Asymptotics of Hankel determinants with a one-cut regular potential and Fisher-Hartwig singularities

Christophe Charlier

Université catholique de Louvain

August, 2017

Hankel matrices and determinants

If w is a function defined on the real line which possesses finite moments

$$w_k = \int_{-\infty}^{+\infty} w(x)x^k dx, \qquad k = 0, 1, 2, ...,$$

then one can associate a Hankel matrix $H_n(w)$ to it:

$$H_n(w) = \begin{pmatrix} w_0 & w_1 & \dots & w_{n-1} \\ w_1 & w_2 & \dots & w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_{n-1} & w_n & \dots & w_{2n-2} \end{pmatrix}.$$

Its associated Hankel determinant will be denoted by
$D_n(w) = \det(H_n(w))$.

Weight of our interest

In this talk we focus on Hankel determinants associated to a weight w of the form

$$w(x) = e^{-nV(x)} e^{W(x)} \omega(x), \qquad \omega(x) = \prod_{j=1}^{m} \omega_{\alpha_j}(x) \omega_{\beta_j}(x), \qquad m \in \mathbb{N},$$

and for each $k \in \{1, ..., m\}$, we have

$$\omega_{\alpha_k}(x) = |x - t_k|^{\alpha_k}, \qquad \omega_{\beta_k}(x) = \begin{cases} e^{i\pi\beta_k}, & \text{if } x < t_k, \\ e^{-i\pi\beta_k}, & \text{if } x > t_k, \end{cases}$$

and $\Re \alpha_k > -1$.

Weight of our interest

The weight depends on

- $m \in \mathbb{N}$ and $n \in \mathbb{N}$,
- $t_1, ..., t_m \in \mathbb{R}$,
- $\alpha_1, ..., \alpha_m \in \{z \in \mathbb{C} : \Re z > -1\}$,
- $\beta_1, ..., \beta_m \in \mathbb{C}$,
- W continuous such that $W(x) = \mathcal{O}(V(x))$ as $|x| \to +\infty$,
- The potential V which satisfies $\lim_{x\to\pm\infty} V(x)/\log|x| = +\infty$.

Notation: we will omit the dependence in m and in $t_1, ..., t_m$ and simply denote $D_n(\alpha, \beta, V, W)$ for the Hankel determinant.

Equilibrium measure

The potentials V we are interested in are described in terms of properties of the equilibrium measure μ_V, which is the unique minimizer of the functional

$$\iint \log|x-y|^{-1} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

among all Borel probability measures μ on $\mathbb{R}$. This measure and its support (denoted $\mathcal{S}$) are completely characterized by the Euler-Lagrange variational conditions (Saff-Totik 1997)

$$2\int_{\mathcal{S}} \log|x-s| d\mu_V(s) = V(x) - \ell, \qquad \text{for } x \in \mathcal{S},$$

$$2\int_{\mathcal{S}} \log|x-s| d\mu_V(s) \leq V(x) - \ell, \qquad \text{for } x \in \mathbb{R} \setminus \mathcal{S}.$$

One-cut regular potentials

A potential V is called one-cut regular if

- $V : \mathbb{R} \to \mathbb{R}$ is analytic.
- $\lim_{x \to \pm\infty} V(x)/\log|x| = +\infty$.
- The Euler-Lagrange inequality is strict.
- The equilibrium measure is supported on $\mathcal{S} = [a, b]$ and is of the form $d\mu_V(x) = \psi(x)\sqrt{(b-x)(x-a)}dx$, where ψ is positive on $[a, b]$.

Without loss of generality, we restrict ourself to the class of one-cut regular potentials whose equilibrium measure is supported on $[-1, 1]$ instead of $[a, b]$.

