Boundary Conditions associated with the Left-Definite Theory for Differential Operators

Matthew Fleeman, Dale Frymark*, and Constanze Liaw

Baylor University

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Overview of Left-Definite Theory

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- The theory requires a self-adjoint operator which is bounded below by some positive constant.

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- The theory requires a self-adjoint operator which is bounded below by some positive constant.
- It allows for the construction of a continuum of operators, defined on subsets of the original Hilbert space.
- These operators can be thought of as simply composing the original operator with itself, and restricting the Hilbert space.

Definition

Suppose A is a self-adjoint operator in the Hilbert space $H = (V, \langle \cdot, \cdot \rangle)$ that is bounded below by kI, where $k > 0$. Let $r > 0$. Define $H_r = (V_r, \langle \cdot, \cdot \rangle_r)$ with

$$V_r = \mathcal{D}(A^{r/2})$$

and

$$\langle x, y \rangle_r = \langle A^{r/2} x, A^{r/2} y \rangle \text{ for } (x, y \in V_r).$$

Then H_r is the rth left-definite space associated with the pair (H, A).

Notable Facts from Left-Definite Theory

- The theory is trivial for bounded operators.
- For unbounded operators, the left-definite spaces are nested in one another; i.e.

$$H \supset H_1 \supset H_2 \supset H_3 \dots$$

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Theorem (Littlejohn, Wellman '02)

If $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthogonal set of eigenfunctions of A in H, then for each $r > 0$, $\{\varphi_n\}_{n=0}^{\infty}$ is a complete set of orthogonal eigenfunctions of the rth left-definite operator A_r in the rth left-definite space H_r.

The Problem

The description of these left-definite spaces is regrettably very complicated. For instance, for the Laguerre differential operator with $\alpha > -1$, $j \in \mathbb{N}_0$, let $L^2_{\alpha+j}(0,\infty)$ be the Hilbert space defined by

$$L^2_{\alpha+n}(0,\infty) := \left\{ f : (0,\infty) \to \mathbb{C} \mid \int_0^\infty |f|^2 t^{\alpha+n} e^{-t} dt < \infty \right\}.$$

Then, the nth left-definite space is defined as

$$V_n := \left\{ f : (0,\infty) \to \mathbb{C} |\ f \in AC^{(n-1)}_{\text{loc}}(0,\infty);\ f^{(n)} \in L^2_{\alpha+n}(0,\infty) \right\}.$$

Main Goal

Question: In what ways can we describe these left-definite spaces as boundary conditions imposed on the original Hilbert space, rather than in terms of integrability conditions?

Motivation

A simple analog of this problem is seen in Sobolev spaces:

$$W^{1,2}(\Omega) = H^1(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha(u) \in L^2(\Omega) \ \ \forall |\alpha| < 1\right\}$$

has the important subspace

$$H_0^1(a,b) = \left\{f \in C[a,b] : f' \in L^2(a,b), f(a) = f(b) = 0\right\}.$$

Our goal is to similarly describe the left-definite domains in terms of boundary conditions.

- Let ℓ be a symmetric (differential) operator on a Hilbert space $\mathcal{H}$. We'll assume that ℓ can be defined on some maximum subspace of $\mathcal{H}$ called the maximal domain, $\mathcal{D}$.

Overview of Extension Theory

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- The spaces

$$\mathcal{D}_+ = \{f \in \mathcal{D} \mid \ell^* f = if\}$$

$$\mathcal{D}_- = \{f \in \mathcal{D} \mid \ell^* f = -if\}$$

are called the defect spaces. The deficiency indices of ℓ are (dim$(\mathcal{D}_+)$,dim$(\mathcal{D}_-)$).

Notable Facts from Extension Theory

- We define a sesquilinear form on the maximal domain by

$$[f,g]\big|_a^b := \int_a^b \ell[f(x)]\overline{g(x)}dx - \int_a^b \ell[\overline{g(x)}]f(x)dx.$$

- The minimal domain is

$$\mathcal{D}_0 := \{f \in \mathcal{D} \mid [f,g]\big|_a^b = 0 \ \forall g \in \mathcal{D}\}$$

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Theorem (Naimark)

The limits $[f,g](b) := \lim_{x\to b^-}[f,g](x)$ *and* $[f,g](a) := \lim_{x\to a^+}[f,g](x)$ *both exist and are finite for all* $f,g \in \mathcal{D}(\ell)$.

