Isomorphisms of $AC(\sigma)$ spaces

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To begin: a well-known relationship

Spectral Decompositions $\leftrightsquigarrow$ Functional calculus

Spectral Decompositions:

$$T = \int \lambda \, dE(\lambda) \quad \text{or} \quad T = \sum_{j=1}^{\infty} \lambda_j P_j.$$

'You can find a suitable family of projections commuting with T from which you can reconstruct the operator'.

Functional calculus:

$\|f(T)\| \leq K \, \|f\|_{\mathcal{A}}$ for all $f \in$ some algebra $\mathcal{A}$.

The bigger $\mathcal{A}$ is, the better the spectral decomposition is.

Classical case

On a Hilbert space, if T is a normal operator then

(i) the map $f \mapsto f(T)$ extends from polynomials to all continuous functions on $\sigma(T)$;

(ii) $\|f(T)\| = \|f\|_\infty$ for $f \in C(\sigma(T))$;

(iii) $C^*(T) \cong C(\sigma(T))$.

(iv) $T = \int_{\sigma(T)} \lambda\, \mathcal{E}(d\lambda)$ (with $\mathcal{E}$ a spectral measure).

In particular, for normal operators T, T',
if $\sigma(T)$ is homeomorphic to $\sigma(T')$ then:

(i) $C(\sigma(T)) \cong C(\sigma(T'))$;

(ii) hence $C^*(T) \cong C^*(T')$.

Our problem

- Replace Hilbert space H by a (reflexive) Banach space X.
- Work with a smaller functional calculus/weaker spectral decomposition.

Why?

Many important bases and decompositions of say $L^2(\mathbb{T})$ are only conditional on $L^p(\mathbb{T})$ ($1 < p < \infty$) and are not associated with spectral measures of the type that appear in the spectral theorem for normal operators.

(eg Fourier series $\{e^{ikt}\}_{k\in\mathbb{Z}}$.)

Semi-classical case

Use the algebra $\mathcal{A} = AC[a,b]$ of absolutely continuous functions on $[a,b]$, with the norm

$$\|f\|_{AC[a,b]} = |f(a)| + \operatorname{var}_{[a,b]} f.$$

Then

$$\|f(T)\| \leq K \|f\|_{AC[a,b]} \text{ for all } f \in AC[a,b]$$
$$\iff T = \int_{[a,b]} \lambda \, dE(\lambda)$$

Here: $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ a uniformly bounded increasing 'spectral family' of projections.

Compact case: $\iff T = \sum_{k=1}^{\infty} \lambda_k P_k$ where the sum may be only conditionally convergent.

Obvious Questions

1. Can you make sense of $AC(\sigma)$ when $\sigma = \sigma(T)$ is any compact subset of $\mathbb{C}$?
2. If ‘Yes’, is there any sort of Banach–Stone Theorem?

The answer to (1) is complicated! Many versions of variation norms exist for functions defined on the plane:

- Vitali–Lebesgue–Fréchet–de la Vallée Poussin
- Hardy–Krause
- Arzelà
- Hahn
- Tonelli
- Berkson–Gillespie

But none was really suitable for spectral theory.

Design parameters

Brenden Ashton's thesis problem (2000):
Can you define Banach algebras $AC(\sigma) \subseteq BV(\sigma)$ for arbitrary compact $\sigma \subseteq \mathbb{C}$ in such a way that

1. it agrees with the usual definition if σ is an interval in $\mathbb{R}$;
2. $AC(\sigma)$ contains all sufficiently well-behaved functions;
3. if $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, then the space $AC(\alpha\sigma + \beta)$ is isometrically isomorphic to $AC(\sigma)$. (and for BV)

(3) is because if we know the structure of T we also know the structure of $\alpha T + \beta I$.

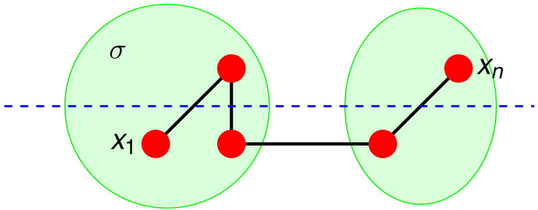
Ashton's $BV(\sigma)$

Fix a compact set $\sigma \subseteq \mathbb{C} = \mathbb{R}^2$ and $f : \sigma \to \mathbb{C}$.
Suppose that $S = [x_0, x_1, \dots, x_n]$ is a finite list of elements of σ (repeats allowed!).

Definition. The *curve variation of f on the set S* is

$$\mathrm{cvar}(f, S) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Let γ_S be the piecewise linear curve joining the points of S.



The *variation factor* of S, $\mathrm{vf}(S)$, is (roughly) the greatest number of times that γ_S crosses any line.