Example

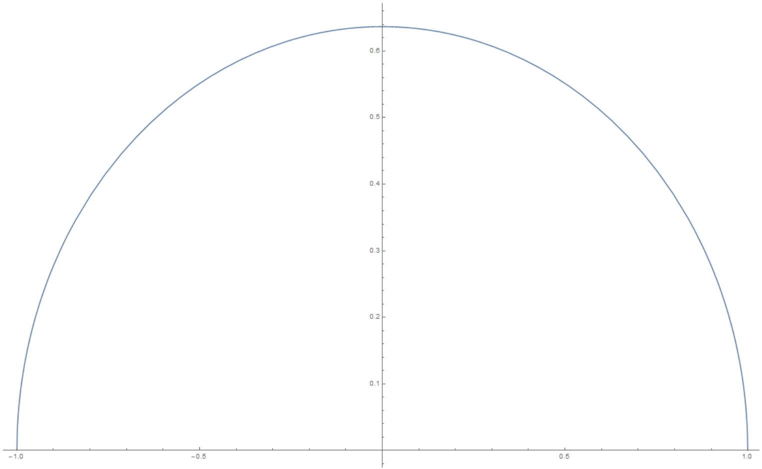


Figure: If $V(x) = 2x^2$, the associated equilibrium measure is given by $d\mu_V(x) = \frac{2}{\pi}\sqrt{1 - x^2}dx$.

Known results for $\alpha = \beta = 0$

- If V is a polynomial, is one-cut regular and such that all zeros of $\psi(x)$ are nonreal, Johansson (1998) obtained rigorously large n asymptotics of $\frac{D_n(0,0,V,W)}{D_n(0,0,V,0)}$.
- For a polynomial one-cut regular potential V, large n asymptotics for $D_n(0,0,V,0)$ have been obtained via the Riemann-Hilbert method by Ercolani-McLaughlin (2003) (if the coefficients of V are sufficiently small), and via deformation equations by Bleher-Its (2005) (under further technical assumptions on V).

Known results for $\alpha \neq 0$

- Large n asymptotics of $D_n(\alpha, 0, 2x^2, 0)$ have been obtained by Krasovsky (2007).

- This result was recently generalized for the class of one-cut regular potentials by Berestycki, Webb and Wong (2017). They obtain large n asymptotics of $D_n(\alpha, 0, V, W)$.

Known results for $\beta \neq 0$

Only limited results are available concerning Hankel determinants with jump discontinuities.

- Large n asymptotics of $D_n(0, \beta_1, 2x^2, 0)$ have been obtained by Its-Krasovsky (2008) with $m = 1$.

Theorem (C '17)

Let $m \in \mathbb{N}$, and let t_j, α_j and β_j be such that

- $t_j \in (-1, 1)$, $t_j \neq t_k$ *for* $1 \leq j \neq k \leq m$,
- $\Re\alpha_j > -1$ *and* $\Re\beta_j \in (\frac{-1}{4}, \frac{1}{4})$, *for* $j = 1, ..., m$.

Let V and W be such that

- *V is a one-cut regular potential whose equilibrium measure is supported on $[-1, 1]$ with density $\psi(x)\sqrt{1 - x^2}$,*
- *$W : \mathbb{R} \to \mathbb{R}$ is analytic in a neighbourhood of $[-1, 1]$, locally Hölder-continuous on $\mathbb{R}$ and such that* $W(x) = \mathcal{O}(V(x))$, *as* $|x| \to \infty$.

As $n \to \infty$, we have

$$\log D_n(\alpha, \beta, V, W) = C_1 n^2 + C_2 n + C_3 \log n + C_4 + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right),$$

with $\beta_{\max} = \max\{|\Re\beta_1|, ..., |\Re\beta_m|\}$ *and*

$$C_1 = -\log 2 - \frac{3}{4} - \frac{1}{2}\int_{-1}^{1} \sqrt{1 - x^2}(V(x) - 2x^2)\left(\frac{2}{\pi} + \psi(x)\right) dx,$$

...