Notable Facts from Extension Theory

- The minimal and maximal operators are adjoints of one another.

Theorem (von Neumann)

Let $\mathcal{D}$ and $\mathcal{D}_0$ be the maximal and minimal domains associated with the differential operator T respectively.

$$\mathcal{D} = \mathcal{D}_0 \dotplus \mathcal{D}_+ \dotplus \mathcal{D}_-$$

Linear Independence Modulo a Vector Space

Definition

Let X_1 and X_2 be subspaces of a vector space X such that $X_1 \subset X_2$. Let $\{x_1, x_2, \ldots, x_n\} \subseteq X_2$. We say that $\{x_1, x_2, \ldots, x_n\}$ is **linearly independent modulo** X_1 if

$$\sum_{i=1}^{n} \alpha_i x_i \in X_1 \text{ implies } \alpha_i = 0,\ i = 1, 2, \ldots n.$$

Glazman–Krein–Naimark Theory

Theorem (GKN)

The domain of definition $\mathcal{D}_L$ *of an arbitrary self-adjoint extension* L *of the operator* L_0 *with the deficiency indices* (m, m) *consists of the set of all functions* $y(x) \in \mathcal{D}$ *which satisfy the conditions*

$$[y, w_k]\Big|_a^b = 0, \; k = 1, 2, \ldots, m \tag{1}$$

where $w_1, \ldots, w_m$ *are certain functions belonging to* $\mathcal{D}$ *and determined by* L *which are linearly independent modulo* $\mathcal{D}_0$ *and for which the relations*

$$[w_j, w_k]\Big|_a^b = 0, \; j, k = 1, 2, \ldots, m \tag{2}$$

hold.

More Intuition

- Self-adjoint extensions of the minimal operator with deficiency indices (m, m) can be put into one-to-one correspondence with the unitary $m \times m$ matrices.
- These unitary matrices act via $u = [u_{jk}] : \mathcal{D}_+ \to \mathcal{D}_-$.
- This defines a self-adjoint extension that includes all functions of the type

$$y(x) = y_0(x) + \sum_{k=1}^{m} u_{kj}\overline{\varphi_k(x)}, \;\; \mu = 1, \ldots, m,$$

where $y_0(x) \in \mathcal{D}_0$, and $\overline{\varphi_k}$ is a basis vector of $\mathcal{D}_-$.

History

- Left-definite theory has been used to analyze a wide range of classical differential operators, including Hermite, Laguerre, Legendre, Fourier, etc.
- GKN boundary conditions for the case $n = 1$ are available for several of these, including Legendre.

History

- Left-definite theory has been used to analyze a wide range of classical differential operators, including Hermite, Laguerre, Legendre, Fourier, etc.
- GKN boundary conditions for the case $n = 1$ are available for several of these, including Legendre.
- The square of the Legendre operator was recently studied and published by Littlejohn and Wicks.
- Cases involving higher values of n are unknown for any classical differential operator.

The Legendre Differential Operator

- Consider the Legendre differential operator on the Hilbert space $L^2(-1,1)$, given by

$$\ell[y](x) = -((1-x^2)y'(x))',$$

which possesses the Legendre polynomials $P_m(x)$, $m \in \mathbb{N}_0$, as eigenfunctions.

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which possesses the Legendre polynomials $P_m(x)$, $m \in \mathbb{N}_0$, as eigenfunctions.

- The polynomial P_m is a solution of the eigenvalue equation

$$\ell[y](x) = m(m+1)y(x).$$

- Let $n = 3$. The deficiency indices of $\ell^3[\cdot]$ are $(3,3)$.

