$ *if and only if*

(i) $A_\xi(z) := \sum_{k\in\mathbf{Z}} a_k(\xi) z^k \neq 0$ *for all* $\xi \in \Delta$ *and* $z \in \mathbf{T}$*;*

(ii) $N_- = N_+$, *where* $N_\pm := \operatorname{ind} \mathcal{A}_\xi(\cdot)$ *for any* $\xi \in M_{\pm\infty}(SO(\mathbf{Z}))$*;*

(iii) *there exists an* $n_0 \in \mathbf{N}$ *such that* $\det \mathcal{D}_{n,0} \neq 0$ *for every* $n > n_0$, *where the* $(2n-1)\times(2n-1)$ *matrices* $\mathcal{D}_{n,0}$ *are identified with the operator*

$$\mathcal{D}_{n,0} := P_n^0 \mathcal{A} P^0_{n-N_-,n+N_+} - \begin{bmatrix} P_n^0 \mathcal{A} P^-_{n-N_-} & P_n^0 \mathcal{A} P^+_{n+N_+} \end{bmatrix} \times \begin{bmatrix} P_n^- \mathcal{A} P^-_{n-N_-} & P_n^- \mathcal{A} P^+_{n+N_+} \\ P_n^+ \mathcal{A} P^-_{n-N_-} & P_n^+ \mathcal{A} P^+_{n+N_+} \end{bmatrix}^{-1} \begin{bmatrix} P_n^- \mathcal{A} P^0_{n-N_-,n+N_+} \\ P_n^+ \mathcal{A} P^0_{n-N_-,n+N_+} \end{bmatrix}$$

acting from the space $P^0_{n-N_-,n+N_+} l^p$ *to the space* $P_n^0 l^p$.

Strict one-sided invertibility of Wiener-type discrete operators

Criteria of the strict one-sided invertibility of the operators $\mathcal{A} \in \mathcal{W}$ on the spaces l^p for $p \in (1, \infty)$ have the following form.

Theorem

The discrete operator $\mathcal{A} = \sum_{k \in \mathbf{Z}} a_k V^k \in \mathcal{W}$ *with coefficients* $a_k \in SO(\mathbf{Z})$ *is strictly left (resp., strictly right) invertible on the space* l^p *with* $p \in (1, \infty)$ *if and only if*

- **(i)** $A_\xi(z) := \sum_{k \in \mathbf{Z}} a_k(\xi) z^k \neq 0$ *for every* $\xi \in \Delta$ *and every* $z \in \mathbf{T}$;
- **(ii)** $N_- > N_+$ *(resp.,* $N_- < N_+$*), where* $N_\pm = \operatorname{ind} \mathcal{A}_\xi(\cdot)$ *for any* $\xi \in M_{\pm\infty}(SO(\mathbf{Z}))$;
- **(iii)** *there exists an* $n_0 \in \mathbf{N}$ *such that the rank of the* $(2n - 1 + N_+ - N_-) \times (2n - 1)$ *matrices* $\mathcal{D}_{n,0}$ *for all* $n \geq n_0$ *equal* $2n - 1 + N_+ - N_-$ *(resp., equal* $2n - 1$*).*

Invertibility of binomial discrete operator: l^∞ coefficients

Theorem

Let $p \in [1, \infty]$ and $a, b \in l^\infty$. The operator $\mathcal{A} := aI - bV$ is invertible on the space l^p if and only if one of the following two alternative conditions holds:

$$\text{(i) } a \in \mathcal{G}l^\infty \ \text{ and } \ r(b/a) < 1, \quad \text{(ii) } b \in \mathcal{G}l^\infty \ \text{ and } \ r(a/b) < 1,$$

where $r(c) := \lim_{n\to\infty} \left(\sup_{k\in\mathbb{Z}} \left|c(k+1)c(k+2)\dots c(k+n)\right|\right)^{1/n}$

for $c \in l^\infty$. If $\mathcal{A}$ is invertible, then its inverse is given by

$$\mathcal{A}^{-1} = \sum_{n=0}^{\infty} \left((b/a)V\right)^n a^{-1} I \quad \text{in case (i)},$$

$$\mathcal{A}^{-1} = -V^{-1} \sum_{n=0}^{\infty} \left((a/b)V^{-1}\right)^n b^{-1} I \quad \text{in case (ii)}.$$

For every $k \in \mathbf{Z}$, we introduce the functions $\chi_k^\pm \in l^\infty$ by

$$\chi_k^+(n) = \begin{cases} 1 & \text{if } n > k, \\ 0 & \text{if } n \le k, \end{cases} \qquad \chi_k^-(n) = \begin{cases} 0 & \text{if } n > k, \\ 1 & \text{if } n \le k. \end{cases} \tag{4}$$

Strict left invertibility of binomial discrete operator

For every $k \in \mathbf{N}$, we also define the functions $\beta_k : \mathbf{Z} \to \mathbf{Z}$ $(k \in \mathbf{N})$ by

$$\beta_k(n) = n + k \quad \text{for all} \quad n \in \mathbf{Z}. \tag{5}$$

Theorem

The operator $\mathcal{A} = aI - bV$ is strictly left invertible on the space l^p with $p \in [1, \infty]$ if and only if the following two conditions hold:

(i) *there exists a number $k \in \mathbf{Z}$ such that $\inf_{n<k} |b(n)| > 0$ and $\inf_{n>k} |a(n)| > 0$;*

(ii) $r\left(\chi_k^- \frac{a \circ \beta_{-1}}{b \circ \beta_{-1}}\right) < 1$ *and* $r\left(\chi_k^+ \frac{b}{a}\right) < 1$,

where the functions $\chi_k^\pm \in l^\infty$ and β_k are given by (4) and (5).