Theorem

$$C_2 = \log(2\pi) - \mathcal{A}\log 2 - \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{V(x)-2x^2}{\sqrt{1-x^2}}dx + \int_{-1}^{1}\psi(x)\sqrt{1-x^2}W(x)dx$$
$$+\sum_{j=1}^{m}\left(\frac{\alpha_j}{2}(V(t_j)-1) + \pi i\beta_j\left(1-2\int_{t_j}^{1}\psi(x)\sqrt{1-x^2}dx\right)\right),$$

$$C_3 = -\frac{1}{12} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right),$$

$$C_4 = \zeta'(-1) + \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}dx - \frac{1}{4\pi^2}\int_{-1}^{1}\frac{W(y)}{\sqrt{1-y^2}}\left(⨍_{-1}^{1}\frac{W'(x)\sqrt{1-x^2}}{x-y}dx\right)dy$$
$$-\frac{1}{24}\log\left(\frac{\pi^2}{4}\psi(1)\psi(-1)\right) + \sum_{1\le j<k\le m}\log\left(\frac{\left(1-t_jt_k-\sqrt{(1-t_j^2)(1-t_k^2)}\right)^{2\beta_j\beta_k}}{2^{\frac{\alpha_j\alpha_k}{2}}|t_j-t_k|^{\frac{\alpha_j\alpha_k}{2}+2\beta_j\beta_k}}\right)$$
$$+\sum_{j=1}^{m}\left(i\mathcal{A}\beta_j\arcsin t_j - \frac{i\pi}{2}\beta_j\mathcal{A}_j + \log\frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\alpha_j)}\right)$$
$$+\sum_{j=1}^{m}\left(\left(\frac{\alpha_j^2}{4}-\beta_j^2\right)\log\left(\frac{\pi}{2}\psi(t_j)\right) - \frac{\alpha_j}{2}W(t_j) + i\frac{\beta_j}{\pi}\sqrt{1-t_j^2}⨍_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}(t_j-x)}dx\right)$$
$$+\sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}-3\beta_j^2\right)\log\left(2\sqrt{1-t_j^2}\right),$$

where G is Barnes' G-function, ζ is Riemann's zeta-function and where we use the notations

$$\mathcal{A} = \sum_{j=1}^{m}\alpha_j, \qquad \mathcal{A}_j = \sum_{l=1}^{j-1}\alpha_l - \sum_{l=j+1}^{m}\alpha_l.$$

Applications in random matrix theory

Random matrix ensembles

Consider the set of $n \times n$ Hermitian matrices M endowed with the probability distribution

$$\frac{1}{\widehat{Z}_n} e^{-n \mathrm{Tr} V(M)} dM, \qquad dM = \prod_{i=1}^{n} dM_{ii} \prod_{1 \le i < j \le n} d\Re M_{ij} d\Im M_{ij},$$

where $\widehat{Z}_n$ is the normalisation constant. This distribution of matrices is invariant under unitary conjugations and induces a probability distribution on the eigenvalues $x_1, ..., x_n$ of M which is of the form

$$\frac{1}{n! Z_n} \prod_{1 \le j < k \le n} (x_k - x_j)^2 \prod_{j=1}^{n} e^{-nV(x_j)} dx_j, \qquad (x_1, ..., x_n) \in \mathbb{R}^n,$$

where Z_n is the partition function.

Applications in random matrix theory if $\alpha = \beta = 0$

- Central limit theorem for the linear statistics:

$$\mathbb{E}\left(e^{\sum_{i=1}^{m} W(x_i)}\right) = \frac{D_n(0,0,V,W)}{D_n(0,0,V,0)}$$

- The partition function is given by $Z_n = D_n(0,0,V,0)$.

Applications in random matrix theory if $\alpha \neq 0$

- Hankel determinants with root-type singularities are related to the statistical properties of the characteristic polynomial $p_n(t) = \prod_{j=1}^{n}(t - x_j)$.

$$\mathbb{E}_{\mathrm{GUE}}\left(\prod_{j=1}^{m}|p_n(t_j)|^{\alpha_j}\right) = \frac{D_n(\alpha, 0, 2x^2, 0)}{D_n(0, 0, 2x^2, 0)}.$$

- B-W-W proved that a sufficiently small power of the absolute value of the characteristic polynomial $p_n(t)$ of a one-cut regular ensemble converges in distribution to a Gaussian multiplicative chaos measure. It was crucial in their analysis to obtain the large n asymptotics of

$$\mathbb{E}_V\left(\prod_{j=1}^{n}e^{W(x_j)}\prod_{j=1}^{m}|p_n(t_j)|^{\alpha_j}\right) = \frac{D_n(\alpha, 0, V, W)}{D_n(0, 0, V, 0)}.$$

Applications in random matrix theory if $\beta \neq 0$

Such determinants appear when we thin the eigenvalues of a random matrix.
We start from the complete spectrum of a matrix:

$$\{x_1, x_2, \ldots, x_n\}$$

We thin the spectrum by deleting each eigenvalue with a given probability $s \in [0, 1]$.