Linear Independence Modulo $\mathcal{D}_0$

- To show that the set $w_1, w_2, \dots, w_6$ is linearly independent modulo the minimal domain $\mathcal{D}_0$, suppose that

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$$\sum_{k=1}^{6} \alpha_k w_k \in \mathcal{D}_0.$$

- By definition, $y \in \mathcal{D}_0$ if and only if $[y, w]_3|_{-1}^{1} = 0$ for all $w \in \mathcal{D}$. Hence

$$\sum_{k=1}^{6} \alpha_k [w_i, w_k]_3|_{-1}^{1} = 0 \text{ for } i = 1, \ldots, 6.$$

  The goal is to show that the only way this can happen is if $\alpha_k = 0$, $k = 1, \ldots, 6$.

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where $w_1, \dots, w_m$ *are certain functions belonging to* $\mathcal{D}$ *and determined by* L *which are linearly independent modulo* $\mathcal{D}_0$ *and for which the relations*

$$[w_j, w_k]\Big|_a^b = 0, \ \ j, k = 1, 2, \dots, m \tag{4}$$

hold.

The problem of showing that $\alpha_k = 0$, $k = 1, \ldots, 6$, is then equivalent to showing the following matrix, M_3, has full rank.

$$\left[\begin{array}{ccc|ccc}
[w_1, w_1]_3 & [w_1, w_2]_3 & [w_1, w_3]_3 & [w_1, w_4]_3 & [w_1, w_5]_3 & [w_1, w_6]_3 \\
[w_2, w_1]_3 & [w_2, w_2]_3 & [w_2, w_3]_3 & [w_2, w_4]_3 & [w_2, w_5]_3 & [w_2, w_6]_3 \\
[w_3, w_1]_3 & [w_3, w_2]_3 & [w_3, w_3]_3 & [w_3, w_4]_3 & [w_3, w_5]_3 & [w_3, w_6]_3 \\
\hline
[w_4, w_1]_3 & [w_4, w_2]_3 & [w_4, w_3]_3 & [w_4, w_4]_3 & [w_4, w_5]_3 & [w_4, w_6]_3 \\
[w_5, w_1]_3 & [w_5, w_2]_3 & [w_5, w_3]_3 & [w_5, w_4]_3 & [w_5, w_5]_3 & [w_5, w_6]_3 \\
[w_6, w_1]_3 & [w_6, w_2]_3 & [w_6, w_3]_3 & [w_6, w_4]_3 & [w_6, w_5]_3 & [w_6, w_6]_3
\end{array}\right]$$

Candidates for Basis Functions

- There are solutions to the Legendre eigenvalue equation that aren't the Legendre polynomials.
- They are often referred to as Legendre functions of the second kind, and are in the maximal domain. They are not orthogonal to each other, and become infinite at the endpoints. For instance,

$$Q_0 = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$
$$Q_1 = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

Simplification of Entries

- There is an easy way to simplify these matrix entries using Green's Formula:

$$
\begin{aligned}
[P_j, Q_k]_3\Big|_{-1}^{1} &= \int_{-1}^{1} \ell^3[P_j]Q_k dx - \int_{-1}^{1} \ell^3[Q_k]P_j dx \\
&= j^3(j+1)^3 \int_{-1}^{1} P_j Q_k dx - k^3(k+1)^3 \int_{-1}^{1} Q_k P_j dx \\
&= [j^3(j+1)^3 - k^3(k+1)^3]\langle P_j, Q_k\rangle.
\end{aligned}
$$

- Formulas for $\langle P_j, Q_k\rangle$ are known for $j, k \in \mathbb{N}$.

Structure of M_n

Proposition (FFL)

The matrix M_n is antisymmetric and takes the form

$$M_n = \left[\begin{array}{c|c} \mathbf{0} & B_n \\ \hline -B_n^T & C_n \end{array}\right]$$

so that the $\det(M_n) = \det(B_n^T B_n) = [\det(B_n)]^2$.

The case $n = 3$

- The following matrix is the result of setting $w_1 = P_0$, $w_2 = P_1$, $w_3 = P_2$, $w_4 = Q_1$, $w_5 = Q_2$, and $w_6 = Q_3$:

$$M_3 = \left[\begin{array}{ccc|ccc} & & & 8 & 0 & 288 \\ & \mathbf{0} & & 0 & 104 & 0 \\ & & & 104 & 0 & 504 \\ \hline -8 & 0 & -104 & 0 & 0 & \frac{860}{3} \\ 0 & -104 & 0 & 0 & 0 & 0 \\ -288 & 0 & -504 & \frac{-860}{3} & 0 & 0 \end{array}\right].$$

- The functions P_0, P_1, and P_2 can hence be used as GKN conditions to generate the left-definite space V_6.