Under these conditions one of the left inverses have the form

$$\mathcal{A}^L := \chi_k^+ \sum_{n=0}^{\infty} ((b/a)V)^n (1/a)I - \chi_k^- V^{-1} \sum_{n=0}^{\infty} ((a/b)V^{-1})^n (1/b)I.$$

Strict right invertibility of binomial discrete operator

Theorem

The operator $\mathcal{A} = aI - bV$ is strictly right invertible on the space l^p with $p \in [1, \infty]$ if and only if the following two conditions hold:

(i) *there exists a $k \in \mathbf{Z}$ such that $\inf_{n \leq k} |a(n)| > 0$ and $\inf_{n > k} |b(n)| > 0$;*

(ii) $r(\chi_k^-(b \circ \beta_{-1})/a) < 1$ *and* $r(\chi_k^+(a \circ \beta_1)/b) < 1$,

where the functions $\chi_k^\pm \in l^\infty$ and β_k are given by (4) *and* (5).
Under these conditions one of the right inverses have the form

$$\mathcal{A}^R := \sum_{n=0}^{\infty} ((b/a)V)^n (\chi_k^-/a)I - V^{-1} \sum_{n=0}^{\infty} ((a/b)V^{-1})^n (\chi_k^+/b)I.$$

Given $\mathbf{R}_+ = (0, \infty)$, let α denote an orientation-preserving homeomorphism of $[0, \infty]$ onto itself, which has only two fixed points 0 and ∞, and its restriction to $\mathbf{R}_+$ is a diffeomorphism. Let $\alpha_0(t) := t$ and $\alpha_n(t) := \alpha[\alpha_{n-1}(t)]$ for all $n \in \mathbf{Z}$ and $t \in \mathbf{R}_+$.

Slowly oscillating functions and shifts on $\mathbf{R}_+$

Let $C_b(\mathbf{R}_+)$ denote the C^*-algebra of all bounded continuous functions on $\mathbf{R}_+ := (0, +\infty)$. Following [Sarason], a function $f \in C_b(\mathbf{R}_+)$ is called slowly oscillating (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0, 1)$,

$$\lim_{r \to s} \sup \big\{ |f(t) - f(\tau)| : t, \tau \in [\lambda r, r] \big\} = 0, \quad s \in \{0, \infty\}.$$

The set $SO(\mathbf{R}_+)$ of all slowly oscillating (at 0 and ∞) functions in $C_b(\mathbf{R}_+)$ is a unital commutative C^*-algebra.

A diffeomorphism $\alpha : \mathbf{R}_+ \to \mathbf{R}_+$ is called a slowly oscillating shift if $\log \alpha' \in SO(\mathbf{R}_+)$. We associate with α the isometric shift operator $U_\alpha \in \mathcal{B}(L^p(\mathbf{R}_+))$ given by $U_\alpha f = |\alpha'|^{1/p}(f \circ \alpha)$.

Let $\mathfrak{A}_W$ be the Banach algebra of Wiener's functional operators

$$A = \sum_{k \in \mathbf{Z}} a_k U_\alpha^k \in \mathcal{B}(L^p(\mathbf{R}_+)) \ \text{ with } \ \|A\|_W := \sum_{k \in \mathbf{Z}} \|a_k\|_{C_b(\mathbf{R}_+)} < \infty,$$

where $a_k \in SO(\mathbf{R}_+)$ for all $k \in \mathbf{Z}$ and α is a slowly oscillating shift on $\mathbf{R}_+$.

Reduction to the one-sided invertibility of discrete operators

If $p \in [1, \infty]$ and $A = \sum_{k \in \mathbf{Z}} a_k U_\alpha^k \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbf{R}_+))$, then for every $t \in \mathbf{R}_+$, we define the discrete operator

$$\mathcal{A}_t := \sum_{k \in \mathbf{Z}} a_{k,t} V^k \in \mathcal{W} \subset \mathcal{B}(l^p),$$

where $a_{k,t}(n) := a_k[\alpha_n(t)]$ for all $k, n \in \mathbf{Z}$ and all $t \in \mathbf{R}_+$, the functions $a_{k,t}$ belong to $SO(\mathbf{Z})$, and

$$\|A\|_{\mathcal{B}(L^p(\mathbf{R}_+))} = \sup_{t \in \mathbf{R}_+} \|\mathcal{A}_t\|_{\mathcal{B}(l^p)} \leq \|A\|_W.$$

Theorem

If $p \in [1, \infty]$, then the functional operator $A = \sum_{k \in \mathbf{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ is invertible on the space $L^p(\mathbf{R}_+)$ if and only if for all $t \in \mathbf{R}_+$ the discrete operators $\mathcal{A}_t \in \mathcal{W}$ are invertible on the space l^p. If $p \in (1, \infty)$, then the left (resp., right) invertibility of the operator A on the space $L^p(\mathbf{R}_+)$ is equivalent to the left (resp., right) invertibility of the operators $\mathcal{A}_t \in \mathcal{W}$ on the space l^p for $t \in \mathbf{R}_+$.