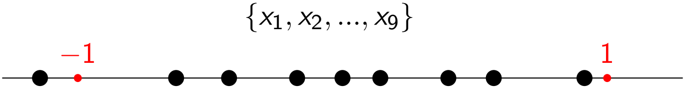
We are left with an incomplete spectrum

$$\{y_1, y_2, \ldots, y_m\},$$

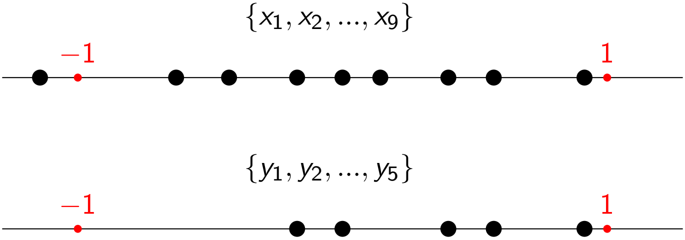
where m is now itself a random variable following the Binomial distribution $B(n, 1-s)$.

Thinning was introduced in random matrice theory by *Bohigas-Pato '04*.

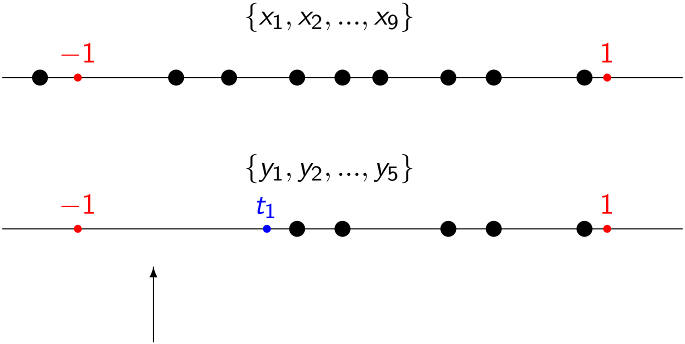
Constant thinning



Constant thinning

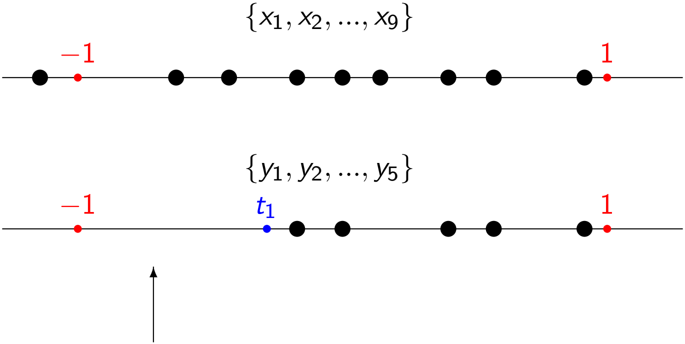


Constant thinning



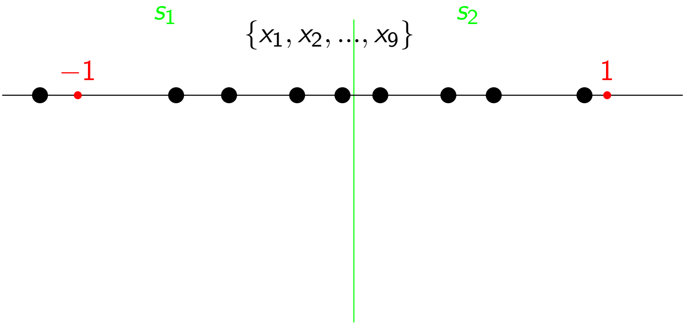
What is $\mathbb{P}\big(\sharp\{y_i : y_i < t_1\} = 0\big)$?