The case $n = 4$

- Choose the basis functions to be $P_0, \ldots, P_3$ and $Q_0, \ldots, Q_3$.

$$B_4 = \begin{pmatrix} 0 & 16 & 0 & 3456 \\ 16 & 0 & 640 & 0 \\ 0 & 640 & 0 & 6480 \\ 3456 & 0 & 6480 & 0 \end{pmatrix}.$$

- The matrix B_n is symmetric when n is even, so showing it has full rank is even easier.

General Basis Functions

Theorem (FFL)

Let M_n define a basis of functions for the space $\mathcal{D}_+^n \oplus \mathcal{D}_-^n$, containing n Legendre polynomials and n Legendre functions of the second kind, all distinct.
Then the total number of even indices for the collection of functions is equal to the number of odd indices.

How far can we go?

Example

Let $n = 4$ and choose the functions $P_{17}, P_{42}, P_{49}, P_{125}$ and $Q_{24}, Q_{82}, Q_{97}, Q_{178}$ as candidates for the basis vectors. The relevant matrix can be computed to be:

$$B_4 = \begin{pmatrix} 8e8 & 6602e8 & 0 & 653190e8 \\ 0 & 0 & 21181e8 & 0 \\ 388e8 & 9686e8 & 0 & 700781e8 \\ 81234e8 & 132802e8 & 0 & 1202916e8 \end{pmatrix}.$$

It can be shown that even this matrix possesses full rank.

Abstract Approach

Theorem (FFL)

Let $\mathcal{L}$ be a self-adjoint operator defined by left-definite theory on $L^2[(a,b),W(x)]$ with domain that includes a complete orthogonal system of eigenfunctions, $\{P_k\}_{n=0}^{\infty}$. Furthermore, let $\mathcal{L}_0$ have deficiency indices (m,m).
Then, the Glazman–Krein–Naimark boundary conditions for the self-adjoint operator $\mathcal{L}$ are given by some $\{P_{k_1},\ldots,P_{k_m}\}$.

The Four Domains

$$\mathcal{A}_n := \Big\{ f : (a,b) \to \mathbb{C} \ \Big| \ f, f', \dots, f^{(2n-1)} \in AC_{\text{loc}}(a,b); \\ (p(x))^n f^{(2n)} \in L^2[(a,b), w] \Big\},$$

$$\mathcal{B}_n := \left\{ f \in \mathcal{D}^n_{\max} \ \Big| \ [f, P_j]_n \Big|_a^b = 0 \text{ for } j = 0, 1, \dots, n-1 \right\},$$

$$\mathcal{C}_n := \left\{ f \in \mathcal{D}^n_{\max} \ \Big| \ [f, P_j]_n \Big|_a^b = 0 \text{ for any } n \text{ distinct } j \in \mathbb{N} \right\}, \text{ and}$$

$$\mathcal{F}_n := \left\{ f \in \mathcal{D}^n_{\max} \ \Big| \ (a_j(x) y^{(j)}(x))^{(j-1)} \Big|_a^b = 0 \text{ for } j = 1, 2, \dots, n \right\}.$$

Main Result

Theorem (FFL)

Let $\mathbf{L}^n$ be a self-adjoint operator defined by left-definite theory on $L^2[(a,b),w]$. Let $\mathbf{L}^n$ operate on its domain, $\mathcal{D}^n_{\mathbf{L}}$, via $\ell^n[\,\cdot\,]$. Let ℓ be a classical Jacobi or Laguerre differential expression. Assume $\mathcal{A}_n = \mathcal{B}_n$ and that $f \in \mathcal{F}_n$ implies that $f'', \ldots, f^{(2n-2)} \in L^2[(a,b),dx]$.
Then $\mathcal{A}_n = \mathcal{B}_n = \mathcal{C}_n = \mathcal{F}_n = \mathcal{D}^n_{\mathbf{L}}$ $\forall n \in \mathbb{N}$.

Thank you!