$BV(\sigma)$

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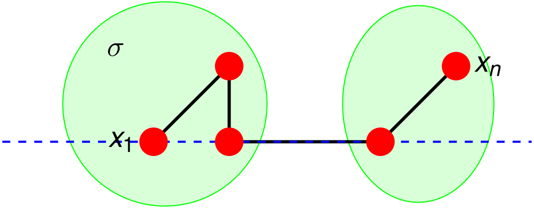

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$BV(\sigma)$

The *two-dimensional variation* of $f : \sigma \to \mathbb{C}$ is

$$\operatorname{var}(f, \sigma) = \sup_S \frac{\operatorname{cvar}(f, S)}{\operatorname{vf}(S)},$$

where the supremum is taken over all finite ordered lists of elements of σ.

The *variation norm* is

$$\|f\|_{BV} = \|f\|_\infty + \operatorname{var}(f, \sigma)$$

and the set of functions of bounded variation on σ is

$$BV(\sigma) = \{f : \sigma \to \mathbb{C} \,:\, \|f\|_{BV} < \infty\}.$$

Theorem. $BV(\sigma)$ is a Banach algebra.

$AC(\sigma)$

$BV(\sigma)$ always contains $\mathcal{P}_2$, the set of polynomials in two variables.

Definition. $AC(\sigma)$ is the closure of $\mathcal{P}_2$ in $BV(\sigma)$.

Theorem. If $\sigma = [a, b]$ then $BV(\sigma)$ and $AC(\sigma)$ give the usual algebras!

Suitably interpreted

$$C^1(\sigma) \subseteq AC(\sigma) \subseteq C(\sigma).$$

$AC(\sigma)$ operators

Definition. $T \in B(X)$ is an $AC(\sigma)$ **operator** if T admits an $AC(\sigma)$ functional calculus.

Historically, the operators with an $AC[a,b]$ functional calculus were called well-bounded operators. **Theorem.**

1. T is well-bounded $\iff$ it is an $AC(\sigma)$ operator with $\sigma \subseteq \mathbb{R}$.
2. T is trigonometrically well-bounded
   $\iff^*$ it is an $AC(\sigma)$ operator with $\sigma \subseteq \mathbb{T}$.
3. If T is an $AC(\sigma)$ operator, then $T = A + iB$ where A and B are commuting well-bounded operators
   (but not conversely!).

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*T&C apply

Banach-Stone type theorems

BS: $C(\sigma_1) \simeq C(\sigma_2) \iff \sigma_1 \sim \sigma_2$. ($\Leftarrow$ easy; $\Rightarrow$ harder!)

Theorem (D-Leinert 2015)

Suppose that $\Phi : AC(\sigma_1) \to AC(\sigma_2)$ is an algebra isomorphism. Then

1. $\|f\|_\infty = \|\Phi(f)\|_\infty$ *for all* $f \in AC(\sigma_1)$.
2. *there exists a homeomorphism* $h : \sigma_1 \to \sigma_2$ *such that* $\Phi(f) = f \circ h^{-1}$ *for all* $f \in AC(\sigma_1)$.
3. Φ *is continuous.*

Here the $\Rightarrow$ direction more or less comes from the BS Theorem.

The $\Leftarrow$ direction isn't true!

A counterexample

Let D be the closed unit disk and $Q = [0, 1] \times [0, 1]$ be the closed unit square.

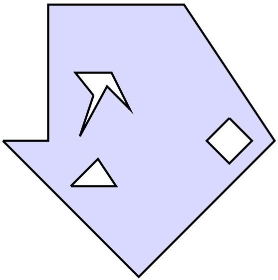
Theorem (D-Leinert 2015)

$AC(D) \not\approx AC(Q)$.

A positive result

Definition. A compact set σ is a **polygonal region of genus** n if there exists a simple polygon P with n nonoverlapping polygonal 'windows' $W_1, \ldots, W_n$ such that

$$\sigma = P \setminus (W_1 \cup \cdots \cup W_n).$$

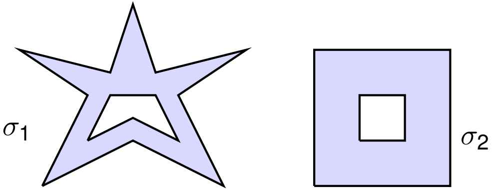


A polygonal region of genus 3.

A positive result

Theorem (D-Leinert)

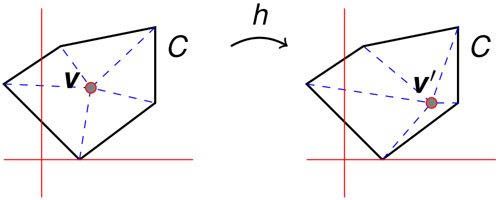
Suppose that σ_1 *and* σ_2 *are polygonal regions of genus* n_1 *and* n_2*. Then*
$AC(\sigma_1)$ *is isomorphic to* $AC(\sigma_2)$
iff $n_1 = n_2$
iff σ_1 *is homeomorphic to* σ_2*.*



$$AC(\sigma_1) \simeq AC(\sigma_2).$$

The Proof: Locally piecewise affine maps

Let C be a convex n-gon in $\mathbb{R}^2$.
Suppose that $\boldsymbol{v}, \boldsymbol{v}'$ lie in the interior of C.