Constant thinning

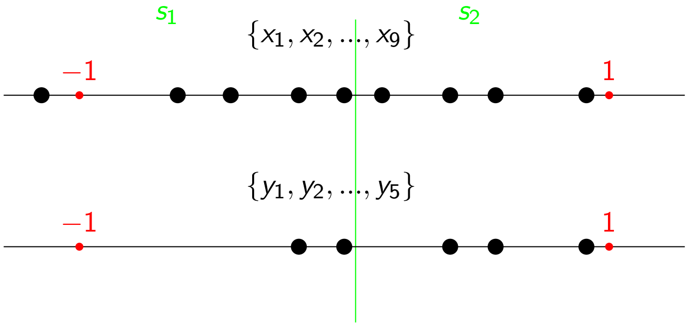


$$\mathbb{P}\left(\sharp\{y_i : y_i < t_1\} = 0\right) = e^{in\pi\beta_1} \frac{D_n(0,\beta_1,2x^2,0)}{D_n(0,0,2x^2,0)}, \text{ with } \beta_1 = \frac{\log s}{2\pi i} \in i\mathbb{R}^+.$$

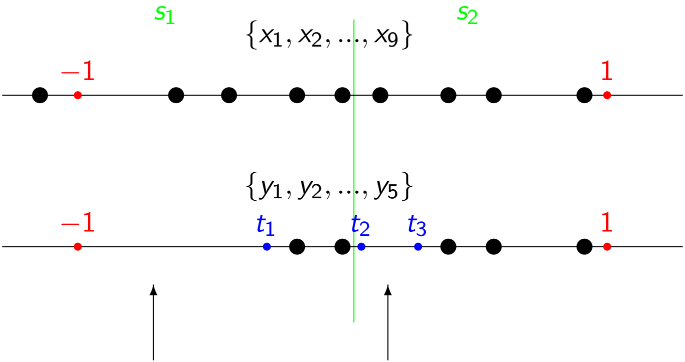
Piecewise constant thinning



Piecewise constant thinning

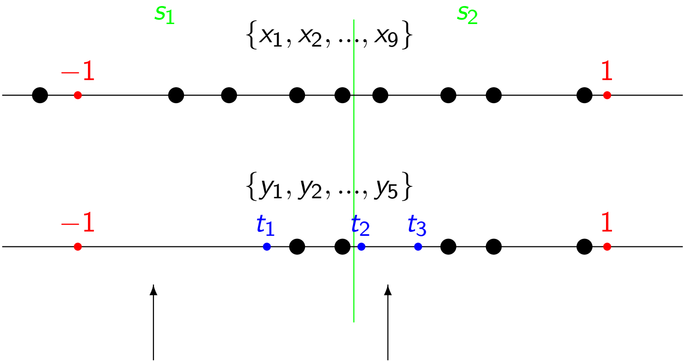


Piecewise constant thinning



What is $\mathbb{P}\big(\sharp\{y_i : y_i \in (-\infty, t_1) \cup (t_2, t_3)\} = 0\big)$?

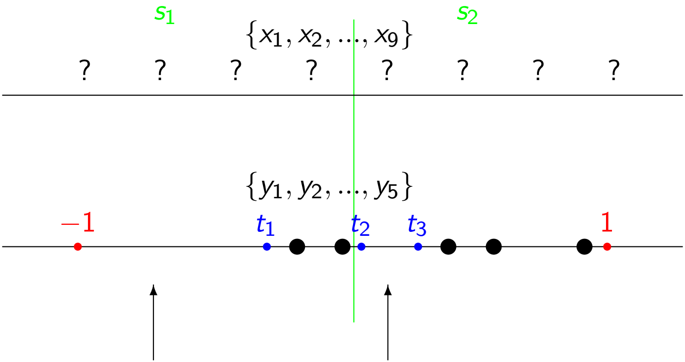
Piecewise constant thinning



$$\mathbb{P}\Big(\sharp\{y_i : y_i \in (-\infty, t_1) \cup (t_2, t_3)\} = 0\Big) = \sqrt{s_1 s_2}\frac{D_n(0,\beta,V,0)}{D_n(0,0,V,0)},$$

where $\beta_1 = \frac{\log s_1}{2\pi i}$, $\beta_2 = \frac{-\log s_2}{2\pi i}$ and $\beta_3 = \frac{\log s_2}{2\pi i}$.