There is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that

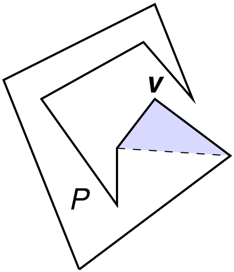
(i) h is the identity outside C, and piecewise affine inside C;

(ii) $h(\boldsymbol{v}) = \boldsymbol{v}', \quad i = 1, \ldots, n$;

(iii) 'h preserves the AC isomorphism class'.

Chopping off ears!

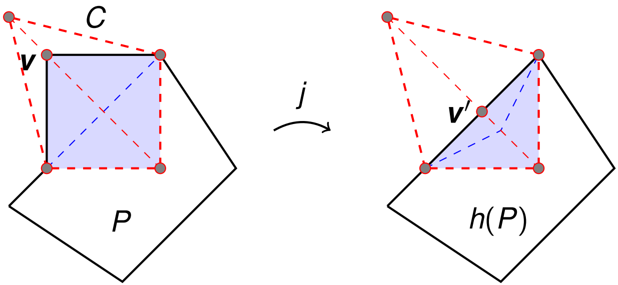
An ear in a polygon P is a vertex so that the line joining the neighbouring vertices lies entirely inside P.



Two Ears Theorem (Meisters). Every polygon with 4 or more vertices has at least two ears.

Chopping off ears!

If you have an ear, such as $\boldsymbol{v}$, you can always find a convex quadrilateral C and a locally piecewise affine map h which flattens out the ear, and hence reduces the number of sides.



Eventually you reduce to a triangle.

Compact operators and countable sets

Trivially, if σ_1 and σ_2 are finite sets then $AC(\sigma_1) \simeq AC(\sigma_2)$ $\iff$ σ_1 and σ_2 have the same number of elements.

For countably infinite sets, things are more complicated.

Definition. We shall say that a set $\sigma \subseteq \mathbb{C}$ is a *C*-set if it is a countably infinite set with unique limit point zero.

All such sets are homeomorphic, but there they produce many non-isomorphic $AC(\sigma)$ spaces.

k-ray sets

Definition. We shall say that a C-set $\sigma \subseteq \mathbb{C}$ is a k-ray set if there are k rays from the origin $r_1, \dots, r_k$ such that

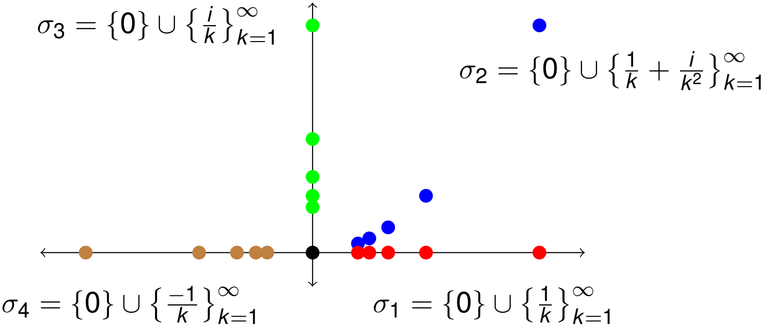
- $\sigma_j = \sigma \cap r_j$ is infinite for $j = 1, \dots, k$ and
- $\sigma_0 = \sigma \setminus (\sigma_1 \cup \dots \cup \sigma_k)$ is finite.

Theorem. Suppose that σ is a k-ray set and that τ is an ℓ-ray set. Then $AC(\sigma) \simeq AC(\tau) \iff k = \ell$.

Thus

- there are infinitely many non-isomorphic $AC(\sigma)$ spaces even among the C-sets,
- up to isomorphism there are precisely two $AC(\sigma)$ spaces for C-sets $\sigma \subseteq \mathbb{R}$.

Some examples



- $AC(\sigma_n) \simeq AC(\sigma_m)$ for all n, m.
- $AC(\sigma_1) \not\simeq AC(\sigma_1 \cup \sigma_4)$.
- $AC(\sigma_1 \cup \{-1\}) \simeq AC(\sigma_1)$.
- $AC(\sigma_1 \cup \sigma_3) \simeq AC(\sigma_1 \cup \sigma_4) \not\simeq AC(\sigma_1 \cup \sigma_3 \cup \sigma_4)$.

References

- B. Ashton and I. Doust, *Functions of bounded variation on compact subsets of the plane*, Studia Math. **169** (2005), 163–188.
- B. Ashton and I. Doust, *A comparison of algebras of functions of bounded variation*, Proc. Edinb. Math. Soc. (2) **49** (2006), 575–591.
- B. Ashton and I. Doust, *$AC(\sigma)$ operators*, J. Operator Theory **65** (2011), 255–279.
- I. Doust and M. Leinert, *Approximation in $AC(\sigma)$*, arXiv:1312.1806v1, 2013.
- I. Doust and M. Leinert, *Isomorphisms of $AC(\sigma)$ spaces*, *Studia Math.* **228** (2015), 7–31.