Conditioning on a gap in the thinned spectrum



Suppose that $\sharp\{y_i : y_i \in (-\infty, t_1) \cup (t_2, t_3)\} = 0$,

what can you say on $\{x_1, ..., x_n\}$?

Conditional point process

From Bayes' formula for conditional probabilities, the conditional point process follows the distribution

$$\frac{1}{n!\widetilde{Z}_n} \prod_{1 \le i < j \le n} (x_j - x_i)^2 \prod_{j=1}^{n} \widetilde{w}(x_j) dx_j, \qquad \widetilde{Z}_n = D_n(0, \beta, V, 0),$$

where $\widetilde{w}(x) = e^{-nV(x)} \prod_{j=1}^{m} \omega_{\beta_j}(x)$. Thus, the generalized correlations of the characteristic polynomial of the conditional point process is expressed as

$$\mathbb{E}\left(\prod_{i=1}^{n} e^{W(x_i)} \prod_{j=1}^{m} |p(t_j)|^{\alpha_j} \right) = \frac{D_n(\alpha, \beta, V, W)}{D_n(0, \beta, V, 0)}.$$

Outline of the proof

We compute the asymptotics for $D_n(\alpha, \beta, V, W)$ in three steps which can be schematized as

$$D_n(\alpha, 0, 2x^2, 0) \mapsto D_n(\alpha, \beta, 2x^2, 0) \mapsto D_n(\alpha, \beta, V, 0) \mapsto D_n(\alpha, \beta, V, W).$$

Each of these steps is subdivided into three parts:

- a differential identity for $\log D_n(\alpha, \beta, V, W)$,
- an asymptotic analysis of a Riemann-Hilbert(RH) problem,
- the integration of the differential identity.

Proof based on orthogonal polynomials

It is known (Szegő 1959) that $D_n(\alpha, \beta, V, W)$ can be expressed in terms of orthogonal polynomials p_k, defined through

$$\int_{\mathbb{R}} p_k(x)p_j(x)w(x)dx = \delta_{jk}, \qquad j = 0, 1, ..., k,$$

and $\kappa_k > 0$ is the leading coefficient of p_k.

Proof based on a Riemann-Hilbert problem

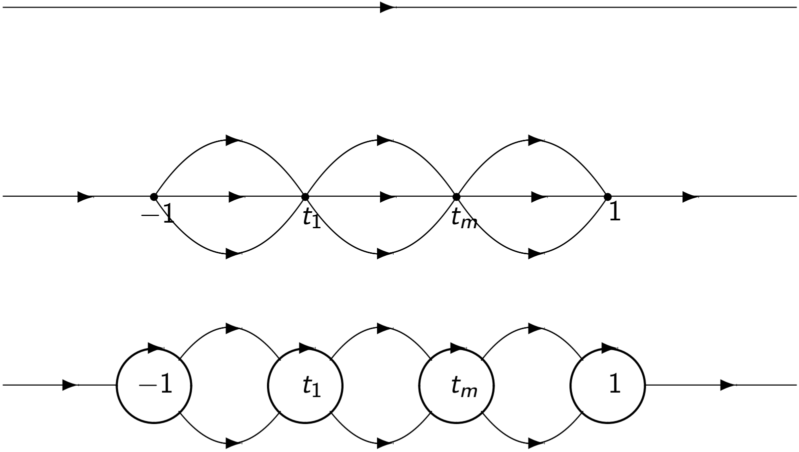
Consider the matrix valued function Y, defined by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \frac{\kappa_n^{-1}}{2\pi i} \int_{\mathbb{R}} \frac{p_n(x)w(x)}{x-z} dx \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathbb{R}} \frac{p_{n-1}(x)w(x)}{x-z} dx \end{pmatrix}.$$

It is known (Fokas-Its-Kitaev) that Y can be characterized as the unique solution of a boundary value problem for analytic functions, called RH problem for Y.

Steepest descent method on Y

$$\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$$



Thank you for your attention